August 15, 2009

Probability Theory: Homework problems

**Homework 1.** Solve each problem and write 1-2 variations on a problem of your choice.

1. A "traditional" three-digit telephone area code is constructed as follows. The first digit is from the set $\{2, 3, 4, 5, 6, 7, 8, 9\}$, the second is either 0 or 1, the last is from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. (a) How many area codes like this are possible? (b) How many such area codes start with 5?

2. In how many ways can three novels, two mathematics books and one chemistry book be arranged on a shelf if (a) any arrangement is allowed; (b) math books must be together and the novels must be together; (c) only the novels must be together.

3. Seven different gifts are distributed among 10 children so that no child gets more than one gift. How many different outcomes are possible.

4. Consider the two-dimensional Cartesian (standard rectangular $x, y$) coordinate system on the plane. You move around the points with integer coordinates in such a way that, at each step you can go either one unit up or one unit to the right (that is, from $(0, 0)$ you can go to either $(1, 0)$ or $(0, 1)$; from $(2, 3)$ you can go either to $(3, 3)$ or $(2, 4)$, etc.) Count the number of different paths from the point $(0, 0)$ to (a) the point $(4, 3)$; (b) the point $(4, 3)$, if you have to visit the point $(2, 2)$; (c) the point $(4, 4)$, if you are not allowed to go above the line $x = y$ (but you are allowed to hit the line, e.g. by visiting point the $(1, 1)$). (Note: I was lazy to draw pictures: it takes too much time in LaTeX, but you should draw the pictures by hand as part of your solution.)

5. You have \$20K to invest, and have a choice of stocks, bonds, mutual funds, or a CD. Investments must be made in multiples of \$1K, and there are minimal amounts to be invested: \$2K in stocks, \$2K in bonds, \$3K in mutual funds, and \$4K in the CD. Count the number of choices in each situation: (a) You want to invest in all four; (b) You want to invest in at least three out of four.

6. Two dice are rolled. Introduce the following events:

(1) $E$ : "the sum is odd"
(2) $F$ : "at least one number is 1"
(3) $G$ : "the sum is 5"

List the elementary outcomes in each of the following events: $EF$, $E \bigcup F$, $FG$, $EF^c$, $EFG$. For this problem, would you care whether the dice are fair?

**Homework 2.** Solve each problem and write 1-2 variations on a problem of your choice.

1. At a certain school, 60% of the students wear neither a ring nor a necklace, 20% wear a ring, 30% wear a necklace. Compute the probability that a randomly selected student wears (a) a ring OR a necklace; (b) a ring AND a necklace.

2. A school offers three language classes: Spanish (S), French (F), and German (G). There are 100 students total, of which 28 take S, 26 take F, 16 take G, 12 take both S and F, 4 take both S and G, 6 take both F and G, and 2 take all three languages.

(1) Compute the probability that a randomly selected student (a) is not taking any of the three language classes; (b) takes EXACTLY one of the three language classes.
(2) Compute the probability that, of two randomly selected students, at least one takes a language class.

3. Two fair dice are rolled. Compute the probability that the number on the first is smaller that the number on the second.

4. Alice and Bob are in a group of $N$ people who are randomly arranged (a) in a row (b) in a circle. In each case, compute the probability that Alice and Bob are next to each other.

5. Four married couples are arranged in a row. Compute the probability that no wife is next to her husband.

6. Two fair dice are rolled. For $i = 2, 3, \ldots, 12$, compute the probability that the first one shows 6 given that the sum is $i$. (Of course, it is zero if $i < 7$).

7. Let $A$ and $B$ be two events such that $P(A) = 0.5$ and $P(B) = 0.6$. (a) Can $A$ and $B$ be mutually exclusive? (b) Assuming that $A$ and $B$ are independent, find $P(A \bigcup B)$.

**Homework 3.** Solve each problem and write 1-2 variations on a problem of your choice.

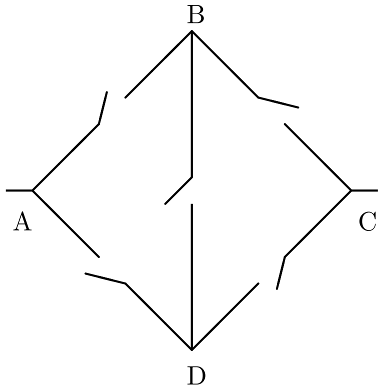


FIGURE 1. A random connection

1. In a certain community, 36% of all the families have a dog and 30% have a cat. Of those families with a dog, 22% also have a cat. Compute the probability that a randomly selected family (a) has both a dog and a cat; (b) has a dog GIVEN that it has a cat.

2. Three fair dice have different colors: red, blue, and yellow. These three dice are rolled and the face value of each is recorded as $R, B, Y$, respectively. Compute the probability that $B < Y < R$. You can proceed as follows: (a) compute the probability that no two dice show the same number; (b) compute the probability that $B < Y < R$ given that all the numbers are different; (c) solve the problem.

3. Suppose that 5% of males and 0.25% of females are colorblind. Compute the probability that a randomly selected colorblind person is male if (a) the proportion of males and females in the population is that same; (b) there are twice as many females as males in the population.

4. English and American spellings are *rigour* and *rigor*, respectively. At a certain hotel, 40% of guests are from England and the rest are from America. A guest at the hotel writes the word (as either *rigour* or *rigor*), and a randomly selected letter from the spelling turns out to be a vowel. Compute the probability that the guest is from England.

5. Two people, $A$ and $B$, are involved in a duel. The rules are simple: shoot at each other once; if at least one is hit, the duel is over, if both miss, repeat (go to the next round), and so on. Denote by $p_A$ and $p_B$ the probabilities that $A$ hits $B$ and $B$ hits $A$ with one shot, and assume that that hitting/missing is independent from round to round. Compute the probabilities of the following events: (a) the duel ends and $A$ is not hit; (b) the duel ends and both are hit; (c) the duel ends after round number $n$; (d) the duel ends after round number $n$ GIVEN that $A$ is not hit; (e) the duel ends after $n$ rounds GIVEN that both are hit; (f) the duel goes on for ever.

6. (For this one I already had a picture...) On Figure 1, each of the five connections can be open or closed independently of other connections. The probability to have a specific connection closed is $p$.
(a) Compute the probability that there is a path of closed connections from A to C.
(b) Compute the conditional probability that the connection along the diagonal $BD$ is closed given that there is a path of closed connections from A to C. (One possible solution: by inclusion-exclusion. To keep track of what you are doing, it might actually be easier to denote the probability of each connection by a different letter).
Whatever answer you get in both parts (a) and (b), check that the result is a function that is monotonically increasing from 0 when p=0 to 1 when p=1 (it might be easier to do it using Matlab). If it is indeed the case, then compare your answer with mine.

**Homework 4.** Solve each problem and write 1-2 variations on a problem of your choice.

1. Five men and five women are ranked according to their performance on a test. Assume that no two test scores are the same and all possible rankings are equally likely. Let $X$ be the highest ranking of a woman. Find the distribution of $X$.

2. A coin is tossed $n$ times. Let $X$ be the difference between the number of heads and the number of tails. Find the possible values of $X$. Do we care whether the coin is fair or not?

3. A fair coin is tossed $n$ times. Let $X$ be the difference between the number of heads and the number of tails. Find the distribution of $X$ when (a) $n = 3$; (b) $n = 4$.

4. Consider the following strategy for paying the roulette. Bet \$1 on red. If red appears (which happens with probability 18/38), then take the \$1 profit and stop playing for the day. If red does not appear, then bet additional \$1 on red each of the following two rounds, and then stop playing for the day no matter the

outcome. Let $X$ be the net gain/loss. (a) Find the distribution of $X$; (b) Compute $P(X > 0)$; (c) Compute the expected value of $X$; (d) Would you consider this a winning strategy?

5. Two teams play a series of games until one of the teams wins $n$ games. In every game, both teams have equal chances of winning and there are no draws. Compute the expected number of the games played when (a) $n = 2$; (b) $n = 3$. (To keep track of what you are doing, it can be easier to use different letters for the probabilities of win for the two teams).

6. A communication channel transmits a signal as sequence of digits 0 and 1. The probability of incorrect reception of each digit is $p$. To reduce the probability of error at reception, 0 is transmitted as 00000 (five zeroes) and 1, as 11111. Assume that the digits are received independently and the majority decoding is used. Compute the probability of receiving the signal incorrectly if the original signal is (a) 0; (b) 101. Evaluate the probabilities when $p = 0.2$.

7. In a certain jurisdiction, it takes at least 9 votes of a 12-member jury to get a conviction. Assume that

(1) 65% of all defendants are guilty;
(2) the probability that a juror will convict an innocent is 0.1;
(3) the probability that a juror will acquit a guilty is 0.2;
(4) each juror votes independently of the rest of the panel;

Compute the probabilities of the following events: (a) the panel renders a correct decision; (b) the defendant is convicted.

**Homework 5.** Solve each problem and write 1-2 variations on a problem of your choice.

1. In a collection of 80000 married couples find approximately the probability that in at least one of them, (a) both husband and wife were born on April 15; (b) both husband and wife were born on the same day.

2. On a certain highway, there are, on average, 2.2 cars abandoned every week. Compute the probability that (a) there will be no cars abandoned next week; (b) there will be at least 5 cars abandoned next month.

3. Suppose that the number of times a person catches cold in a year is a Poisson random variable with parameter $\lambda = 5$. A new drug is claimed to reduce this parameter $\lambda$ to 3 for 75% of the population and has no effect on the rest of the population. Somebody takes the drug and gets two colds in a year. Compute the probability that the drug was beneficial for that person. (Note: this is a classical Bayes rule problem).

4. At time $t_0 = 0$, a fair coin is tossed and lands heads. Then, at a random time $T > 0$, the coin is tossed again. Given a $t > 0$, compute the probability that the coin shows heads at time $t$, if $T$ is the moment of the first event in a Poisson process with parameter $\lambda$. How will the answer change if the coin is not fair?

5. An urn contains four black and four white balls. Four balls are taken out of the urn. If two are black and two are white, the experiment ends. Otherwise, the balls are returned to the urn and the experiment is repeated. Denote by $X$ the number of experiments conducted. Find the probability distribution of $X$. (Note: the probability of success is 18/35; start by verifying this).

6. Let $X$ be a distribution function with the probability distribution $F$ and let $\alpha, \beta$ be real numbers and $\alpha \neq 0$. Find the distribution function of the random variable $\alpha X + \beta$. (Keep in mind that $F(x) = P(X \leq x)$ and you cannot assume that $F$ is continuous.)

**Homework 6.** Solve each problem and write 1-2 variations on a problem of your choice.

1. Consider the function

$$f(x) = \begin{cases} C(2x - x^2) & 0 < x < 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Could $f$ be a distribution function? If so, find $C$.
(b) Could $f$ be a probability density function? If so, find $C$.

2. A stick is broken into two pieces at random. Compute the probability that the ratio of the longer part to the shorter is at least $a$, where $a > 1$. (The length of the stick does not matter; put it equal to 1 if you want).

3. Given a normal random variable $X$ with mean 10 and variance 36, compute the following probabilities: (a) $P(X > 5)$; (b) $P(4 < X < 16)$; (c) $P(X < 8)$; (d) $P(X < 20)$; (e) $P(X > 16)$. Please use a table of the standard normal distribution.

4. Compute the variance of the normal random variable $X$ if $E(X) = 5$ and $P(X > 9) = 0.2$.

5. Let $X$ be binomial random variable with parameters $n = 100$ and $p = 0.65$. Use normal approximation with the continuity correction to find the following probabilities: (a) $P(X \geq 50)$; (b) $P(60 \leq X \leq 70)$; (c) $P(X < 75)$.

6. The number of miles a certain car can drive before dying is a random variable $X$. The car has been driven for 10000 miles. Compute the probability that the car will drive another 20000 if the distribution of $X$ is (a) exponential, with average value 20000; (b) unform on $(0, 40000)$.

7. Let $X$ be exponential random variable with mean 1. Find the probability density function of $\ln X$.

8. Let $X$ be uniform on $(0, 1)$. Find the probability density function of $e^X$.

**Homework 7.** Solve each problem and write 1-2 variations on a problem of your choice.

1. Two fair dice are rolled. Define the following random variables: $X$, the value of the first die; $Y$, the sum of the two values; $Z$, the larger of the two values; $V$, the smaller of the two values. Find the joint distribution of (a) $Z$ and $Y$; (b) $X$ and $Y$; (c) $Z$ and $V$.

2. The joint probability density function of two random variables $X$ and $Y$ is

$$f(x, y) = c(y^2 - x^2)e^{-y}, \ -y \leq x \leq y, \ 0 < y < +\infty.$$

Find (a) the value of $c$; (b) the marginal densities of $X$ and $Y$; (c) expected value of $X$.

3. A man and a woman decide to meet at a certain location. The arrival time of the man is uniformly distributed between 12:15pm and 12:45pm. The arrival time of the woman is uniformly distributed between 12pm and 1pm. The man and the woman arrive independently of each other. (a) Compute the probability that the first to arrive will wait at most five minutes. (b) Compute probability that the man arrives first. (Use the picture).

4. Compute the probability that $n$ points, selected randomly and independently on a circle, will be in the same semi-circle. (Suggestion: Fix a point $P_i$ and denote by $A_i$ the event that all points starting with $P_i$ and going clockwise are in the same semi-circle. Argue that $A_i$ and $A_j$ are mutually exclusive and that the probability of each $A_i$ is $2^{-n+1}$.)

5. Two points $X, Y$ are selected at random on the interval $[0, 1]$ so that $X$ is uniform on $(0, 1/2)$, $Y$ is uniform on $(1/2, 1)$ and $X, Y$ are independent. Compute $P(Y - X > 1/3)$.

6. Given the joint density $f = f(x, y)$ of two random variables $X, Y$, decide whether the random variables are independent:

$$(a) \quad f(x) = \begin{cases} xe^{-(x+y)}, & x > 0, y > 0; \\ 0, & \text{otherwise} \end{cases} \qquad (b) \quad f(x) = \begin{cases} 2, & 0 < x < y, \ 0 < y < 1; \\ 0, & \text{otherwise} \end{cases}$$

7. Player A's bowling score in one game is approximately normal with mean 170 and standard deviation 20; player B's score in one game is approximately normal with mean 160 and standard deviation 15. Assuming that the scores are independent, compute the probability that, in one game, (a) Player A scores higher than player B; (b) The total score is over 350.

**Homework 8.** Solve each problem and write 1-2 variations on a problem of your choice.

1. The joint probability density function of two random variables $X$ and $Y$ is

$$f(x, y) = c(x^2 - y^2)e^{-x}, \ 0 < x < +\infty, \ -x \leq y \leq x$$

Find the conditional distribution of $Y$ given $X = x$.

2. Three cars break down on a road of length $L$, randomly and independently of one another. Given a $d < L/2$, find the probability that the distance between any two of the cars is at least $d$. (Keep in mind that there are six possible arrangements of the cars).

3. The random variables $X, Y$ have the joint density

$$f(x, y) = \begin{cases} \frac{1}{\pi}, & x^2 + y^2 \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

In other words, the vector $(X, Y)$ is uniform in the unit disk. Find the joint density of $\sqrt{X^2 + Y^2}$ and $\tan^{-1}(Y/X)$.

4. Let $U, Z$ be independent random variables such that $U$ is uniform on $(0, 2\pi)$ and $Z$ is exponential with mean 1. Show that $X = \sqrt{2Z} \cos U$ and $Y = \sqrt{2Z} \sin U$ are independent standard normal random variables.

5. Let $X, Y$ be independent random variables, both uniform on $(0,1)$. Find the joint density of the following random variables; (a) $X+Y,\ X/Y$; (b)$X,\ X/Y$, (c) $X+Y,\ X/(X+Y)$.

**Homework 9.** Solve each problem and write 1-2 variations on a problem of your choice.

1. Compute the expected winning in the following game. A fair die is rolled and a fair coin is tossed. If the coin lands heads, the winning amount is the twice the number on the die. If the coin lands tails, the winning amount is half the number on the die. (Apparently this is not a casino game...)

2. A fair die is rolled 10 times. Compute the expected value of the sum.

3. Two people, A and B, choose randomly and independently three objects out of 10. Compute the expected number of objects (a) chosen by both A and B; (b) chosen by neither A nor $B$; (c) chosen by exactly one person.

4. 1000 cards with numbers from 1 to 1000 randomly distributed among 1000 people. Compute the expected number of people whose age is the same as the number on the card they got. (Do not assume an upper bound on the age).

**Homework 10.** Solve each problem and write 1-2 variations on a problem of your choice.

1. For a group of 100 people, compute the expected value of (a) the number of days in a 365-day year that are birthdays of exactly three people; (b) the number of distinct birthdays.

2. Compute the expected number of rolls of a fair die before all sides appear at least once.

3. Let $X_1, X_2, \ldots$ be independent identically distributed continuous random variables. Define the random variable $N$ as follows:

$$X_1 > X_2 > \ldots > X_{N-1} < X_N$$

Compute the expected value of $N$. (Hint: start by showing that $P(N \geq n) = 1/n!$).

4. Let $X_1, \ldots, X_4$ be random variables such that $EX_i = 0,\ EX_i^2 =,\ i = 1, \ldots, 4;\ E(X_iX_j) = 0,\ i \neq j$. Compute the correlation of (a) $X_1 + X_2$ and $X_1 + X_3$; (b) $X_1 + X_2$ and $X_3 + X_4$.

5. Consider a graph on $n$ vertices. Define the degree $D_i$ of the vertex $i$ as the number of edges coming out of the vertex. Assume that between any two different vertices, an edge is preset with probability $p$, independent of all other edges. Compute the distribution of $D_i$ and the correlation between $D_i$ and $D_j$.

6. A fair die is rolled repeatedly. Denote by $X$ the number of rolls necessary to get a 6 for the first time. Denote by $Y$ the number of rolls necessary to obtain 5 for the first time. Compute (a) $E(X)$; (b) $E(X|Y=1)$; (c) $E(X|Y=5)$.

**Homework 11.** Solve each problem and write 1-2 variations on a problem of your choice.

1. The joint density of $X$ and $Y$ is

$$f(x,y) = \frac{e^{-x/y}e^{-y}}{y},\ 0 < x < \infty,\ 0 < y < \infty.$$

Compute $E(X^2|Y=y)$.

2. A prisoner is trapped in a cell containing three doors. The first door leads to a tunnel that would lead back to the cell after two days of travel. The second door leads to a tunnel that would lead back to the cell after 4 days of travel. The third door leads to a tunnel that would lead to freedom after one day of travel. The prisoner cannot label doors and will always select the doors randomly with probabilities $0.5, 0.3$ and $0.2$, respectively. Compute the expected number of days before the prisoner reaches freedom.

3. The number of people entering the elevator on the ground floor is a Poisson random variable with mean 10. There are $N$ floors above the ground floor, and everybody in the elevator is equally likely to exit on any of the $N$ floors, independently of everybody else. Nobody enters the elevator above the ground floor. Compute the expected number of stops the elevator makes before everybody is out.

4.The expected number of accidents at an industrial facility is 5 per week. The average number of injured people in each accident is 2.5. Assuming all the independence you need, compute the expected number of injured workers per week.

5. The life time of the light bulb is characterized by the mean value $\mu$ and standard deviation $\sigma$. There are two types of light bulbs in a box, with the corresponding parameters $\mu_1, \sigma_1$ and $\mu_2, \sigma_2$. The proportion

of type-1 bulbs in the box is $p$. A bulb is selected at random. Denote by $X$ the life time of the this bulb. Compute the expected value and the variance of $X$.

6. The number of accident a person has in a year is a Poisson random variable with parameter $\lambda$. For 60% of the population, $\lambda = 2$; for the rest, $\lambda = 3$. Compute the probability that, in one year, a randomly selected person will have (a) no accidents; (b) three accidents; (c) three accidents given hat there were no accidents the previous year. (The answers for (b) and (c) are different because this is not a standard Poisson process and so the numbers of accidents from year to year are not independent, although they become independent if you fix $\lambda$).

**Homework 12.** Solve each problem and write 1-2 variations on a problem of your choice.

1. Given independent identically distributed Poisson random variables $X_1, \ldots, X_{20}$ with parameter 1, compute an approximation of $P(X_1 + \ldots + X_{20} > 15)$ using (a) Markov's inequality; (b) the central limit theorem.

2. A fair die is rolled until the total sum exceeds 300. Compute approximately the probability that at least 80 rolls will be necessary.

3. Assume that the amount of weight, in units of 1000 pounds, a bridge can hold without collapsing, is normally distributed with mean 400 and standard deviation 40. Assume that the weight of a car, in the same units, is a random variable with mean 3 and standard deviation 0.3. How many cars should there be on the bridge for the probability of collapse to exceed 0.1?

4. The daily price $Y_n$ of a certain stock is modeled by the relation

$$Y_n = Y_{n-1} + X_n, n \geq 1,$$

where $X_k$ are independent identically distributed standard normal random variables. Suppose that the current stock price is \$100. Compute the probability that the price will exceed \$105 after 10 days.

**Homework 13.** Solve each problem and write 1-2 variations on a problem of your choice.

1. Customers arrive at a bank according to a Poisson process. Two customers arrived during one hour. Compute the probability that (a) both arrived during the first 20 min; (b) at least one arrived during the first 20 min. Note that the rate $\lambda$ of the process is not necessary to solve the problem.

2. Cars cross a certain point in the highway following a Poisson process, with 3 cars per minute on average. A dog runs straight across the road and will get injured on the encounter of two or more cars. Compute the probability that the dog crosses the highway unhurt if it takes the dog $s$ seconds to cross the road.

3. Consider an ergodic Markov chain with transition probability matrix $(P_{ij},\ i, j = 0, \ldots, M)$, and assume that $\sum_{i=1}^{M} P_{ij} = 1$ for all $j$. Show that the invariant distribution assigns the same probability $1/(M + 1)$ to every state.

4. Jon goes for a run every morning. When he leaves his house for a run, he is equally likely to go out of the front or the back door. When he returns, he is again equally likely to get in through either the front or the back door. When he comes back, he leaves the shoes at the door. For the next run, he selects a random pair from those at the door from which he is leaving. If there are no shoes at the door from which he leaves, then he runs barefoot. Jon has five pairs of shoes. Compute the percentage of time he runs barefoot.

**Homework 14.** Solve each problem and write 1-2 variations on a problem of your choice.

1. Describe a procedure to simulate a random variable with the distribution function

$$F(x) = \begin{cases} 0 & x \leq -3 \\ \frac{1}{2} + \frac{x}{6} & -3 < x < 0 \\ \frac{1}{2} + \frac{x^2}{32} & 0 < x \leq 4 \\ 1 & x > 4. \end{cases}$$

2. Describe a procedure to simulate a random variable with the distribution function $F(x) = 1 - e^{-ax^b}$, $a > 0,\ b > 0$.

3. Describe a procedure to simulate a random variable with the following failure rate function (a) $\lambda(t) = c$; (b) $\lambda(t) = ct$; (c) $\lambda(t) = ct^2$; (d) $\lambda(t) = ct^3$.

4. Given an integer $n > 1$, describe a procedure to simulate a random variable with the distribution function $F(x) = x^n,\ 0 < x < 1$, using (a) one uniform on $(0, 1)$ random variable; (b) $n$ independent uniform on $(0, 1)$ random variables (but this time without extracting any roots).