Theory of Probability: The Project

Experimental part

Investigate how the shuffle function works on a digital music player. (All the details are up to you.)

Theoretical part

Problem 1. Let V be a random variable uniformly distributed on the unit sphere. That is, V is the random variable taking values on the surface of the unit ball and the probability that V takes the value in a particular region on the surface is proportional to the area of that region. In the Cartesian coordinates, we have $V = (X, Y, Z)$, where X, Y, Z are random variables and $X^2 + Y^2 + Z^2 = 1$.

(a) Find the distribution of X.

(b) Find the joint distribution of X and Y.

(c) Use computer to generate a thousand points that are random, independent, and uniform on the unit sphere, and print the resulting picture.

Problem 2. Let $U_1, U_2, \dots$ be independent random variables, all having the same distribution F, and let $x > 0$ be a real number. Define the random variable $N(x)$ as the smallest value of n such that $\sum_{k=1}^{n} U_k > x$. Find the expected value of $N(x)$, when the distribution F is

(1) Uniform on $(0, L)$, $L > 0$;
(2) Exponential with parameter $\lambda > 0$;
(3) Gamma distribution with parameters α and λ;
(4) Beta distribution with parameters p, q;
(5) Normal distribution with mean μ and variance σ.

For each distribution, start by getting an approximate graph of the function $y = N(x)$ using Monte-Carlo simulations. Then see if you can get a closed-form analytic expression for $N(x)$.

Problem 3. Describe a procedure for distinguishing a randomly generated sequence of zeroes and ones from a cooked-up one.

Problem 4. Here, you can establish the connection between two famous equalities:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} dx = 1$$

and

$$\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots 2n \cdot 2n \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)(2n+1) \cdots};$$

the second is known as the *Wallis formula.* If you know one equality, you can easily derive the other. The key relation is

$$\int_0^{\sqrt{n}} \left(1 - \frac{x^2}{n}\right)^n dx = \sqrt{n} \int_0^{\pi/2} \cos^{2n+1} t \, dt.$$

Your task is to fill in the details.