Truthful Double Auction Mechanisms

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Following the multistage design approach, we propose two asymptotically efficient truthful double auction mechanisms, the BC-LP mechanism and the MBC mechanism, for an exchange market with many buyers and sellers. In this market, each buyer wants to procure a bundle of commodities and each seller supplies one unit of a commodity. Furthermore, various transaction-related costs will be incurred when a buyer trades with a seller. We prove that under both mechanisms, truthful bidding is a dominant strategy for all buyers and sellers when the buyers’ bundle information and the transaction cost information are common knowledge. The BC-LP mechanism can be implemented by just solving two linear programs, whereas the MBC mechanism has a higher complexity. The empirical study shows that the MBC mechanism achieves higher efficiency over the BC-LP mechanism and that both outperform the KSM-TR mechanism, the only known truthful mechanism for a more restrictive exchange market.

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1. Introduction

Transactions of hundreds of billions of dollars take place through online business-to-business markets because of the low transaction costs involved and the accessibility to buyers and sellers (Blackmon 2000). In particular, industry e-marketplaces that provide hosted procurement and sourcing services are indispensable.

Many of these marketplaces involve the sale/purchase of a variety of distinct assets. For example, in industrial procurement auctions, a buyer may want to purchase different components needed to produce the final product. To have a better understanding of the current practice, let us look at the popular procurement auction mechanism used by FreeMarkets (merged with Ariba) when a large industrial buyer seeks to procure a bundle with specified amounts of several components (Gallien and Wein 2005): (1) the component types are ordered, and each component is sequentially auctioned off; (2) after the entire bidding process is over, the buyer considers the nonprice factors associated with the purchase of each component, including the lead time of delivery and the relationship with the sellers; and (3) based on these factors, the buyer makes the final decision on whether or not to award a contract to each winning seller.

Potential directions for improvement arise from this practice. How can a mechanism explicitly take into account the nonprice factors? How does a mechanism decide the optimal order of the component types? Furthermore, because there are thousands of procurement auctions taking place with different buyers, how can a mechanism help sellers decide which buyer’s auction they should participate in?

There are some initial attempts to address these issues. In the mechanism design literature, Beil and Wein (2003) study the mechanism design problem that takes into account the nonprice attributes. They construct a multiround open-ascending auction mechanism for the exchange environment in which the monopsonistic buyer knows the parametric form of the sellers’ cost functions in terms of the nonprice attributes, and tries to determine to which seller to award a contract. The mechanism maximizes the buyer’s utility under the assumption that sellers submit their myopic best-response bids in the last round, and do not distort their bids in the earlier rounds.

Elmaghraby (2003) shows that if the components are auctioned off sequentially, the ordering of the components is important because the final allocation may be far from optimal in that the buyer may end up paying more than he/she needs to. Gallien and Wein (2005) propose a multi-item procurement auction that solves the ordering problem by trading all the components simultaneously in multiple iterations. In each iteration, sellers know the current prices of all the components, and this helps sellers decide which components to bid on. By using a linear programming-based decision-support system, sellers can figure out their myopic best responses and are likely to provide true private information during each auction iteration. Compared to the existing procedure, this mechanism simplifies sellers’ decisions about which components they
should bid on and is likely to achieve a final allocation that is more efficient, which benefits both the monopsonistic buyer and the sellers. Parkes and Kalagnanam (2005) propose a family of iterative primal-dual based Vickrey auctions that can deal with multiple nonprice attributes and achieve high market efficiency.

However, the above papers only deal with a single buyer with several sellers. In practice, each seller may also want to contact several buyers to reach the best deal. Extensive negotiations are typically involved to establish an exchange relationship in the market with many buyers and many sellers, and the process can be very time consuming. In an effort to design better exchange markets, we propose double auction mechanisms that model the bidding behaviors of both buyers and sellers in an industrial procurement setting in which each buyer wants to purchase bundles of different items and each supplier (seller) produces only one product. The proposed mechanisms explicitly take into account multiple buyers and hidden costs, such as costs due to transportation, buyer-vendor relationships, and timing of deliveries. Furthermore, these mechanisms achieve asymptotic efficiency; almost all feasible social welfare will be realized if we have a sufficient number of buyers and sellers.

Babaioff and Walsh (2005) and Chu and Shen (2006) offer pioneer work in strategyproof double auction mechanism design. Babaioff and Walsh (2005) consider a bilateral exchange environment, in which each buyer wants to acquire a bundle of commodities and each seller provides a single unit of one commodity. They assume no transaction costs and propose the known-single minded trade reduction (KSM-TR) mechanism. Chu and Shen (2006) propose the first strategyproof budget-balanced double auction mechanisms for environments with transaction costs. They design the buyer competition (BC) mechanism and the seller competition (SC) mechanism for a simple exchange environment where each buyer and each seller exchange one unit of the same commodity.

In this paper, we consider a bilateral exchange environment with transaction costs. Similar to Babaioff and Walsh (2005), we assume that each buyer wants to acquire a bundle of commodities and each seller provides a single unit of one commodity. The resulting exchange environment is a generalization for both Babaioff and Walsh (2005) and Chu and Shen (2006). Unfortunately, no naive extension/combination of the KSM-TR mechanism and/or the BC/SC mechanisms provides a strategyproof double auction mechanism. In this paper, we propose two novel strategyproof mechanisms, the buyer competition (LP) mechanism and the modified buyer competition (MBC) mechanism, for this generalized environment. Our mechanisms adopt the multistage design approach proposed in Chu and Shen (2006). Specifically, we first make the pricing decision for the buyers: For each buyer, we generate a buying price for her bundle. The buyers who bid lower than their corresponding buying prices will be eliminated from the auction. We then take an efficient allocation among the original sellers and the remaining buyers as the allocation decision. Finally, we decide the selling price for each winning seller using a Vickrey-Clarke-Groves (VCG) mechanism among the original sellers and the remaining buyers. Note that a seller’s bid may influence all the buying prices and consequently the final allocation, so that a seller may find it profitable to manipulate his bid and hide his true valuation. To guarantee the truthfulness of the mechanism, we will set relatively high buying prices during the first stage, so that only a (small) group of competitive buyers will remain in the auction. As a result, during the allocation decision, all the remaining buyers win their bundles and the selling prices can be determined by the competition among the sellers. By carefully designing the buying prices, the sellers will truthfully reveal their valuations and we can obtain asymptotically efficient truthful mechanisms. Both mechanisms follow this framework with differences in the pricing decisions. Due to these differences, the BC-LP mechanism has advantage in computational complexity and can be implemented by solving two linear programs, whereas the MBC mechanism shows higher market efficiency and captures more welfare.

The remainder of this paper is organized as follows. In §2, we describe a bilateral exchange model and review literature related to double auctions. In §3, we propose a buyer competition (LP) mechanism and study its implementation and properties. In §4, we propose a modified buyer competition mechanism. We conduct computational tests in §5 to compare the efficiencies of the mechanisms, and we conclude in §6.

2. Model and Background

2.1. Model

Let \( I \) denote the group of buyers, and \( J \) the group of sellers, hereafter both called “agents.” We refer to a buyer as “she” and a seller as “he.” Let \( C \) denote the set of indivisible commodities. In this market, buyer \( i \) \( (i \in I) \) wants to procure a bundle of goods \( q_i = (q_{i,c})_{c \in C} \). All \( q_{i,c} \) \( (i \in I, c \in C) \) are nonnegative integers because the commodities are indivisible. We assume that the bundle information is common knowledge, that is, we follow the “known single-minded” assumption (Mudlem and Nisan 2002). In this market, each seller only produces a single unit of one commodity. The assumption we make about the sellers is called the single output restriction (Babaioff and Walsh 2005). To facilitate the mathematical presentation and the shadow price interpretation, we also use a vector representation \( q_j = (q_{j,c})_{c \in C} \) for seller \( j \)’s supply. \( q_{j,c} = 1 \) if seller \( j \) supplies one unit commodity \( c \), and \( q_{j,c} = 0 \) otherwise. The single output restriction is important in this model. In §6, we will discuss its impact on the mechanisms when this restriction is relaxed. Throughout the main body of the paper, we will assume this restriction to gain insight and establish strategyproof double auction mechanisms.
Transactions in this marketplace may have transaction costs, including costs associated with transportation, quality, lead time, customization, and the buyer-vendor relationship. When buyer \( i \) purchases commodity \( c \) from seller \( j \), a transaction cost \( d_{i,j,c} \) is incurred. Thus, even if two sellers provide the same commodities, they may still be heterogeneous due to these transaction-related costs. The transaction costs are assumed to be common knowledge. Nevertheless, the auctioneer and each agent may or may not know the number of agents involved, the statistical joint distribution of the valuations, or any other relevant information.

We assume a private value model, where each agent’s valuation of his/her bundle is private information. Suppose that all agents have quasilinear utility, that is, if an agent makes no transaction, his or her utility (payoff) is zero; otherwise, the payoff is the difference between the agent’s valuation of the bundle and the amount of money transferred. We use the quasilinear assumption to facilitate the understanding of the proposed mechanism, and the results in §§3 and 4 do not depend on this assumption on individual preference given the known single-minded assumption.

Without loss of generality, we focus on one-shot sealed-bid auction mechanisms. We use the term “bid” to denote both a buyer’s and a seller’s declaration, although some authors prefer to use the term “ask” to denote a seller declaration. Let \( f_i \) be the bid price of buyer \( i \) for her bundle, and \( g_j \) be the bid price of seller \( j \).

If all agents bid truthfully, the maximum feasible social welfare can be formulated as the following mixed-integer program:

\[
\begin{align*}
\max \quad & \sum_{i \in I} \sum_{j \in J} f_i x_{i} - \sum_{j \in J} g_j y_{j} \\
\text{subject to } & \sum_{j \in J} z_{i,j,c} = q_i^c x_{i} \quad \text{for each } i \in I, c \in C, \\
& \sum_{i \in I} z_{i,j,c} = q_j^c y_{j} \quad \text{for each } j \in J, c \in C, \\
& 0 \leq z_{i,j,c} \quad \text{for each } i \in I, j \in J, c \in C, \\
& x_{i} \in \{0, 1\} \quad \text{for each } i \in I, \\
& y_{j} \in \{0, 1\} \quad \text{for each } j \in J,
\end{align*}
\]

where \( x_i \) and \( y_j \) denote whether an agent trades in the auction, and \( z_{i,j,c} \) specifies the quantity of commodity \( c \) buyer \( i \) acquires from seller \( j \). The variables \((x_{i,c}, y_{j,c})\) specify the resource allocation.

However, the main challenge in designing a truthful mechanism is to induce agents to report their true valuations when agents are heterogeneous and may not know the number of agents involved and/or the valuation distributions. Before discussing this issue, we first go over some auction notation and review literature on double auctions.

### 2.2. Terminology

An auction is an institution widely used to allocate goods when the valuations of these goods are unknown. Because each agent may not know the number of agents involved and/or the statistical joint distribution of valuations, we seek a strategyproof or dominant-strategy incentive-compatible mechanism, where truthful revelation is a dominant strategy for each agent.

To make an auction practical, the mechanism must be individual-rational and budget balanced. A mechanism is (interim) individual-rational if an agent’s expected utility from participation is no less than his or her utility from nonparticipation, after the agent knows his or her own valuation of the bundle. A mechanism is (weakly) budget balanced if the auctioneer’s expected payoff (total payments from the buyers, less the revenues of the sellers and the needed transaction costs) is nonnegative. Thus, individual rationality draws the potential buyers and sellers, whereas budget balance motivates the auctioneer to hold the auction. There are also stronger versions of these properties: ex post individual rationality means that an agent’s utility from participation is no less than his or her utility from nonparticipation for all possible outcomes; ex post budget balance means that the auctioneer’s payoff is nonnegative for all possible outcomes.

As suggested by Milgrom (2000) and Wise and Morrison (2000), we focus on auction efficiency, which compares the social welfare achieved by the mechanism with the maximum feasible social welfare with complete information. A mechanism is efficient if it implements an allocation that maximizes social welfare. However, Hurwicz (1972) shows that in a simple exchange environment, in which buyers and sellers exchange single units of the same good, it is impossible to implement an efficient, budget-balanced, and strategyproof mechanism. This result is further strengthened by Myerson and Satterthwaite (1983). They show the impossibility of having an efficient, individual-rational, incentive-compatible, and budget-balanced mechanism. Asymptotic efficiency means that the welfare loss under the mechanism compared to the maximum feasible social welfare converges to zero as the number of buyers and/or sellers approaches infinity; thus, as the auction becomes large enough, almost all the feasible social welfare will be realized.

Our goal here is to design a strategyproof, individual-rational, budget-balanced, and highly efficient double auction mechanism for the bilateral exchange environment with the known single-minded assumption and the single output restriction.

### 2.3. Related Research

The literature on strategyproof mechanism design starts from the classic results by Vickrey (1961), Clarke (1971), and Groves (1973), under which the unique seller decides the buyers’ trading prices according to their marginal
contributions to the system. Recently, Chen et al. (2005) considered multiunit Vickrey auctions for procurement in supply chain settings and were the first to incorporate transportation costs into auctions. The VCG mechanism is strategyproof, (ex post) individual-rational, and efficient. Nevertheless, the VCG mechanism is not budget balanced for exchange environments with many buyers and sellers.

Few papers deal with strategyproof budget-balanced double auction mechanisms, and most of them do not consider transaction costs. McAfee (1992) presents a strategyproof budget-balanced double auction mechanism for a simple exchange environment, in which buyers and sellers exchange single units of the same good. Huang et al. (2004) extend this mechanism and agents may exchange multiunits of the same good. Using the same design approach, Babaioff and Walsh (2005) propose a strategyproof budget-balanced double auction mechanism for a bilateral exchange environment with the known single-minded assumption and the single output restriction, where each buyer wants to acquire a bundle of commodities. All these mechanisms assume no transaction costs. Babaioff et al. (2004) design a strategyproof budget-balanced double auction mechanism for an exchange environment with transaction costs in which buyers and sellers exchange single units of the same good. Note that both exchange environments in Babaioff and Walsh (2005) and Babaioff et al. (2004) are the special cases of the exchange environment studied in this paper.

To facilitate the design of the strategyproof double auction mechanism, Babaioff and Walsh (2005) propose a trade reduction approach under which the mechanisms select a subset of trades from the efficient allocation by removing the least profitable trade(s) and setting the trading prices according to the bids in the removed trade(s). Bredin and Parkes (2005) present a method to design truthful double auctions in dynamic multiagent systems, such that no agent can benefit from misreporting its arrival time, duration, or value.

The most related literature is Chu and Shen (2006), in which a multistage design approach (shown in Figure 1) is proposed to handle the transaction costs in an exchange environment. Under this approach, a mechanism: (1) begins with a pricing decision for one side that eliminates some agents from the auction, (2) makes the allocation decision, and (3) makes the pricing decision for the other side. For example, a mechanism can first set the buying price for each buyer and remove the buyers who bid lower than their buying prices. Then, the allocation decision can be made by choosing an efficient allocation among the remaining buyers and the original sellers. Finally, the mechanism makes the pricing decision on the seller side, that is, sets the selling prices for each trading seller. The mechanisms under the multistage design approach may offer different final allocations compared with the mechanisms under the trade reduction approach even in a simple exchange environment.3

For a simple exchange environment where buyers and sellers exchange single units of the same good, Chu and Shen (2006) design a buyer competition mechanism, or BC mechanism for short, under the multistage design approach.

Unfortunately, the BC mechanism fails to be strategyproof for the sellers in the bilateral exchange environment studied here. The sellers may have incentives to manipulate their bids and, consequently, improve their own payoffs. To address this problem, we propose a linear programming-based buyer competition mechanism in the next section.

3. BC-LP Mechanism

3.1. The Mechanism

Because the mechanism proposed here is linear programming-based, we name this mechanism the buyer competition (LP) mechanism or BC-LP mechanism in short. The BC-LP mechanism is strategyproof, (ex post) individual-rational, (ex post) weakly budget balanced, and asymptotically efficient for the simple exchange environment.

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The BC-LP mechanism follows the multistage design approach illustrated in Figure 1. The key difference between the BC mechanism and the BC-LP mechanism is that the pricing decisions in the BC-LP mechanism are based on the linear relaxation of the social welfare:

$$ \hat{v}(I', J') = \max \sum_{i \in I'} p_i x_i - \sum_{j \in J'} g_j y_j $$

$$ - \sum_{i \in I', j \in J', c \in C} d_{i,j,c} z_{i,j,c} $$

Figure 1. Diagram for the buyer competition mechanism under the multistage approach from Chu and Shen (2006).
subject to $\sum_{j\in J'} z_{i,j,c} = q_i' x_i$ for each $i \in I'$, $c \in C$, 
$\sum_{j\in J} z_{i,j,c} = q_j' y_j$ for each $j \in J$, $c \in C$, 
$0 \leq z_{i,j,c}$ for each $i \in I'$, $j \in J'$, $c \in C$, 
$0 \leq x_i \leq 1$ for each $i \in I'$, 
$0 \leq y_j \leq 1$ for each $j \in J'$,

where $I' \subseteq I$ and $J' \subseteq J$. For simplicity of representation, we may drop the parameters $(I', J')$ when the references to the buyer set and the seller set are obvious.

Throughout the paper, we will use $\bar{e}$ and $V$ to represent the MIP formulation and its objective value, and use $\bar{\hat{e}}$ and $\bar{\hat{V}}$ to represent the linear relaxation formulation and corresponding objective value.

To determine the pricing decisions of the BC-LP mechanism, we need information from the following two formulations.

$\hat{\hat{e}}_k(I, J)$ Maximize $\bar{\hat{V}}_k(I, J) = \sum_{i\in I} f_i x_i - \sum_{j\in J} g_j y_j - \sum_{i\in I, j\in J, c\in C} d_{i,j,c} z_{i,j,c}$ subject to $\sum_{j\in J} z_{i,j,c} = q'_i x_i$ for each $i \in I'$, $c \in C$, 
$\sum_{i\in I} z_{i,j,c} = q'_j y_j$ for each $j \in J$, $c \in C$, 
$0 \leq z_{i,j,c}$ for each $i \in I$, $j \in J$, $c \in C$, 
$0 \leq x_i \leq 1$ for each $i \in I \setminus \{k\}$, 
$0 \leq y_j \leq 1$ for each $j \in J$, 

where $k \in J$.

$\hat{\hat{e}}_k(I, J)$ is the linear relaxation formulation of the social welfare when we have one more agent who is identical to buyer $k$. $\bar{\hat{V}}_k$ is the corresponding objective value. We use $\hat{\hat{e}}_{-k}$ and $\bar{\hat{V}}_{-k}$ to represent the formulation and its objective value when seller $k$ is absent from the system.

$\hat{\hat{e}}_{-k}(I, J)$

Maximize $\bar{\hat{V}}_{-k}(I, J) = \sum_{i\in I} f_i x_i - \sum_{j\in J} g_j y_j - \sum_{i\in I, j\in J, c\in C} d_{i,j,c} z_{i,j,c}$ subject to $\sum_{j\in J} z_{i,j,c} = q'_i x_i$ for each $i \in I'$, $c \in C$, 
$\sum_{i\in I} z_{i,j,c} = q'_j y_j$ for each $j \in J$, $c \in C$, 
$0 \leq z_{i,j,c}$ for each $i \in I$, $j \in J$, $c \in C$, 
$0 \leq x_i \leq 1$ for each $i \in I$, 
$y_k = 0$, 
$0 \leq y_j \leq 1$ for each $j \in J \setminus \{k\}$,

where $k \in J$.

Using these formulations, we define the critical prices that will be used in the pricing decision. We use $p^b$ to denote the buying price and $p^s$ to denote the selling price.

$p^b_i(I', J') = \inf \{ f_i | \bar{\hat{V}}_i(I', J') > \bar{\hat{V}}(I', J') \}, \quad i \in I,$

$p^s_j(I', J') = \sup \{ g_j | \bar{\hat{V}}_j(I', J') > \bar{\hat{V}}_j(I', J') \}, \quad j \in J.$

Now we are ready to present the BC-LP mechanism under the multistage design approach:

The Procedure of the BC-LP Mechanism

- Each agent submits one sealed bid.
- For buyer $i \in I$, if her bid $f_i$ is no more than $p^b_i(I, J)$, she is eliminated from the auction. Let $\bar{I}$ denote the remaining buyer set $\bar{I} = \{ i | f_i > p^b_i(I, J), i \in I \}$.
- The items are allocated among the remaining agents ($\bar{I}$ and $J$) according to the optimal solution to $\hat{\hat{e}}(\bar{I}, J)$.
- The trading buyer $i$ pays $p^b_i(I, J)$, and the trading seller $j$ receives $p^s_j(\bar{I}, J)$.

3.2. The Polynomial-Time Implementation

Given the BC-LP mechanism, two natural questions arise: (1) What is the economic explanation for $p^b$ and $p^s$? (2) How can we calculate these prices efficiently?

If $p^b$ and $p^s$ are finite, they can be interpreted and calculated via the shadow prices of formulation $\hat{\hat{e}}$. Generally, the shadow price can take any value in the interval: [minimum shadow price, maximum shadow price]. We next show that $p^b$ and $\bar{\hat{V}}_i$ are closely related to the minimum shadow price of the constraint associated with agent $i$, while $p^s$ and $\bar{\hat{V}}_j$ are closely related to the maximum shadow price of the constraint associated with agent $j$. Let $p_i$ denote the minimum shadow price of the constraint associated with buyer $i$, $x_i \leq 1$. Similarly, let $\bar{p}_j$ denote the maximum shadow price of the constraint associated with seller $j$, $y_j \leq 1$. When the minimum shadow price $p_i$ is positive, if buyer $i$ trades at $p^b$, her utility equals the minimum shadow price $p_i$. The following propositions formalize this relationship.

Proposition 1. For buyer $i \in I$, $f_i > p^b_i(I, J)$ if and only if $p_i(\bar{I}, J) > 0$.

Proposition 2. For buyer $i \in I$, if $p^b_i(I, J) > 0$, then $p^b_i(I, J) = f_i - \bar{p}_i(\bar{I}, J)$.

We next show that $p^s_j$ and $\bar{p}_j$ are also closely related for each trading seller.

Proposition 3. In the remaining system $\bar{I} \cup J$, $p^s_j(\bar{I}, J) = g_j + \bar{\hat{V}}(\bar{I}, J) - \bar{\hat{V}}_j(\bar{I}, J) = g_j + \bar{p}_j(\bar{I}, J)$ for trading seller $j$.

Propositions 1, 2, and 3 (whose proofs are available in the appendix) enable us to calculate $p^b$ and $p^s$ using the shadow prices, which can be obtained in polynomial time by solving linear programs. Nevertheless, to implement the
BC-LP mechanism, we still need to solve an MIP formulation \( \epsilon(\bar{I}, J) \) optimally. Fortunately, we prove that \( \epsilon(\bar{I}, J) \) can be solved efficiently and that the BC-LP mechanism can be implemented in polynomial time. In fact, we can solve \( \epsilon(\bar{I}, J) \) by solving its linear relaxation \( \tilde{\epsilon}(\bar{I}, J) \) due to the following theorem:

**Theorem 1.** All the optimal extreme point solutions to formulation \( \tilde{\epsilon}(\bar{I}, J) \) are integer valued.

Theorem 1 and Propositions 1, 2, and 3 lead to a polynomial implementation.

**The Polynomial Implementation of the BC-LP Mechanism**

- Collect one sealed bid from each agent.
- Solve the linear program \( \tilde{\epsilon}(\bar{I}, J) \).
- For each buyer \( i \), calculate \( \tilde{p}_i(\bar{I}, J) \), the minimum shadow price of constraint \( x_i \leq 1 \) in \( \tilde{\epsilon}(\bar{I}, J) \). If \( \tilde{p}_i(\bar{I}, J) > 0 \), \( \tilde{p}_i(\bar{I}, J) = f_i - \tilde{p}_i(\bar{I}, J) \). Otherwise, buyer \( i \) is eliminated.
- Solve the linear program \( \tilde{\epsilon}(\bar{I}, J) \), where \( \bar{I} \) is the remaining buyer set, and pick an optimal extreme point solution.
- For each trading buyer \( j \), calculate \( \tilde{p}_j(\bar{I}, J) \), which equals \( g_j + \tilde{p}_j(\bar{I}, J) \), where \( \tilde{p}_j(\bar{I}, J) \) is the maximum shadow price of constraint \( y_j \leq 1 \) in \( \tilde{\epsilon}(\bar{I}, J) \).
- Conduct transactions according to the optimal solution to \( \tilde{\epsilon}(\bar{I}, J) \). The transaction price for trading buyer \( i \) is \( \tilde{p}_i(\bar{I}, J) \), and the transaction price for trading seller \( j \) is \( \tilde{p}_j(\bar{I}, J) \).

Thus, the BC-LP mechanism is (ex post) individual-rational for all buyers and sellers.

**Theorem 3.** The BC-LP mechanism is strategyproof in the bilateral exchange environment with the known single-minded assumption and the single output restriction.

**Theorem 4.** The BC-LP mechanism is (ex post) weakly budget balanced in the bilateral exchange environment with the known single-minded assumption and the single output restriction.

The proofs of Theorems 3 and 4 are available in the appendix. In the proof of Theorem 3, we notice that for each buyer, \( p^b_i(\bar{I}, J) \) is the critical threshold price for buyer \( i \) such that she trades her bundle if she bids above \( p^b_i(\bar{I}, J) \) and does not trade if she bids below. A similar result holds for the sellers. This means that Theorems 2, 3, and 4 hold without the assumption that each agent’s utility is quasi-linear because each agent only faces a “take-it-or-leave-it” situation given the known single-minded assumption.

Furthermore, the BC-LP mechanism is asymptotically efficient. To evaluate the efficiency, let us assume that there are finite types of commodities and there exists a number \( M \) such that \( \sum_{c \in C} q_c^i < M \) for every buyer \( i \). That is, \( M \) is the limit on how many units of commodities a buyer can acquire. Let each buyer’s valuation independently follow \( F_{(q_c^i)_{c \in C}} \), where \( (q_c^i)_{c \in C} \) is the bundle buyer \( i \) acquires, and let each seller’s valuation independently follow \( G_{(q'_c)_{c \in C}} \), where \( (q'_c)_{c \in C} \) is what seller \( j \) provides. We assume that the transaction cost \( d_{i,j,c} \) is proportional to the distance between buyer \( i \) and seller \( j \), who are independently distributed according to some continuous distribution \( U \) on some compact domain \( H \). Let us also assume the cost function \( d \) is continuous with respect to the agents’ locations.

**Theorem 5.** With bounded continuous valuation distributions and continuous transaction costs, the BC-LP mechanism is asymptotically efficient if every agent randomly offers or acquires a certain bundle according to the same distribution in the bilateral exchange environment with the known single-minded assumption and the single output restriction.

The asymptotic efficiency result in Theorem 5 is based on the assumption that each agent draws his or her valuation independently from some continuous distribution. Can the result still hold if the continuity assumption is relaxed? In the next section, we will discuss a perturbation technique that improves the efficiency of the mechanism and achieves asymptotic efficiency without the continuity assumption.

**3.3. The Economic Properties**

In this section, we show that the BC-LP mechanism possesses the desired auction properties. It is strategyproof, individual-rational, budget balanced, and highly efficient.

**Theorem 2.** The BC-LP mechanism is (ex post) individual-rational.

**Proof.** Trading buyer \( i \) bids \( f_i \) and pays \( p^b_i(\bar{I}, J) \). Because she survives the buyer elimination phase, \( f_i > p^b_i(\bar{I}, J) \). Thus, the difference between the bid price and the transaction price for trading buyer \( i \) is positive, and the BC-LP mechanism is therefore (ex post) individual-rational for each trading buyer.

If seller \( j \) trades in the remaining system consisting of \( \bar{I} \) and \( J \), he receives \( p^s_j(\bar{I}, J) \). For any \( \epsilon > 0 \), if seller \( j \) bids \( g_j + \epsilon \) instead of \( g_j \), \( \tilde{V}(\bar{I}, J) \) will increase, and we will have \( \tilde{V}(\bar{I}, J) < \tilde{V}(\bar{I}, J) \). That is, \( g_j(\bar{I}, J) \geq g_j + \epsilon \) for any \( \epsilon > 0 \). Thus, \( p^s_j(\bar{I}, J) \geq g_j \); this means that seller \( j \) receives no less than his bid price, and the BC-LP mechanism is (ex post) individual-rational for each trading seller.

Because the payoffs of nontrading buyers and sellers are zero, all the buyers and sellers get nonnegative payoffs.

**3.4. An Enhanced BC-LP Mechanism**

Let us first look at the example in Figure 2, in which there are five sellers, two buyers, and one commodity. Each buyer wants two units of the commodity and each seller supplies...
The implementation of the Enhanced BC-LP Mechanism

- Collect one sealed bid from each agent.
- Generate an arbitrary lexicographic order.
- Solve the linear program $\widehat{\ell}(I, J)$, and pick the optimal solution based on the lexicographic order.
- For each buyer $i$, check whether this optimal solution changes if constraint $x_i \leq 1$ is removed. If so, $p_b^i(I, J) = f_i - \widehat{p}_i(I, J)$, where $\widehat{p}_i(I, J)$ is the minimum shadow price of the above constraint in $\widehat{\ell}(I, J)$; if not, buyer $i$ is eliminated.
- Solve the linear program $\widehat{\ell}(\bar{I}, J)$, where $\bar{I}$ is the remaining buyer set, and pick the optimal solution based on the lexicographic order.
- For each trading seller $j$, calculate $p_j^i(\bar{I}, J)$, which equals $g_j + \bar{p}_j(\bar{I}, J)$, where $\bar{p}_j(\bar{I}, J)$ is the maximum shadow price of constraint $y_j \leq 1$ in $\widehat{\ell}(\bar{I}, J)$.
- Conduct transactions according to the optimal solution to $\widehat{\ell}(\bar{I}, J)$. The transaction price for trading buyer $i$ is $p_b^i(I, J)$, and the transaction price for trading seller $j$ is $p_j^i(\bar{I}, J)$.

To implement the enhanced BC-LP mechanism, we only need to solve two linear programs $\widehat{\ell}(I, J)$ and $\widehat{\ell}(\bar{I}, J)$, and calculate the associated shadow prices.

The enhanced BC-LP mechanism inherits the desirable properties of the BC-LP mechanism. Formally, we have:

**Theorem 6.** The enhanced BC-LP mechanism is strategy-proof, (ex post) individual-rational, and (ex post) weakly budget balanced in the bilateral exchange environment with the single output restriction.

**Proof.** We first show that the enhanced BC-LP mechanism is (ex post) weakly budget balanced. Because the BC-LP mechanism is (ex post) weakly budget balanced, that is, the auctioneer’s payoff is nonnegative for all possible bid combinations. By viewing the enhanced BC-LP mechanism as a BC-LP mechanism with bids $f_i + \epsilon_i$ and $g_j - \epsilon_j$, the auctioneer’s payoff under the enhanced BC-LP mechanism is always nonnegative. That is, the enhanced BC-LP mechanism is (ex post) weakly budget balanced.

For the strategy-proofness and individual-rationality properties on the sellers, it turns out that by treating buyers’ bids as $f_i + \epsilon_i$ instead of $f_i$, the original proof for the BC-LP mechanism remains valid.

We now prove the strategy-proofness and individual-rationality properties for the buyers. Consider two scenarios of buyer $i$’s bid price:

1. Buyer $i$ bids higher than $p_b^i(I, J)$: We show that buyer $i$ trades her bundle at $p_b^i(I, J)$ under this scenario. Note that if we remove the constraint associated with buyer $i$, $x_i \leq 1$, the optimal solution changes as the optimal objective function value increases. Thus, buyer $i$ survives according to the procedure of the enhanced BC-LP mechanism. Because all the surviving buyers trade in the BC-LP mechanism, buyer $i$ acquires her bundle at $p_b^i(I, J)$.
Thus, there exists no optimal solution with restriction.

Theorem 8. With bounded valuation distributions and continuous transaction costs, the enhanced BC-LP mechanism is asymptotically efficient if every agent independently draws the agent bundle type from the same distribution in the bilateral exchange environment with the single output restriction.

Because the enhanced BC-LP mechanism dominates the original BC-LP mechanism in all aspects, we will only focus on the enhanced version and use “the BC-LP mechanism” to refer to the enhanced BC-LP mechanism in the rest of the paper.

3.5. An Example

In this section, we go through a detailed example of how the BC-LP mechanism makes the pricing decision and the allocation decision.

Let us consider an example with three buyers, seven sellers, and one commodity. Buyer 1 wants five units of the commodity, buyers 2 and 3 each want three units of the commodity. Each of the seven sellers provides one unit of the commodity. To keep the example simple, let the transaction costs be zero for all possible trades.

Let us assume that buyer 1 is willing to pay $550 for his five-unit bundle; buyers 2 and 3 are willing to pay $315 and $300 for the three-unit bundle, respectively; sellers 1 through 7 all demand 50 for each unit. This information is illustrated in Figure 3.

If we hold a double auction applying the BC-LP mechanism, all the agents submit their bids in the first step. Because the BC-LP mechanism is strategyproof, it is in the agents’ best interests to submit their true valuations: 550, 315, and 300 for the buyers, and 50 for each seller.

Given these bids, we solve \( \tilde{\mathcal{E}}(I, J) \). In the optimal solution, buyer 1 acquires her bundle; buyer 2 acquires 2/3 of her bundle; buyer 3 acquires nothing; and all the sellers sell their units. To determine \( p_b^S \), we check the shadow price for \( x_i \leq 1 \). This constraint is not binding for buyer 2 and buyer 3, and they are eliminated from the auction. For buyer 1, this constraint is tight, and the shadow price for relaxing this constraint is 25. Thus, the buying price for buyer 1 is 550 − 25 = 525, whereas the remaining buyer set \( \tilde{I} \) consists of only buyer 1.

Then, we solve \( \tilde{\mathcal{E}}(\tilde{I}, J) \). In this optimal solution, buyer 1 acquires her bundle; five of the seven sellers sell their units. Let us assume that the trading sellers are sellers 1 through 5 according to some lexicographic order. In our final allocation, buyer 1 will trade with sellers 1 through 5.

In the final step, we decide the selling price for each trading seller. It turns out the shadow prices for all of their constraints are zero because \( \tilde{\mathcal{V}}(\tilde{I}, J) \) will not change if a
single seller quits the auction. Thus, the trading prices for all trading sellers are \(50 + 0 = 50\).

The payoff for buyer 1 is 25, whereas the payoffs for other buyers and sellers are zero. The payoff of the auctioneer is \(525 - 5 \times 50 = 275\). The social welfare is \(550 - 5 \times 50 = 300\), and the maximum social welfare is \(315 + 300 - 6 \times 50 = 315\), achieved when buyers 2 and 3 trade with six of the seven sellers. The BC-LP mechanism fails to achieve the efficient allocation in this example.

Note that in this example, buyer 1, who does not trade in the efficient allocation, trades in the final allocation of the BC-LP mechanism. Will excluding such buyers improve the efficiency? This question motivates us to design a different trading mechanism, the modified buyer competition mechanism, or MBC mechanism for short. The idea of the MBC mechanism is to have a screen stage before the first pricing decision. By doing so, we can remove those buyers who do not trade in the efficient allocation, that is, to eliminate any buyer who bids lower than or equal to her VCG price.

### 4. Modified Buyer Competition Mechanism

Figure 4 illustrates the diagram for the MBC mechanism. \(p_{iVCG}^k\) is agent \(k\)’s VCG price, the price at which agent \(k\) becomes part of the efficient allocation to \(\mathcal{E}\). For buyer \(i\) who bids higher than her \(p_{iVCG}^k(I, J)\), she must trade in the efficient allocation and her \(p_{iVCG}^k(I, J) = f_i - (V(I, J) - V(I \setminus \{i\}, J))\). Similarly, for trading sellers, \(p_{jVCG}^k(I, J) = g_j + (V(I, J) - V(I, J \setminus \{j\}))\). To ensure the strategyproofness of the mechanism, it turns out that we also need to incorporate these VCG prices into the pricing decisions. The following is the detailed implementation.

**The Implementation of the Modified Buyer Competition Mechanism**

- Collect one sealed bid from each agent.
- Generate an arbitrary lexicographic order.
- Calculate the optimal solution to \(\mathcal{E}(I, J)\) based on the lexicographic order and the VCG price \(p_{iVCG}^k(I, J)\) for each agent.
- Remove all buyers who are not involved in the optimal solution. Let \(\mathcal{I}\) denote the set of trading buyers in the optimal solution.
- Solve linear program \(\mathcal{E}(\mathcal{I}, J)\) and pick the optimal solution based on the lexicographic order.
- For each buyer \(i\), check whether this optimal solution changes if constraint \(x_i \leq 1\) is removed. If so, \(p_{iVCG}^k(\mathcal{I}, J) = f_i - p_{iVCG}(\mathcal{I}, J)\); if not, buyer \(i\) is eliminated.
- Solve the linear program \(\mathcal{E}(\mathcal{I}, J)\), where \(\mathcal{I}\) is the remaining buyer set, and pick the optimal solution based on the lexicographic order.
- For each trading seller \(j\), calculate \(p_{jVCG}(\mathcal{I}, J)\) by solving \(p_{jVCG}(\mathcal{I}, J) = g_j + p_{iVCG}(\mathcal{I}, J)\).
- Conduct transactions according to the optimal solution to \(\mathcal{E}(\mathcal{I}, J)\). The transaction price for trading buyer \(i\) is \(\max\{p_{iVCG}^k(I, J), p_{iVCG}(\mathcal{I}, J)\}\); and the transaction price for trading seller \(j\) is \(\min\{p_{jVCG}^k(I, J), p_{jVCG}(\mathcal{I}, J)\}\).

The MBC mechanism also has the desired properties. Formally, we have:

**THEOREM 9.** The MBC mechanism is strategyproof, (ex post) individual-rational, and (ex post) weakly budget balanced in the bilateral exchange environment with the known single-minded assumption and the single output restriction.

The proof of Theorem 9 is available in the appendix. Furthermore, the MBC mechanism is asymptotically efficient. The proof is similar to the proof of Theorem 5; thus, it is omitted here due to space limitation.

**THEOREM 10.** With bounded valuation distributions and continuous transaction costs, the MBC mechanism is asymptotically efficient if every agent independently draws the agent bundle type from the same distribution in the bilateral exchange environment with the known single-minded assumption and the single output restriction.

Now we apply the MBC mechanism to the example studied in the previous section. Recall that we have three buyers, seven sellers, and one commodity. Buyer 1 wants five units of the commodity, buyers 2 and 3 each want three units of the commodity. Buyers’ valuations are 550, 315, and 300, respectively, and all the sellers’ valuations are 50.

Under the MBC mechanism, all agents will still submit their true valuations because the MBC mechanism is strategyproof. Given these bids, we solve \(\mathcal{E}(I, J)\) and calculate \(p_{iVCG}\). In the optimal solution to \(\mathcal{E}(I, J)\), buyers 2 and 3 trade with six of the seven sellers. The corresponding social welfare is \(315 + 300 - 6 \times 50 = 315\). Let us assume that the
trading sellers are buyers 1 through 6 according to some lexicographic order. The set of trading buyers in the optimal solution, \( I \), consists of buyer 2 and buyer 3.

We also need to calculate the VCG price \( p_{VCG} \). Note that buyer 1 does not trade in the optimal solution to \( \mathcal{E}(I, J) \), and will be eliminated from the auction, so we need not calculate buyer 1’s VCG price. For buyers 2 and 3, we calculate their VCG prices using \( p_{VCG}(I, J) = f_i - (V(I, J) - V(\hat{I}(\{i\}), J)) \). When buyer 2 or buyer 3 is absent from the system, the optimal solution is to let buyer 1 trade with five of the seven sellers with social welfare \( 550 - 5 * 30 = 300 \). Therefore, the VCG price for buyer 2 is \( 315 - (315 - 300) = 300 \), whereas the VCG price for buyer 3 is \( 300 - (315 - 300) = 285 \). Similarly, we calculate the VCG prices of the sellers using \( p_{VCG}(I, J) = g_j + (V(I, J) - V(I, J \setminus \{j\}) \). It turns out that all the sellers’ VCG prices are 50 because the optimal social welfare will not change if a single seller quits the auction.

Now we eliminate buyer 1, and solve \( \mathcal{E}(I, J) \). Using the same lexicographic order, we find that buyers 2 and 3 trade with sellers 1 through 6 in the optimal solution. To determine \( p_{VCG}^2 \), we check the shadow price for \( x_i \leq 1 \). For buyer 2, the shadow price for relaxing this constraint is 165. For buyer 3, the shadow price for relaxing this constraint is 150. Thus, \( p_{VCG}^2 \) for both buyers 2 and 3 are \( 165 = 300 - 150 = 150 \), whereas the remaining buyer set \( \hat{I} \) consists of buyer 2 and buyer 3.

We then solve \( \mathcal{E}(I, J) \). In this optimal solution, buyers 2 and 3 trade with sellers 1 through 6. In our final allocation, buyers 2 and 3 trade with sellers 1 through 6.

In the final step, we determine \( p_{VCG}^j \) for each trading seller. It turns out that the shadow price for each constraint is zero because \( \hat{V}(I, J) \) will not change if a single seller quits the auction. Thus, \( p_{VCG}^j \) for all trading sellers are \( 50 + 0 = 50 \).

Therefore, the trading price for buyer 2 is \( \max\{300, 150\} = 300 \); the trading price for buyer 3 is \( \max\{285, 150\} = 285 \); and the trading prices for all the sellers 1 through 6 are \( \min\{50, 50\} = 50 \).

The payoffs for both buyer 2 and buyer 3 are 15, whereas the payoffs for buyer 1 and the sellers are zero. The payoff of the auctioneer is \( 300 + 285 - 6 * 50 = 285 \). The social welfare is \( 315 + 300 - 6 * 50 = 315 \). The MBC mechanism achieves the efficient allocation in this example.

5. Computational Comparison

We conduct a computational test to study the efficiencies of the BC-LP and MBC mechanisms. We investigate a bilateral exchange environment without transaction costs so that we can compare their performances against the performance of the KSM-TR mechanism (Babaioff and Walsh 2005), the only known truthful mechanism applicable to a bilateral exchange environment in which buyers acquire bundles.\(^7\)

To keep the setting simple, we will have only three types of commodities, and each bundle acquired by a buyer will be represented by an integer triple \((i, j, k)\), where \( i, j, \) and \( k \) each corresponds to the demand for one commodity. We call this integer triple a bundle type. To generate a bundle type, we let \( i, j, \) and \( k \) be independently drawn from uniform 0 to 10.

To execute the KSM-TR mechanism, we need to group the buyers according to their bundle types. Therefore, we investigate the performances of these mechanisms as parameters vary on the following three dimensions: (1) the number of bundle types, which can be large \((M_\ell = 10)\) or small \((M_\ell = 5)\); (2) the number of buyers per bundle type, which can be large \((N_s = 10)\) or small \((N_s = 5)\); and (3) the standard deviation in the valuation distribution, which can be large \((\sigma_L = 20)\) or small \((\sigma_L = 10)\). The number of sellers for each commodity is set to equal the expected total demand of each commodity. The valuations of the agents are independent random variables. The valuation of a seller is normally distributed with mean 100 and variance \( \sigma^2 \).

The valuation of a buyer is normally distributed with mean \((i + j + k)100 \) and variance \((i + j + k)\sigma^2 \), where \((i, j, k)\) is the bundle she wants.

Parameters and notation used in the computational test are summarized in Table 1.

In Table 2, we summarize the finding from this computational test. We solve MIP formulation \( \mathcal{E} \) to find the maximum social welfare to be our benchmark. \( A(\%) \) represents the average efficiency, the average of social welfare achieved by a mechanism over the maximum feasible social welfare; \( E(\%) \) denotes the percentage of instances in which a mechanism yields efficient allocation; and \( B(\%) \) indicates the percentage of instances in which a mechanism gives the best efficiency result among the three mechanisms. It is observed that, for all scenarios, MBC achieves the highest efficiency. BC-LP is also highly desirable because it involves only solving linear programming problems and still achieves over 95% efficiency in all scenarios. The computational test seems to support the viewpoint that we can improve average efficiency by excluding buyers that do not trade in the efficient allocation to formulation \( \mathcal{E} \). The computational results also show that the performance of the KSM-TR mechanism is sensitive to the number of buyers.

<table>
<thead>
<tr>
<th>Table 1. Parameter settings.</th>
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<tbody>
<tr>
<td><strong>Variable name</strong></td>
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<tr>
<td>Number of bundle types</td>
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<tr>
<td>Number of buyers for each bundle type</td>
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<tr>
<td>Standard deviation of seller’s valuation ( \sigma )</td>
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<tr>
<td>Bundle formation ((i, j, k))</td>
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<td>Number of sellers</td>
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<td>Valuation of sellers</td>
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<td>Valuation of buyers</td>
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for each bundle type, and its efficiency can be substantially lower than that of both the MBC mechanism and the BC-LP mechanism. In fact, we prove that the MBC mechanism dominates the KSM-TR mechanism for all instances. For details, please refer to Chu and Shen (2007).

As the number of agents increases, all the mechanisms become more efficient because they are all asymptotically efficient in the current computational test scheme. For all three mechanisms, the increase of the standard deviation in valuation distributions seems to have only trivial impact.

6. Conclusions

In this paper, we propose the BC-LP mechanism and the MBC mechanism for a bilateral exchange environment with the known single-minded assumption and the single output restriction. These mechanisms can be applied to online procurement marketplaces where various transaction costs exist. The BC-LP and MBC mechanisms extend the reach of truthful double auctions to handle known transaction costs, while outperforming the KSM-TR mechanism in terms of efficiency on environments without transaction costs.

We also want to emphasize the important contribution in terms of implementation. One of the major concerns in implementing procurement auctions is that almost every such auction involves solving some sort of NP-hard optimization problem (\(c(I, J)\)) is an NP-hard formulation), which can be very time consuming. We show that the computation in our BC-LP mechanism is time efficient because the mechanism is based on two linear programming problems.

One important future research question is to relax the single output restriction, that is, to study an exchange environment where a seller can provide multiple units of a single commodity. One way to apply the BC-LP or the MBC mechanisms to this setting is to treat a seller with multiple units of supply as multiple copies of a seller with single unit output. If we do so, these mechanisms remain (ex post) weakly budget balanced. Furthermore, each buyer will still find that bidding true valuation is her best strategy; thus, these mechanisms are strategyproof and (ex post) individual-rational for the buyers. These mechanisms are also (ex post) individual-rational for the sellers. Nevertheless, the sellers may find it profitable to overstate their valuations. Therefore, these mechanisms are not strategyproof for the sellers. A truthful mechanism for a setting without the single output restriction is an intriguing and challenging task for the near future.

Appendix A. Proofs of Propositions 1 and 2

Propositions 1 and 2 explore the relationship between \(p_i^b\) and the shadow price \(p_i\). The following linear program representation will be used extensively to facilitate the presentation.

\[
\hat{c}(I, J)(\lambda, \mu): \quad \hat{V}(I, J)(\lambda, \mu) = \text{Maximize} \sum \sum_{j \in J} f_j x_j - \sum g_j x_j - \sum d_{i_1,j,c} z_{i_1,j,c} \quad \text{subject to} \sum z_{i_1,j,c} = q_i f_j \quad \text{for each } i \in I, c \in C,
\]

\[
0 \leq z_{i_1,j,c} \quad \text{for each } i \in I, j \in J, c \in C,
\]

\[
0 \leq x \leq 1 + \lambda,
\]

\[
0 \leq y \leq 1 + \mu.
\]

where \(\lambda\) is a column vector with \(|I|\) entries and \(\mu\) is a vector with \(|J|\) entries. \(x\) is a column vector with \(|I|\) entries, and the \(i\)th entry is \(x_i\); \(y\) is a column vector with \(|J|\) entries, and the \(j\)th entry is \(y_j\). Notations \(0\) and \(e_i\) denote a column vector where all entries are zero and a column vector where the \(k\)th entry is one and all other entries are zero, respectively. For the remainder of this section, we may use \(\hat{V}(\lambda, \mu)\) instead of \(\hat{V}(I, J)(\lambda, \mu)\) if the agent sets \(I\) and \(J\) are the original buyer set and seller set. Note that as long as \(\lambda + 1 \geq 0\) and \(\mu + 1 \geq 0\), the linear program has feasible solution \(\{0\} \). Furthermore, the objective value function \(\hat{V}(\lambda, \mu)\) is concave, as well as piecewise linear nondecreasing for each component of \(\lambda\) and \(\mu\).

**Lemma 1.** For all \(i \in I, f_i > p_i^b(I, J)\) if and only if \(\hat{V}(e_i, 0) > \hat{V}(0, 0)\).

**Proof.** Recall that \(p_i^b\) is the infimum of bid prices for buyer \(i\) that makes \(\hat{V}(e_i, 0) > \hat{V}(0, 0)\).

If \(\hat{V}(e_i, 0) > \hat{V}(0, 0)\), \(f_i \geq p_i^b\) according to the definition of \(p_i^b\). Because both \(\hat{V}(e_i, 0)\) and \(\hat{V}(0, 0)\) are continuous functions of \(f_i\), there exists some \(\varepsilon > 0\) such that \(\hat{V}(e_i, 0) > \hat{V}(0, 0)\) if the coefficient in the objective function is \(f_i - \varepsilon\) instead of \(f_i\). Then, \(f_i - \varepsilon \geq p_i^b\) according to the definition of \(p_i^b\). Thus, \(f_i > p_i^b\) if \(\hat{V}(e_i, 0) > \hat{V}(0, 0)\).

If \(f_i > p_i^b\), according to the definition of \(p_i^b\), there exists some bid price \(f_i'\left( p_i^b \leq f_i' \leq f_i\right)\) such that if we replace the coefficient \(f_i\) with some \(f_i'\), \(\hat{V}(e_i, 0) > \hat{V}(0, 0)\). Therefore,
the optimal solution for \( \hat{E}(e, 0) \) must have \( x_i > 1 \) under coefficient \( f_i' \). If we increase the coefficient from \( f_i' \) to \( f_i \), \( \tilde{V}(0, 0) \) increases by at most \((f_i - f_i') \cdot 1 = (f_i - f_i')\), and \( \hat{V}(e, 0) \) increases by at least \((f_i - f_i')x_i\), which is greater than \((f_i - f_i')\). Thus, \( \tilde{V}(e, 0) > \hat{V}(0, 0) \) if \( f_i > p_i^b \). □

Because \( \hat{V}(\lambda, \mu) \) is piecewise linear nondecreasing concave for each component of \( \lambda \) and \( \mu \), \( \hat{V}(re_i, 0) \) is a piecewise-linear nondecreasing concave function of \( r \) \((r > 0)\), and the right and left derivatives of \( \hat{V}(re_i, 0) \) must exist. The minimum shadow price \( p_i \) is the right derivative of \( \hat{V}(re_i, 0) \) at \( r = 0 \). Similarly, the maximum shadow price \( \bar{p}_j \) is the left derivative of \( \tilde{V}(0, re_j) \) at \( r = 0 \). Now, we are ready to prove Propositions 1 and 2.

**Proof of Proposition 1.** By Lemma 1, \( f_i > p_i^b \) if and only if \( \tilde{V}(e_i, 0) > \hat{V}(0, 0) \). Because \( \hat{V}(re_i, 0) \) is a nondecreasing concave function of \( r \), \( \hat{V}(e_i, 0) > \hat{V}(0, 0) \) if and only if the right derivative of \( \hat{V}(re_i, 0) \) with respect to \( r \) is positive; that is, if and only if \( p_i > 0 \). Thus, for all \( i \in I \), \( f_i > p_i^b \) if and only if \( p_i > 0 \). □

**Proof of Proposition 2.** By Proposition 1, if \( p_i > 0 \), \( f_i > p_i^b \), and \( p_i^b \) is a finite value that is independent of bid price \( f_i \). Also by Proposition 1, if \( f_i = p_i^b \), the corresponding \( p_i \) is \( 0 \). Because \( \hat{V}(re_i, 0) \) is an increasing function of \( r \), we always have \( p_i > 0 \). Thus, \( p_i = 0 \) if \( f_i = p_i^b \).

To prove \( p_i^b = f_i - p_i \) when \( f_i > p_i^b \), it suffices to show that if \( f_i \) increases from \( p_i^b \) to \( p_i^b + \Delta \) for some \( \Delta > 0 \), the corresponding \( p_i \) increases from zero to \( \Delta \). We prove this result by investigating how \( p_i \) may change with \( f_i \) when \( f_i \) is greater than \( p_i^b \).

If \( f_i \) is greater than \( p_i^b \), by Lemma 1 we have \( \hat{V}(e_i, 0) > \hat{V}(0, 0) \); thus, if \( x_i \leqslant 1 \), the constraint associated with buyer \( i \) in formulation \( \hat{V}(0, 0) \) must be tight. As \( f_i \) increases from \( p_i^b \) to \( p_i^b + \Delta \), \( \hat{V}(0, 0) \) increases by \( \Delta \). For any \( r > 0 \), as \( f_i \) increases from \( p_i^b \) to \( p_i^b + \Delta \), \( \hat{V}(e_i, 0) \) increases by at most \((1 + r)\Delta r = \Delta \). To show that \( p_i \) increases from 0 to \( \Delta \), it suffices to show that for any small \( \epsilon > 0 \), we can find a \( r > 0 \) such that \( \hat{V}(re_i, 0) > (\Delta - \epsilon) \).

Now set \( f_i = p_i^b + \epsilon \). Because \( \hat{V}(re_i, 0) > \hat{V}(0, 0) \), we can pick \( r > 0 \) such that \( \hat{V}(re_i, 0) > \hat{V}(0, 0) \) and constraint \( x_i \leqslant 1 + r \) in formulation \( \hat{V}(re_i, 0) \) is tight. As \( f_i \) increases from \( p_i^b + \epsilon \) to \( p_i^b + \Delta \), \( \hat{V}(re_i, 0) \) increases by \((1 + r)\Delta 0 \), whereas \( \hat{V}(0, 0) \) increases by \( \Delta \). Therefore, \( \hat{V}(re_i, 0) - \hat{V}(0, 0) > (\Delta - \epsilon) \), and \( p_i \) is at least \( \Delta - \epsilon \). Because \( \epsilon \) can be arbitrarily small, \( p_i \) equals \( \Delta \) as \( f_i \) increases from \( p_i^b + \epsilon \) to \( p_i^b + \Delta \). We can conclude that for all \( i \in I \), if \( p_i > 0 \), \( p_i^b = f_i - p_i \). □

**Appendix B. Proof of Theorem 1**

Propositions 1 and 2 illustrate the fundamental relationship between the threshold price \( p_i^b \) and the shadow price \( p_i \) and enable us to implement the first pricing stage efficiently. Now we will prove Theorem 1, which enables us to find the final allocation efficiently.

**Lemma 2.** If \( f_i > p_i^b \), then \( x_i = 1 \) in every optimal solution to \( \hat{E}(I, J) \).

**Proof.** By Proposition 1, if buyer \( i \) bids higher than \( p_i^b(I, J) \), the corresponding shadow price is positive. Therefore, constraint \( x_i \leqslant 1 \) is tight and \( x_i \) must equal one in every optimal solution to \( \hat{E}(I, J) \). □

Now we show that if buyer \( i \) bids \( f_i > p_i^b(I, J) \), then she will trade in the final allocation. We first prove \( x_i = 1 \) in every optimal solution to \( \hat{E}(I, J) \), where \( I = \{i \mid f_i > p_i^b(I, J), i \in I\} \) is the remaining buyer set.

To establish this result, it is helpful to view the solutions of \( \hat{E}(I, J) \) and \( \hat{E}(I, J) \) as flows sent from sellers to buyers (by the flow decomposition theorem; Ahuja et al. 1993). Let us consider the difference between an optimal solution \( [\hat{x}, \hat{y}, \hat{z}] \) to \( \hat{E}(I, J) \) and an optimal solution \( [\tilde{x}, \tilde{y}, \tilde{z}] \) to \( \tilde{E}(I, J) \). We know the difference can be represented as finite path flows and cycle flows; that is, if we send a certain amount of flows along these paths and cycles, we can get \( [\tilde{x}, \tilde{y}, \tilde{z}] \) from \( [\hat{x}, \hat{y}, \hat{z}] \). Moreover, if we obtain these paths and cycles according to the flow decomposition theorem, we can send partial amounts of these flows along the paths and cycles without violating the nonnegative constraints of the arcs because the flow of each arc will be between \( \hat{z} \) and \( \tilde{z} \). Each path connects a source to a sink. Note that no buyer can be a sink because \( \hat{x} \geqslant \tilde{x} \); thus, there are two possibilities for a path flow: (1) a seller to a seller; (2) a buyer to a seller.

Figure B.1 illustrates the flow representations of solutions to \( \hat{E}(I, J) \) and \( \hat{E}(I, J) \), and their difference. In this example, only the middle buyer is eliminated. The difference is represented by two flows: One goes from a seller to a seller, and the other goes from a buyer to a seller.

**Lemma 3.** If \( f_j > p_j^b(I, J) \), then \( x_j = 1 \) in every optimal solution to \( \hat{E}(I, J) \).

**Proof.** We prove by contradiction. Consider the flow decomposition of the difference between an optimal solution \( [\hat{x}, \hat{y}, \hat{z}] \) to \( \hat{E}(I, J) \) and an optimal solution \( [\tilde{x}, \tilde{y}, \tilde{z}] \) to \( \tilde{E}(I, J) \). Let \( \mathcal{F} \) denote the flow set in the difference. If \( f_j > p_j^b(I, J) \), \( x_j = 1 \) in every optimal solution to \( \hat{E}(I, J) \).

![Figure B.1. Illustration of the flows.](image-url)
by Lemma 2. Therefore, if \( x_i < 1 \), we have flows starting from buyer \( i \) in the difference of the two optimal solutions. We can obtain another feasible solution to the original problem \( \tilde{\mathcal{E}}(I, J) \) by adding a proportion of all flows in \( T \) to \( \{\hat{x}, \hat{y}, \hat{z}\} \). The generated solution cannot be optimal for \( \tilde{\mathcal{E}}(I, J) \) because \( x_i < 1 \) in the generated solution and \( x_i = 1 \) in every optimal solution to \( \tilde{\mathcal{E}}(I, J) \) by Lemma 2; thus, the objective function value must decrease. Therefore, if \( x_i < 1 \), by removing a proportion of all flows in \( T \) from \( \{\hat{x}, \hat{y}, \hat{z}\} \), we can get a feasible solution to \( \tilde{\mathcal{E}}(I, J) \) with higher objective function value compared with the optimal solution \( \{X, \hat{y}, \hat{z}\} \). This is a contradiction. Thus, we must have \( x_i = 1 \) in every optimal solution to \( \tilde{\mathcal{E}}(I, J) \). \( \square \)

**Proof of Theorem 1.** By Lemma 3, \( x_i = 1 \) \((i \in \hat{I})\) in every optimal solution to \( \tilde{\mathcal{E}}(I, J) \). Given that the optimal \( x_i \) is integer valued and that the single output restriction applies to the sellers, the linear program \( \tilde{\mathcal{E}}(I, J) \) is a network flow problem. Thus, all the extreme points of the optimal solution set to \( \tilde{\mathcal{E}}(I, J) \) are integer valued. \( \square \)

Therefore, we can solve the integer program \( \mathcal{E}(I, J) \) by solving its linear relaxation \( \tilde{\mathcal{E}}(I, J) \) and picking an extreme point optimal solution.

**Appendix C. Proof of Theorem 3**

With Propositions 1 and 2 and Theorem 1, we are now able to establish our main strategyproof result. We first show that truthful revelation is a dominant strategy for each buyer.

**Theorem 11.** Bidding truthfully is a (weakly) dominant strategy for each buyer.

**Proof.** Note that \( p^b_i(I, J) \) is determined by the agent set \( \hat{I} \cup J \setminus \{i\} \), and it is independent of the bid price \( f_i \). Also, if buyer \( i \) trades, her payment is \( p^b_i(I, J) \). Thus, if buyer \( i \)’s valuation is higher than \( p^b_i(I, J) \), she prefers to trade, which will happen if she bids her valuation by Lemma 3. If her valuation is lower than \( p^b_i(I, J) \), she prefers not to trade, which can also be achieved by bidding her valuation. If her valuation equals \( p^b_i(I, J) \), she is indifferent between trading and not trading. Thus, bidding truthfully is a (weakly) dominant strategy for each buyer. \( \square \)

Now we need the following lemmas to show that truthful revelation is a dominant strategy for each seller. Let \( \hat{p}(j) \) denote the maximum bid price for seller \( j \) such that there exists an optimal solution to \( \tilde{\mathcal{E}}(I, J) \) satisfying \( y_j = 1 \). (The case when such \( \hat{p}(j) \) does not exist is trivial and seller \( j \) will never trade.) We first show that if seller \( j \) bids higher than \( \hat{p}(j) \), he does not trade.

**Lemma 4.** If seller \( j \in J \) bids higher than \( \hat{p}(j) \), then he does not trade.

**Proof.** By Lemma 2, \( x_i = 1 \) for \( i \in \hat{I} \) in every optimal solution to \( \tilde{\mathcal{E}}(I, J) \). By Lemma 3, \( x_i = 1 \) for \( i \in \hat{I} \) in every optimal solution to \( \tilde{\mathcal{E}}(I, J) \). Now, let us apply the flow decomposition to the difference of the optimal solutions to \( \tilde{\mathcal{E}}(I, J) \) and \( \tilde{\mathcal{E}}(I, J) \). The flows are either from a seller to a seller or from a buyer to a seller. Because seller \( j \in J \) bids higher than \( \hat{p}(j) \), no optimal solution to \( \tilde{\mathcal{E}}(I, J) \) satisfies \( y_j = 1 \). Using the flow decomposition representation, we can show that no optimal solution to \( \tilde{\mathcal{E}}(I, J) \) satisfies \( y_j = 1 \). Because all of the extreme points of the optimal solution set to \( \tilde{\mathcal{E}}(I, J) \) are integer valued by Theorem 1, \( y_j = 0 \) in every optimal solution. That is, seller \( j \) does not trade in the remaining system consisting of \( \hat{I} \) and \( J \). \( \square \)

**Lemma 5.** No remaining buyer will be eliminated if seller \( j \) increases his bid, as long as his bid is no greater than \( \hat{p}(j) \).

**Proof.** Note that \( f_i > p^b_i(I, J) \) is equivalent to \( \hat{V}(e_i, 0) - \hat{V}(0, 0) > 0 \) by Lemma 1. If seller \( j \) increases his bid price by \( \Delta \), as long as he bids no more than \( \hat{p}(j) \), \( \hat{V}(0, 0) \) decreases by \( \Delta \), whereas \( \hat{V}(e_i, 0) \) decreases by at most \( \Delta \). Thus, the difference \( \hat{V}(e_i, 0) - \hat{V}(0, 0) \) is still positive, and buyer \( i \) is not eliminated. \( \square \)

Now let us consider the remaining buyer set as the bid price \( g_j \) of seller \( j \) goes from \( \hat{p}(j) \) to negative infinity. By Lemma 5, the remaining buyer set becomes smaller and smaller as the bid price decreases. We use \( \hat{I} \) to denote the limit of the remaining buyer set. Note that because there is only a finite number of possible remaining buyer sets, if seller \( j \) bids low enough, the corresponding set of the remaining buyers becomes the limit set, \( \hat{I} \). We use \( \tilde{p}_j \) to denote \( p^b_j(I, J) \), the supremum of bid prices of seller \( j \) satisfying \( \hat{V}(\hat{I}, J) > \hat{V}(\hat{I}, J) \).

In the following, we prove that \( \tilde{p}_j \) is indeed the critical price for seller \( j \). If he bids lower than \( \tilde{p}_j \), he trades. If he bids higher than \( \tilde{p}_j \), he does not trade. Also, when seller \( j \) trades, the remaining buyer set is \( \hat{I} \).

**Lemma 6.** For all \( j \in J \), \( \tilde{p}_j \leq \hat{p}(j) \).

**Proof.** We prove this by contradiction. Assume that for some seller \( j \in J \), \( \hat{p}(j) < \tilde{p}_j \). By Lemma 5, if seller \( j \) bids \( \hat{p}(j) \), all buyers in \( \hat{I} \) survive the elimination phase; that is, \( \hat{V}(e_i, 0) - \hat{V}(0, 0) > 0 \) for all \( i \in \hat{I} \). Because \( \hat{V}(0, 0) \) and \( \hat{V}(e_i, 0) \) are continuous functions of bid price \( g_j \), there exists some \( \epsilon > 0 \) such that \( \hat{p}(j) + \epsilon < \tilde{p}_j \) and \( \hat{V}(e_i, 0) - \hat{V}(0, 0) > 0 \) for all \( i \in \hat{I} \). If seller \( j \) bids \( \hat{p}(j) + \epsilon \). Now, assume that seller \( j \) bids \( \hat{p}(j) + \epsilon \). By Lemma 2 and the fact that \( \hat{I} \subseteq \hat{I} \), \( x_i = 1 \) for \( i \in \hat{I} \) in every optimal solution to \( \tilde{V}(I, J) \). Therefore, if we apply the flow decomposition to the difference of the optimal solutions to \( \tilde{\mathcal{E}}(I, J) \) and \( \tilde{\mathcal{E}}(I, J) \), we can show \( x_i = 1 \) for \( i \in \hat{I} \) in every optimal solution to \( \tilde{\mathcal{E}}(I, J) \). Because seller \( j \) bids higher than \( \hat{p}(j) \), no optimal solution to \( \tilde{\mathcal{E}}(I, J) \) satisfies \( y_j = 1 \). Using the flow decomposition representation, we can show that no optimal solution to \( \tilde{\mathcal{E}}(I, J) \) satisfies \( y_j = 1 \). Given that optimal \( x_i = 1 \) for \( i \in \hat{I} \) and that the single output restriction applies to the sellers, \( \tilde{\mathcal{E}}(I, J) \) is a network flow problem; thus, all the extreme points for the optimal solution set of...
Lemma 7. If seller $j$ bids lower than $\hat{p}_j$, he trades.

Proof. As shown in the proof of Lemma 6, all the extreme points for the optimal solution set to $\hat{\mathcal{E}}(I, J)$ are integer valued. Because seller $j$ bids lower than $\hat{p}_j$, $y_j > 0$ in every optimal solution to $\hat{\mathcal{E}}(I, J)$; therefore, $y_j = 1$ in every optimal solution to $\hat{\mathcal{E}}(I, J)$. By Lemma 5, we know that the remaining buyer set $\hat{I}$ contains $\hat{I}_j$. By Lemma 3, $x_i = 1$ for $i \in \hat{I}_j \subseteq I$ in every optimal solution to $\hat{\mathcal{E}}(I, J)$. Therefore, if we apply the flow decomposition to the difference of the optimal solutions to $\hat{\mathcal{E}}(I, J)$ and $\hat{\mathcal{E}}(\hat{I}, J)$, we can show that $x_i = 1$ for $i \in \hat{I}_j$ in every optimal solution to $\hat{\mathcal{E}}(\hat{I}, J)$. Because $y_j = 1$ in every optimal solution to $\hat{\mathcal{E}}(\hat{I}, J)$, we can show that the optimal solution to $\hat{\mathcal{E}}(\hat{I}, J)$ must also have $y_j = 1$ by using the flow decomposition representation. Therefore, seller $j$ trades if he bids lower than $\hat{p}_j$. □

Lemma 8. If $\hat{I} \neq \hat{I}_j$, seller $j$ does not trade.

Proof. By Lemma 4, if seller $j$ bids higher than $\hat{p}(j)$, he does not trade. Now consider the case in which the bid price $g_j$ is no more than $\hat{p}(j)$.

By Lemma 5, the remaining buyer set $\hat{I}$ always contains $\hat{I}_j$. If $\hat{I} \neq \hat{I}_j$, there must be some buyer $i \in I \setminus \hat{I}_j$ in the remaining buyer set $\hat{I}$. For each remaining buyer $i \in \hat{I}$, by Lemma 1, $\hat{V}(\epsilon e_i, 0) > \hat{V}(0, 0)$. $\hat{V}(\lambda, \mu)$ is a continuous function. Therefore, there exists some $\epsilon_j > 0$ such that $\hat{V}(\epsilon e_i + \epsilon_j e, 0) > \hat{V}(\epsilon e_i, 0)$ for all $0 < \epsilon < \epsilon_j$. Because there are a finite number of buyers, there exists an $\epsilon_j > 0$ such that $\hat{V}(\epsilon e_i + \epsilon_j e, 0) > \hat{V}(\epsilon e_i, 0)$ for all the remaining buyers $\hat{I}$. This means that in the optimal solutions to $\hat{\mathcal{E}}(\epsilon e_i, 0)$, $x_i = 1$ for all $i \in I \setminus \hat{I}$. Because buyer $l$ survives the elimination phase, Proposition 1 implies that $\hat{\mathcal{V}}(\epsilon e_i, 0) > \hat{\mathcal{V}}(0, 0)$, and the optimal $x_i$ in every optimal solution to $\hat{\mathcal{E}}(\epsilon e_i, 0)$ is greater than one.

Thus, $x_i \geq 1$ for all $i \in \hat{I}$ in the optimal solutions to $\hat{\mathcal{E}}(\epsilon e_i, 0)$. Furthermore, $x_i = 1$ for all $i \in \hat{I}$ in the optimal solutions to $\hat{\mathcal{E}}(\hat{I}, J)$ by Lemma 3. Now let us apply the flow decomposition to the difference of the optimal solutions to $\hat{\mathcal{E}}(\epsilon e_i, 0)$ and $\hat{\mathcal{E}}(\hat{I}, J)$—the flows are either from a seller to a buyer or from a buyer to a seller.

Note that if seller $j$ bids low enough, buyer $l$ is eliminated and $\hat{\mathcal{V}}(\epsilon e_i, 0) = \hat{\mathcal{V}}(0, 0)$. If the bid price $g_j$ increases, as long as $g_j < \hat{p}(j)$, $\hat{\mathcal{V}}(\epsilon e_i, 0)$ decreases at the same rate, and $\hat{\mathcal{V}}(\epsilon e_i, 0)$ must decrease at a slower rate to have $\hat{\mathcal{V}}(\epsilon e_i, 0) > \hat{\mathcal{V}}(0, 0)$. Thus, $y_j < 1$ in every optimal solution to $\hat{\mathcal{E}}(\epsilon e_i, 0)$.

Now recall the flow decomposition representation to the difference of the optimal solutions to $\hat{\mathcal{E}}(\epsilon e_i, 0)$ and $\hat{\mathcal{E}}(\hat{I}, J)$; we can prove that $y_j < 1$ in every optimal solution to $\hat{\mathcal{E}}(\hat{I}, J)$ because $y_j < 1$ in every optimal solution to $\hat{\mathcal{E}}(\epsilon e_i, 0)$. Because all the extreme points of the optimal solution set to $\hat{\mathcal{E}}(\hat{I}, J)$ are integer valued by Theorem 1,
and \( \{x', y', z'\} \) to \( \hat{\mathcal{L}}(I, J) \), where \( y'_j = 0 \) and \( y''_j = 1 \), and 
\[
\hat{V}(I, J) = \hat{V}_J(I, J) = \hat{V}_I(I, J)(0, r_e) \text{ for } -1 \leq r \leq 0.
\]

As seller \( j \) lowers his bid price from \( \hat{p}_j^r \) to \( \hat{p}_j^r - \Delta \), \( I = \hat{I}, \nabla(I, J) \) increases by \( \Delta \), and \( \hat{\nu}(I, J) \) remains the same. The constraint associated with seller \( j \) in \( \hat{\mathcal{L}}(I, J)(0, r_e) \) \(-1 \leq r \leq 0\) is tight as long as the bid price is lower than \( \hat{p}_j^r \), so \( \hat{V}(I, J)(0, r_e) \) increases by \((1 + r)\Delta \) for \(-1 \leq r \leq 0 \) as seller \( j \) lowers his bid price. Thus, \( \hat{p}_j^r(I, J) = \Delta \), and \( \hat{p}_j^r = p_j^r(I, J) = g_j + \hat{V}(I, J) - \hat{\nu}_.(I, J) = g_j + \hat{p}_j(I, J) \) for trading seller \( j \). □

Appendix E. Proof of Theorem 4

In this section, we show that the BC-LP mechanism is (ex post) weakly budget balanced in the bilateral exchange environment with the single output restriction. We show this result by letting each agent’s bid price converge to his/her trade price and proving that the final allocation is still efficient in the remaining system. This is accomplished by the following lemmas.

**Lemma 9.** For all \( j \in J, x_i = 1 \) for all \( i \in I \) in every optimal solution to \( \hat{\mathcal{L}}_.(I, J) \).

**Proof.** If \( y_j = 0 \) in the optimal solution to \( \hat{\mathcal{L}}(I, J) \), then 
\[
\hat{V}_J(I, J) = \hat{V}(I, J).
\]
Because \( x_i = 1 \) for all \( i \in I \) in every optimal solution to \( \hat{\mathcal{L}}(I, J) \) by Lemma 3, the same is true for every optimal solution to \( \hat{\mathcal{L}}_.(I, J) \).

Now consider the case where seller \( j \) trades in \( \hat{\mathcal{L}}(I, J) \). By Theorem 11, seller \( j \) bids no more than \( \hat{p}_j^r \), and the remaining buyer set \( \hat{I} = \hat{I}_j \). By Lemma 5, no remaining buyer is eliminated if seller \( j \) increases his bid, as long as his bid is no greater than \( \hat{p}(j) \). So if seller \( j \) bids \( \hat{p}_j^r \), no remaining buyer is eliminated. That is, 
\[
\hat{V}(e_j, 0) > \hat{V}(0, 0) \text{ for each buyer } i \in I \text{ by Lemma 1 when } g_j = \hat{p}_j^r.
\]

Now assume that seller \( j \) raises his bid to above \( \hat{p}_j^r \). Because \( \hat{V}(\lambda, p) \) is continuous, there exists some \( \epsilon > 0 \) such that \( \hat{V}(e_j, 0) > \hat{V}(0, 0) \) for all \( i \in I \) as long as \( g_j < \hat{p}_j^r + \epsilon \). Thus, if seller \( j \) bids between \( \hat{p}_j^r \) and \( \hat{p}_j^r + \epsilon \), \( x_i = 1 \) for all \( i \in I \) in every optimal solution to \( \hat{V}(0, 0) \). Therefore, if we apply the flow decomposition to the difference of the optimal solutions to \( \hat{\mathcal{L}}(I, J) \) and \( \hat{\mathcal{L}}(I, J) \), we can show \( x_i = 1 \) for all \( i \in I \) in every optimal solution to \( \hat{V}(I, J) \) when \( \hat{p}_j^r < g_j < \hat{p}_j^r + \epsilon \). Because we have set seller \( j \)'s bid price above \( \hat{p}_j^r \), \( y_j = 0 \) in every optimal solution to \( \hat{\mathcal{L}}(I, J) \); that is, 
\[
\hat{V}(I, J) = \hat{\nu}_.(I, J).
\]
Note that the optimal solution set to \( \hat{\mathcal{L}}_.(I, J) \) is independent of bid price \( g_j \). Therefore, we have \( x_i = 1 \) for all \( i \in I \) in every optimal solution to \( \hat{\mathcal{L}}_.(I, J) \). □

**Lemma 10.** If seller \( k \) bids lower than \( \hat{p}_k^r \), then \( y_k = 1 \) in every optimal solution to \( \hat{\mathcal{L}}_.(I, J) \).

**Proof.** We prove this lemma by contradiction. Consider the flow decomposition of the difference between an optimal solution to \( \hat{\mathcal{L}}(I, J) \) and an optimal solution to \( \hat{\mathcal{L}}_.(I, J) \) in which \( y_k < 1 \). By Lemmas 3 and 9, \( x_i = 1 \) for all \( i \in I \) in every optimal solution to \( \hat{\mathcal{L}}(I, J) \), and every optimal solution to \( \hat{\mathcal{L}}_.(I, J) \). Thus, all the flows of the difference must go from a seller to a seller.

Because \( k \) bids lower than \( \hat{p}_k^r \), \( y_k = 1 \) in every optimal solution to \( \hat{\mathcal{L}}(I, J) \), and \( y_k < 1 \) in an optimal solution to \( \hat{\mathcal{L}}_.(I, J) \). Therefore, there exists a flow starting from seller \( k \) in the flow decomposition. Starting from the optimal solution to \( \hat{\mathcal{L}}(I, J) \), sending this flow in the reverse direction gives us another feasible solution to \( \hat{\mathcal{L}}(I, J) \) because each seller only provides one type of commodity. Furthermore, because \( y_k < 1 \) in this feasible solution, the objective function value must decrease. Thus, starting from the optimal solution to \( \hat{\mathcal{L}}_.(I, J) \), this flow gives us another feasible solution to \( \hat{\mathcal{L}}_.(I, J) \) with a higher objective function value than its optimal solution. This is a contradiction. Thus, \( y_k = 1 \) in every optimal solution to \( \hat{\mathcal{L}}_.(I, J) \). □
(iii) Because seller $k$ trades in every efficient allocation to $\hat{\mathcal{E}}(I, J)$ if he bids lower than $\hat{p}_k^0$ by Lemma 7, the amount of the social welfare decreased is the same as the amount by which he increases his bid. (iiii) follows. □

**Lemma 12.** Suppose that buyer $k$ bids $f_k > p_k^0(I, J)$, then lowers her bid to $p_k^0(I, J) + \epsilon_k \in (0, \infty)$. We show that (i) the remaining buyer set and $p_k^0(I, J)$ for buyer i in the set do not change; (ii) the original efficient allocation to $\hat{\mathcal{E}}(I, J)$ is still efficient.

Proof. (i) If buyer $k$ bids higher than $p_k^0(I, J) + \epsilon_k$, by Lemma 1, $\tilde{V}(e_k, 0) > \tilde{V}(0, 0)$. Because $\tilde{V}(\lambda, \mu)$ is continuous, there exists some $\delta > 0$ such that for all $i \in I$, $\tilde{V}(e_i + e_k, 0) > \tilde{V}(e_i, 0)$ as long as $0 < \epsilon < \delta$. Thus, the constraint associated with buyer $k$ must be tight in $\tilde{V}(e_i, 0)$, and $x_i = 1$ in every optimal solution to $\hat{\mathcal{E}}(e_i, 0)$.

Thus, if buyer $k$ lowers her bid price $f_k$ to $p_k^0(I, J) + \epsilon_k$, $\tilde{V}(e_i, 0)$, and $\tilde{V}(0, 0)$ decrease the same amount, $\tilde{V}(e_i, 0) - \tilde{V}(0, 0)$ remains the same, and $p_k(I, J)$ remains the same. By Proposition 1, the remaining buyer set does not change. By Proposition 2, for all $i \in I$, $p_k^0(I, J) = f_i - p_i(I, J)$; thus, $p_k^0(I, J)$ for buyer $i$ remains the same.

(ii) By Theorem 11, if buyer $k$ bids higher than $p_k^0(I, J)$, she trades in every efficient allocation to $\hat{\mathcal{E}}(I, J)$, and the social welfare decreases by an amount equal to the change in the bid. (iiii) follows. □

**Proof of Theorem 4.** By Lemma 11, if we let all the trading sellers who bid lower than $\hat{p}_j^0$ raise their bids to $\hat{p}_j^0 + \epsilon_j$ (for some $\epsilon_j > 0$) one by one, the remaining buyer set does not change and the original allocation in the remaining system is still efficient. Then, by Lemma 12, we can let all the trading buyers who bid higher than $p_k^0(I, J)$ lower their bids to $p_k^0(I, J) + \epsilon_k$ (for some $\epsilon_j > 0$) one by one while keeping the original allocation in the remaining system efficient. Because the original allocation in the remaining system is still efficient, the objective function value must be nonnegative because zero is a feasible solution. We have $\sum(p_k^0(I, J) + \epsilon_j)x_j - \sum(\hat{p}_j^0 - \epsilon_j)y_j - \sum d_{i,j} + c_{i,j,i,j} \geq 0$ for the optimal allocation $(x, y, z)$ in the remaining system. Because $\epsilon_j$ and $\epsilon_k$ can take arbitrary positive numbers, $\sum p_k^0(I, J)x_j - \sum \hat{p}_j^0y_j - \sum d_{i,j} + c_{i,j,i,j} \geq 0$, i.e., the BC-LP mechanism is (ex post) weakly budget balanced. □

**Appendix F. Proof of Theorem 5**

In this section, we prove the asymptotic efficiency result of the BC-LP mechanism.

Assume that there are finite types of commodities, and there exists a number $M$ such that $\sum_{c \in C} q^c < M$ for every buyer $i$. That is, $M$ is the limit on how many units of commodities a buyer can acquire. Let each buyer's valuation independently follow $F_{\alpha, b, c}$, where $(q^c_{\alpha, b, c})_{c \in C}$ is the bundle buyer $i$ acquires, and let each seller's valuation independently follow $G_{\alpha, c, r}$, where $(q^c_{\alpha, c, r})_{c \in C}$ is what seller $j$ provides. We assume that the transaction cost $d_{i,j}$ is proportional to the distance between buyer $i$ and seller $j$, who are independently distributed according to some continuous distribution $U$ on some compact domain $H$. That is, the transaction cost $d$ depends on $i$'s and $j$'s locations, $x$ and $y$, respectively ($x, y \in H$). Furthermore, the distance function $d(x, y)$ is a metric: $d$ is symmetric, $d$ satisfies the triangle inequality, and $d(x, y) = 0$ if and only if $x = y$. Let us assume that distance function $d$ is continuous with respect to agents’ locations. We further assume that valuations are generated in such a way that the probability to make a profitable transaction is positive; otherwise, the probability of the social welfare being positive would be zero.

**Proof of Theorem 5.** Let $\alpha_i$ denote the probability that an agent is a seller providing commodity $c$. Without loss of generality, we assume that $\alpha_i$ is positive for all $c \in C$. For simplicity, we slightly expand the notation $c$ to represent the bundle containing only one unit of commodity $c$. We use $b$ to denote a bundle of a buyer, and $b_i$ to denote the amount of commodity $c$ that the buyer wants to acquire. Let $\alpha_i$ be the probability of an agent being a buyer acquiring bundle $b$, and let $B$ be the set of bundles with $\alpha_i > 0$. We assume that $B$ is nonempty; otherwise, the probability of making a transaction is zero.

We first calculate an upper bound of the maximum feasible social welfare per agent. Let $\alpha_i$ and $\alpha_j$ be the decision variables, denoting the transaction percentage for agents acquiring bundle $b$ and supplying bundle $c$, respectively. To get this upper bound, we ignore the transaction costs and solve the following problem:

\[
\hat{\mathcal{E}}: \text{Maximize } \sum_{b \in B} \tilde{V}F_b(\alpha_b) - \sum_{c \in C} \tilde{V}G_c(\alpha_c) \\
\text{subject to } \alpha_i \alpha_i \sum_{b \in B} \alpha_i \beta_b \text{ for all } c \in C, \\
0 \leq \alpha_i \leq 1 \text{ for all } c \in C, \\
0 \leq \alpha_i \leq 1 \text{ for all } b \in B,
\]

where $\beta_b$ and $\beta_c$ are the realized proportions of the agents acquiring bundle $b$ and supplying bundle $c$, respectively. $\tilde{V}F_b(\alpha_b) \equiv \int_{1-a} f_{b-1}(x) dx$, and $\tilde{V}G_c(\alpha_c) \equiv \int_0 \hat{G}_{c-1}(x) dx$, where $\hat{F}_b$ and $\hat{G}_c$ are the realized valuation distributions of bundle $b$ and bundle $c$, respectively. $\tilde{V}F_b$ and $\tilde{V}G_c$ are the welfare contributions per agent that come from acquiring bundle $b$ and supplying bundle $c$, respectively.

Let $V F_b(\alpha) \equiv \int_{1-a} f_{b-1}(x) dx$ and $V G_c(\alpha) \equiv \int_0 \hat{G}_{c-1}(x) dx$. As the number of agents approaches infinity, $\beta_b \rightarrow p_b$, $\beta_i \rightarrow p_i$, $\hat{F}_b \rightarrow F_b$, and $\hat{G}_c \rightarrow G_c$. $\tilde{V}F_b(\alpha) \rightarrow V F_b(\alpha)$ and $\tilde{V}G_c(\alpha) \rightarrow V G_c(\alpha)$. Because $V F_b$ and $V G_c$ are all continuous with respect to their parameters, we can obtain an upper limit of the maximum feasible social welfare per agent by solving the following problem:

\[
\hat{\mathcal{E}}: \text{Maximize } \sum_{b \in B} V F_b(\alpha_b) - \sum_{c \in C} V G_c(\alpha_c) \\
\text{subject to } \alpha_i \alpha_i \sum_{b \in B} \alpha_i \beta_b \text{ for all } c \in C, \\
0 \leq \alpha_i \leq 1 \text{ for all } c \in C, \\
0 \leq \alpha_i \leq 1 \text{ for all } b \in B.
\]
Note that the feasible region is a compact convex set because if we have two feasible solutions \( \alpha' \) and \( \alpha'' \), then \((\alpha' + \alpha'')/2\) is also a feasible solution. Moreover, the objective function is strictly concave because \( V_{F_b} \) is strictly concave and \( V_G \) is strictly convex. Thus, problem \( \mathcal{E} \) has a unique optimal solution \( \alpha^* \), and the optimal objective function value gives the upper limit \( \bar{C} \) of the maximum feasible social welfare per agent. Also, as the number of agents approaches infinity, \( \alpha_k \to \alpha^* \) and \( \alpha_r \to \alpha^* \). Therefore, \( F_{b}^{-1}(\alpha_{k}^{*}) \) and \( G_{c}^{-1}(\alpha_{r}^{*}) \) are the limit equilibrium prices for bundle \( b \) and bundle \( c \), respectively.

Now we show that with transaction costs, the maximum feasible social welfare per agent converges to this limit \( \bar{C} \). Because \( H \) is compact, there exists a finite \( \epsilon \)-partition \( A_1, A_2, \ldots, A_k \) of \( H \) (i.e., a partition such that \( A_i \) has a radius less than \( \epsilon \)) for any \( \epsilon > 0 \). If we restrict transactions to be within each partition, the social welfare per agent is no less than \( \bar{C} - \epsilon \) because the transaction cost per agent is less than \( \epsilon \). If the number of agents is large enough, we can get a feasible solution where the welfare per agent is no less than \( \bar{C} - \epsilon \). Thus, \( \bar{C} - \epsilon \) is a lower bound for the limit maximum feasible social welfare per agent. Because \( \epsilon \) can be arbitrarily small, the maximum feasible social welfare per agent converges to the limit \( \bar{C} \).

Then, because all the valuation distributions are continuous, we can prove that as the number of agents approaches infinity and the maximum feasible social welfare per agent approaches \( \bar{C} \), the percentage of the buyers who bid higher than the limit equilibrium price for each bundle, but fail to get a transaction, must approach zero in the efficient allocation by contradiction, because otherwise the social welfare would converge to a number strictly smaller than \( \bar{C} \). Therefore, for each bundle in each partition \( A_i \), some buyer with valuation no more than \( \epsilon \) above the equilibrium price trades in the efficient allocation almost surely. Also note that the transaction costs of an agent within each partition \( A_i \) are bounded by \( M\epsilon \) because \( M \) is the limit on how much a buyer can acquire. Now consider buyer \( i \) in partition \( A_i \) who bids above the limit equilibrium price by \( (M + 1)\epsilon \). There exists another buyer \( k \) in partition \( A_i \) who acquires the same bundle as buyer \( i \) and trades in the efficient allocation with valuation no more than \( \epsilon \) above the equilibrium price. This tells us that \( \bar{V}(I, J) > V(I, J) \) because we can improve the social welfare by replacing buyer \( k \) in the efficient allocation with an additional buyer who is identical to \( i \). That is, when we apply the BC-LP mechanism, it is almost sure that each buyer who bids above the limit equilibrium price by \( (M + 1)\epsilon \) survives the elimination phase. By Lemma 3, all the surviving buyers trade in the final allocation. Thus, we can obtain a lower bound of social welfare per agent under the BC-LP mechanism by calculating the social welfare of the system consisting of the surviving buyers and the original sellers. Also, as \( \epsilon \) goes to zero, the lower bound of the social welfare per agent under the BC-LP mechanism converges to \( \bar{C} \) because all the valuation distributions are continuous. Because \( \bar{C} \) is also an upper bound for the social welfare per agent, the social welfare per agent achieved by the BC-LP mechanism converges to \( \bar{C} \).

Because both the maximum feasible social welfare per agent and the social welfare per agent under the BC-LP mechanism converge to \( \bar{C} \), the ratio between the welfare under the BC-LP mechanism and the maximum feasible social welfare converges to one as the number of agents approaches infinity. Thus, the BC-LP mechanism is asymptotically efficient if every agent randomly offers or acquires a certain bundle according to the same distribution. □

**Appendix G. Proof of Theorem 9**

We need the following lemma to prove Theorem 9:

**Lemma 13.** Trading seller \( l \) under the MBC mechanism also trades in the optimal solution to \( \mathcal{E} \).

**Proof.** We prove by contradiction. Consider the flow representation of the difference between the optimal solution to \( \mathcal{E} \) and the final allocation. Assume that seller \( l \) trades in the final allocation and does not trade in the optimal solution to \( \mathcal{E} \). Then, there must be a flow sending from seller \( l \) to some seller \( j \) in the difference. Because both solutions are integer valued, the flow sends exactly one unit of the commodity seller \( l \) supplies. Note that starting from the optimal solution to \( \mathcal{E} \), sending this unit flow gives us a feasible solution to \( \mathcal{E} \). Due to the perturbation, \( \mathcal{E} \) has a unique optimal solution, so this feasible solution must be inferior. Now, starting from the final allocation, sending this flow in the reverse direction improves the final allocation. This is contradictory. Thus, under the MBC mechanism, each trading seller is involved in the transaction under the optimal solution to \( \mathcal{E} \). □

**Proof of Theorem 9.** We prove that the MBC mechanism is (ex post) weakly budget balanced, (ex post) individual rational, and strategyproof here.

(Ex post) weakly budget balanced. We compare the following two scenarios: (1) we have buyer set \( I \) and seller set \( J \), and we apply the MBC mechanism; (2) we have buyer set \( I \) and seller set \( J \), and we apply the BC-LP mechanism. Note that both scenarios produce the same allocation. Trading buyer \( i \) in scenario 1 pays at least the price in scenario 2 because \( \max[p_{i}^{\text{VCG}}(I, J), p_{i}^{\text{BC-LP}}(I, J)] \geq p_{i}^{\text{BC-LP}}(I, J) \), and trading seller \( j \) in scenario 1 receives at most the price in scenario 2 because \( \min[p_{j}^{\text{VCG}}(I, J), p_{j}^{\text{BC-LP}}(I, J)] \leq p_{j}^{\text{BC-LP}}(I, J) \). Because the BC-LP mechanism is (ex post) weakly budget balanced, the MBC mechanism is also weakly budget balanced.

(Ex post) individual-rational. Consider the scenario in which we have buyer set \( I \) and seller set \( J \), and apply the BC-LP mechanism. Because the BC-LP mechanism is (ex post) individual-rational, trading buyer \( i \) must bid no less than her \( p_{i}^{\text{BC-LP}}(I, J) \), whereas trading seller \( j \) must bid no more than his \( p_{j}^{\text{BC-LP}}(I, J) \). Now, consider the scenario in which we have buyer set \( I \) and seller set \( J \), and
we apply the MBC mechanism. Trading buyer $i$ must bid no less than $p^\text{VCG}_{ij}(I, J)$ because she trades in the optimal solution to $\ell_i$. Thus, trading buyer $i$’s bid price $f_i \geq \max\{p^\text{VCG}_{ij}(I, J), p^\text{VCG}_{ij}(I, J), p^\text{VCG}_{ij}(I, J)\}$, and each buyer’s payoff is nonnegative. By Lemma 13, trading seller $j$ also trades in the optimal solution to $\ell_j$, and must bid no more than $p^\text{VCG}_{ij}(I, J)$. Thus, trading seller $j$’s bid price $g_j \leq \min\{p^\text{VCG}_{ij}(I, J), p^\text{VCG}_{ij}(I, J)\}$, and each seller’s payoff is nonnegative.

Strategicproofness for the buyer side. For each buyer $i$, $p^\text{VCG}_{ij}(I, J)$ is determined by the agent set $I \cup J \setminus \{i\}$. If buyer $i$ trades in the optimal solution to $\ell_i$, $I$ is independent of the bid price $f_i$. Thus, if buyer $i$ trades in the final allocation, both $p^\text{VCG}_{ij}(I, J)$ and $p^\text{VCG}_{ij}(I, J)$ are independent of her bid price $f_i$. By the strategicproofness of the BC-LP mechanism, if buyer $i$ bids lower than $\max\{p^\text{VCG}_{ij}(I, J), p^\text{VCG}_{ij}(I, J)\}$, she does not trade; if she bids higher than $\max\{p^\text{VCG}_{ij}(I, J), p^\text{VCG}_{ij}(I, J)\}$, she acquires the bundle at $\max\{p^\text{VCG}_{ij}(I, J), p^\text{VCG}_{ij}(I, J)\}$; if she bids $\max\{p^\text{VCG}_{ij}(I, J), p^\text{VCG}_{ij}(I, J)\}$, either she trades at $\max\{p^\text{VCG}_{ij}(I, J), p^\text{VCG}_{ij}(I, J)\}$ or she does not trade. Thus, if buyer $i$’s valuation is higher than $\max\{p^\text{VCG}_{ij}(I, J), p^\text{VCG}_{ij}(I, J)\}$, she prefers to trade, which can be achieved by bidding her valuation. If buyer $i$’s valuation is lower than $\{p^\text{VCG}_{ij}(I, J), p^\text{VCG}_{ij}(I, J)\}$, she prefers not to trade, which can also be achieved by bidding her valuation. If buyer $i$’s valuation is equal to $\max\{p^\text{VCG}_{ij}(I, J), p^\text{VCG}_{ij}(I, J)\}$, she is indifferent between trading and not trading. Thus, bidding truthfully is a (weakly) dominant strategy for each buyer.

Strategicproofness for the seller side. For every seller $j$, $p^\text{VCG}_{ij}(I, J)$ is determined by the agent set $I \cup J \setminus \{j\}$. If seller $j$ trades in the optimal solution to $\ell_j$, $I$ is independent of the bid price $g_j$. By Lemma 13, trading seller $j$ must trade in the optimal solution to $\ell_i$. Furthermore, if seller $j$ trades in the final allocation, $p^\text{VCG}_{ij}(I, J)$ is independent of the bid price $g_j$ because the BC-LP mechanism is a strategyproof deterministic mechanism. Thus, if seller $j$ trades, both $p^\text{VCG}_{ij}(I, J)$ and $p^\text{VCG}_{ij}(I, J)$ are independent of the bid price $g_j$. By Lemma 13 and the strategicproofness of the BC-LP mechanism, if seller $j$ bids higher than $\min\{p^\text{VCG}_{ij}(I, J), p^\text{VCG}_{ij}(I, J)\}$, he does not trade; if seller $j$ bids lower than $\min\{p^\text{VCG}_{ij}(I, J), p^\text{VCG}_{ij}(I, J)\}$, he trades the bundle at $\min\{p^\text{VCG}_{ij}(I, J), p^\text{VCG}_{ij}(I, J)\}$; if he bids $\min\{p^\text{VCG}_{ij}(I, J), p^\text{VCG}_{ij}(I, J)\}$, he either trades at $\min\{p^\text{VCG}_{ij}(I, J), p^\text{VCG}_{ij}(I, J)\}$ or he does not trade. Therefore, if seller $j$’s valuation is lower than $\min\{p^\text{VCG}_{ij}(I, J), p^\text{VCG}_{ij}(I, J)\}$, he prefers not to trade, which can also be achieved by bidding his valuation. If seller $j$’s valuation is equal to $\min\{p^\text{VCG}_{ij}(I, J), p^\text{VCG}_{ij}(I, J)\}$, he is indifferent between trading and not trading. Thus, bidding truthfully is a (weakly) dominant strategy for each seller.

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