Decision Support

Trade reduction vs. multi-stage: A comparison of double auction design approaches

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Abstract

With the growth of electronic markets, designing double auction mechanisms that are applicable to emerging market structures has become an important research topic. In this paper, we investigate two truthful double auction design approaches, the Trade Reduction Approach and the Multi-Stage Design Approach, and compare their resulting mechanisms in various exchange environments. We find that comparing with the Trade Reduction Approach, the Multi-Stage Design Approach offers mechanisms applicable to more complicated exchange environments. Furthermore, for the known trade reduction mechanisms, we prove that the corresponding mechanisms under the multi-stage design approach are superior in terms of both social efficiency and individual payoffs, in each exchange environment of interest. Our computational tests show that the mechanisms under the multi-stage design approach achieve very high efficiency in various scenarios. © 2006 Published by Elsevier B.V.

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1. Introduction

Transactions of hundreds of billions of dollars take place through online auction markets. As a result of this phenomenon, considerable research has addressed issues on the design of double auction mechanism, under which the auctioneer, acting as a broker, matches up the buyers with the sellers and decides the transfer prices. Furthermore, the current research devotes much attention on the behavior of the self-interested agents, and focuses on the design of truthful double auction mechanisms, in which the self-interested agents voluntarily reveal their true private information. For examples of (non-truthful) double auction mechanisms, Babaioff and Nisan (2004) provide an excellent review.

In this paper, we investigate two recent truthful double auction design approaches, one in Babaioff and Walsh (2003) and one in Chu and Shen (in press). Different from Deshmukh et al. (2002), in which the auctioneer tries to maximize his/her own payoffs, these two design approaches try to maximize the social welfare. These two approaches offer mechanisms suitable for various exchange environments, and we compare the effectiveness of
the resulting mechanisms. Each mechanism discussed in this paper has been proven to hold the following economics properties: (1) strategy-proofness: truthful revelation of private information is a dominant strategy for each agent; (2) (ex post) individual-rationality: each agent’s payoff from participation is no less than his or her payoff from non-participation; (3) (ex post) weakly budget balance: the auctioneer’s payoff is non-negative.

The strategy-proofness property assures each agent’s behavior under the mechanisms, while individual-rationality and weakly budget balance properties guarantee the participation of both the agents and the auctioneer. In this paper, we first compare the flexibility of the two approaches; that is, we examine which design approach can offer mechanisms for more general exchange environments. Then, for each exchange environment of interest, we compare all the applicable mechanisms based on the following three criteria:

- **Individual payoff**: The higher the expected payoffs are, the more likely a mechanism is to attract individual buyers and sellers.
- **Social efficiency**: The higher the efficiency is, the more likely a mechanism is to generate higher revenues for the auctioneer or the auction marketplace in the long run as pointed by Milgrom (2000) and Wise and Morrison (2000).
- **Implementation complexity**: The level of complexity to implement these mechanisms determines whether they can be applied in some practical situations.

The remainder of the paper is organized as follows. In Section 2, we describe various exchange environments of interest. In Section 3, we review the existing mechanisms under the two approaches. Specifically, in Section 3.1, we present the Trade Reduction Approach and the resulting mechanisms, and we describe the Multi-Stage Design Approach and its corresponding mechanisms in Section 3.2. We then evaluate the mechanisms in Section 4, with Section 4.1 devoted to the implementation and applicability comparison, and Section 4.2 focused on the efficiency and payoff comparisons. In Section 5, we conclude.

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1 We use agent to refer both seller and buyer.
\( i \) trades with seller \( j \), transaction cost \( d_{ij} \) is incurred.

- **Environment C**: Bilateral exchange environment with the single output restriction. There are multiple indivisible commodities in this environment. Each buyer wants to purchase a bundle of goods and each seller can produce only a single unit of one commodity.

- **Environment D**: Bilateral exchange environment with the single output restriction and transaction costs. Same setting as in Environment C, except when buyer \( i \) buys commodity \( g \) from seller \( j \), transaction cost \( d_{ijg} \) is incurred.

As pointed by Chu and Shen (in press), Environment \( B \), the simple exchange environment with transaction costs, can be used to model the current customer-to-customer online market, while Environments \( C \) and \( D \), the bilateral exchange environments with the single output restriction, can be used to model the procurement auction environment (Babaioff and Walsh, 2003; Chu and Shen, Working paper).

We use \( X \subset Y \) to denote Environment \( Y \) contains Environment \( X \) as a special case. It is easy to see that \( A \subset B \), \( C \subset D \), \( A \subset C \), and \( B \subset D \). Furthermore, \( A \subset D \). For Environments \( B \) and \( D \), we assume that the transaction costs are common knowledge.

Let us first study Environments \( A \) and \( B \). If all agents bid truthfully, the maximum feasible social welfare for Environments \( A \) and \( B \) can be formulated as follows \((d_{i,j} = 0 \ \forall i, j \) in Environment \( A \)):

\[
\mathcal{P} : \text{Maximize } \sum_{i \in I} f_i x_i - \sum_{j \in J} g_j y_j - \sum_{i, j} d_{i,j} z_{i,j} \\
\text{Subject to } \sum_j z_{i,j} = x_i \text{ for each } i \in I, \\
\sum_i z_{i,j} = y_j \text{ for each } j \in J ,
\]

where \( x_i \) and \( y_j \) denote whether an agent trades in the auction, and \( z_{i,j} \) specifies whether seller \( j \) trades with buyer \( i \).

Let \( \mathcal{P} \) denote the linear relaxation of \( \mathcal{P} \). Note that \( \mathcal{P} \) is a network formulation, thus \( \mathcal{P} \) has an integer-valued optimal solution, and \( \mathcal{P} \) and \( \mathcal{P} \) have the same optimal objective value.

For Environments \( C \) and \( D \), we assume there are multiple indivisible commodities in this environment. Let \( G \) denote the set of indivisible commodities. Each buyer \( i \) \((i \in I)\) wants to purchase a bundle of goods \( q_i = \{q_{ij}\}_{j \in G} \), where \( q_{ij} \)'s are non-negative integers since the commodities are indivisible. Due to the single output restriction, each seller \( j \) \((j \in J)\) can produce only a single unit of one commodity; thus, seller \( j \) supplies the bundle \( q_j = \{q_{ij}\}_{i \in I} \) satisfying the conditions that \( q_{ij} \)'s are non-negative integers and \( \sum_{g \in G} q_{ij}^g = 1 \).

The problem of finding the maximum feasible social welfare can be formulated as the following mixed integer programming, if all agents bid truthfully \((d_{i,jg} = 0 \ \forall i, j, g \) in Environment \( C \)):

\[
\hat{\mathcal{P}} : \text{Maximize } \sum_{i \in I} f_i x_i - \sum_{j \in J} g_j y_j - \sum_{i, j, g \in G} d_{i,jg} z_{i,jg} \\
\text{Subject to } \sum_j z_{i,jg} = q_i^g x_i \text{ for each } i \in I, \ g \in G, \\
\sum_i z_{i,jg} = q_j^g y_j \text{ for each } j \in J, \ g \in G, \\
0 \leq z_{i,jg} \text{ for each } i \in I, \ j \in J, \ g \in G, \\
x_i \in \{0, 1\} \text{ for each } i \in I, \\
y_j \in \{0, 1\} \text{ for each } j \in J, \\
z_{i,jg} \in \{0, 1\} \text{ for each } i \in I \text{ and } j \in J ,
\]

where \( x_i \) and \( y_j \) denote whether an agent trades in the auction, and \( z_{i,jg} \) specifies whether seller \( j \) trades commodity \( g \) with buyer \( i \).

Let \( \hat{\mathcal{P}} \) denote the linear relaxation of \( \hat{\mathcal{P}} \). Note that \( \hat{\mathcal{P}} \) is a network formulation, thus \( \hat{\mathcal{P}} \) has an integer-valued optimal solution, and \( \hat{\mathcal{P}} \) and \( \mathcal{P} \) have the same optimal objective value.

Before we present the two auction design approaches that are applicable to the above four environments, we first introduce the following notation.

**Notation**

\( V(I', J') \) The maximum feasible social welfare regarding to the bids of buyer set \( I' \) and seller set \( J' \).
3. Overview of existing mechanisms

3.1. Under trade reduction approach

Inspired by McAfee (1992), one approach to double auction design is to first make the allocation decision, and then the pricing decision. Under this approach, a mechanism typically selects a subset of trades from the efficient allocation. In order to be strategy-proof and budget-balanced, the subset is generally selected by removing the least profitable trade(s) from the efficient allocation. Since each agent has other agents who are perfect substitutes in these models, those bids in the removed trade(s) become reference prices, and the pricing decision can be made by setting the transaction prices equal to the reference prices. This is the Trade Reduction Approach summarized by Babaioff and Walsh (2003).

3.1.1. KSM-TR mechanism

Babaioff and Walsh (2003) propose a known single-minded trade reduction (KSM-TR) mechanism. This mechanism is strategy-proof, (ex post) individually-rational, and (ex post) weakly budget-balanced when applied in Environment C: the bilateral exchange environment with the single output restriction.

In the KSM-TR mechanism, agents are classified according to the bundles they request/provide. A term market is used to represents a set of agents who acquire or supply the same bundle. When we say buyer i’s market, we mean the set of all buyers who acquire the same bundle as buyer i; when we say seller j’s market, we mean the set of all seller who supply the same commodity as seller j.

The first step of the KSM-TR mechanism is to pick an optimal solution to formulation 2. In case of multiple optimal solutions, a perturbation technique is adopted to break the tie and select a unique optimal solution. To present the perturbation technique, let us index the buyers by ik, k = 1, 2, ..., |I|; and index the sellers by jk, k = 1, 2, ..., |J|. A perturbation factor ε is added into each agent’ bid price, that is, we treat their bids as fik + εik and gjk − εjk instead of fik and gjk for each buyer and seller, where 1 ≫ εi ≫ εj ≫ ... ≫ εi ≫ εj ≫ εj ≫ ... ≫ εj > 0. After adding the perturbation factors, formulation 2 has one unique solution. Since ε ≪ 1, this solution indeed is an efficient allo-

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2 In Babaioff and Walsh (2003), the target exchange environment is a class of supply chain formation problem with Unique Manufacturing Technologies (UMT) and single output restriction, which is equivalent to a bilateral exchange environment with the single output restriction. The following interpretation of KSM-TR mechanism is also in its equivalent form instead of its original narration.
cation under the original bid prices. The perturbation factor essentially specifies a lexicographic order among the multiple optimal solutions to formulation $\mathcal{P}$.

Now, let us present the procedure of the KSM-TR mechanism using the lexicographic order instead of the perturbation technique:

- Collect one sealed bid from each agent.
- Generate an arbitrary lexicographic order.
- Calculate the VCG payment $p_{-s}$ for all the agents.
- Calculate the optimal solution to $\mathcal{P}$ based on the lexicographic order.
- Remove all the buyers who are not involved in the transactions specified by the optimal solution.
- For each buyer's market, rank the remaining buyers by their bid prices from high to low and break the tie based on the lexicographic order; set this market's reference price $p_r$ equal to the bid price of the buyer who ranks last, and remove this buyer from the exchange system.
- Calculate the total demand of the remaining buyers and the demand $D_g$ for each commodity.
- For each seller's market, rank the sellers by their bid prices from low to high and break the tie based on the lexicographic order; set this market's reference price $p_r$ equal to the bid price of the $(D_g + 1)$th buyer in this market.
- Conduct transactions between all the remaining buyers and the first $D_g$ sellers of commodity $g$. The transaction price for a buyer is the higher of her VCG payment and her market's reference price, i.e. $\max\{p_{-r}, p_r\}$. The transaction price for a trading seller is the lower of his VCG payment and his market's reference price, i.e. $\min\{p_{-r}, p_r\}$.

3.1.2. Trade reduction mechanism

Babaioff et al. (2004) propose a trade reduction mechanism (TRM), which is strategy-proof, (ex post) individual-rational, and (ex post) weakly budget-balanced when applied in Environment B: the simple exchange environment with transaction costs.

The trade reduction mechanism needs to specify an optimal solution to formulation $\mathcal{P}$, and it is implied that the perturbation technique is applied. In the TRM, it is assumed that agents reside at only finite locations and the transaction costs are only location dependent. The term market refers a set of agents residing at the same location.

Babaioff et al. (2004) implement a directed graph formulation. In the graph, each market is formulated as a node and each possible transaction is formulated as an uncapacitated edge with a transaction cost. A sink node is also introduced which connects all other nodes with edges. The capacity of a edge from the sink to a market is number of the sellers in this market. The cost of sending one unit flow on this edge is the lowest seller's bid in this market, and the cost of sending one more unit flow on this edge is the next lowest seller's bid in this market; therefore, the cost of this edge is a convex function with respect to the flow quantity. The capacity of a edge from a market to the sinker is the number of the buyers in this market. The cost of sending one unit flow on this edge is the negative of the highest buyer's bid in this market, and the cost of sending one more unit flow on this edge is the negative of the next highest buyer's bid in this market. Therefore, the cost of this edge is also a convex function. At this point, maximization problem $\mathcal{P}$ is represented as a convex minimal cost flow problem, and an allocation is represented by an integer flow in the graph. After perturbation, the formulation $\mathcal{P}$ has a unique optimal solution.

We need the following definitions from Babaioff et al. (2004) in order to represent the mechanism:

Definition 1. Markets $M_i$ and $M_j$ are in a direct commercial relationship (CR) if there is trade between $M_i$ and $M_j$ in the efficient allocation.

Definition 2. Markets $M_i$ and $M_j$ are in an indirect commercial relationship if there are markets $M_{k_1}, M_{k_2}, \ldots, M_{k_l}$, and trades between $M_i$ and $M_{k_1}$, between $M_{k_1}$ and $M_{k_2}$, $\ldots$, and between $M_{k_{l-1}}$ and $M_j$ in the efficient allocation.

Definition 3. The commercial relationship component (CRC) of a market $M_i$ refers to a subset of markets, each of which and $M_i$ are in a direct or indirect commercial relationship.

Definition 4. The reduced residual graph (RRG) is a graph consisting of all nodes of the residual graph and the following subset of the edges, each with its cost in the residual graph serving as the edge length.

- For each edge $(M_i, M_j)$ such that there is flow on the edge, we add the edge with its cost and its reversed residual edge with the negated cost.
3.2. Under multi-stage design approach

Chu and Shen (in press) propose a multi-stage design approach to handle the transaction costs. Different from the trade reduction approach, under which the mechanisms make the allocation decision first, the multi-stage design approach first make a partial pricing decision for one side of the market, say the buyers side, to remove some unqualified buyers who bid lower than their threshold prices. The allocation decision is then made based on the most efficient allocation among the remaining buyers and original sellers. The transaction prices for the trading sellers are set in the end.

3.2.1. AC-DA mechanism

Chu and Shen (in press) propose the agent competition double auction (AC-DA) mechanism, which is strategy-proof, (ex post) individually-rational, and (ex post) weakly budget-balanced when applied in Environment B: the simple exchange environment with transaction costs.

The AC-DA mechanism has two different versions: Buyer Competition Mechanism and Seller Competition Mechanism. See Fig. 2 for illustration.

- **Buyer Competition (BC) Mechanism:**
  - Each agent submits one sealed bid.
  - For buyer \( i \in I \), if her bid \( f_i \) is less than \( p_+(i)(I,J) \), she is eliminated from the auction. Let \( I \) denote the remaining buyer set \( \{ i | f_i < p_+(i)(I,J) \} \).
  - The items are allocated among the remaining agents (\( I \) and \( J \)) in the most efficient way.
  - The trading buyer \( k \) pays \( p_+(k)(I,J) \), and the trading seller \( l \) receives \( p_-(l)(I,J) \).

- **Seller Competition (SC) Mechanism:**
  - Each agent submits one sealed bid.
  - For seller \( j \in J \), if his bid \( g_j \) is greater than \( p_+(j)(I,J) \), he is eliminated from the auction. Let \( J \) denote the remaining seller set \( \{ j | g_j > p_+(j)(I,J) \} \).
  - The items are allocated among the remaining agents (\( I \) and \( J \)) in the most efficient way.
  - The trading buyer \( k \) pays \( p_-(k)(I,J) \), and the trading seller \( l \) receives \( p_+(l)(I,J) \).

Similar as in the TRM, if agent \( k \) trades in some efficient allocation, \( p_-(k) \) and \( p_+(k) \) can be interpreted as the distance in the residue graph. This observation leads to the following interpretation of the Buyer Competition Mechanism:

![Fig. 2. Diagram for the buyer competition mechanism under multi-stage approach.](image-url)
Collect one sealed bid from each agent.

Calculate an optimal solution to $\mathcal{P}$.

For each buyer $i \in I$, set $p_+(i)(I,J)$ prohibitively high if buyer $i$ does not trade in any optimal solution to $\mathcal{P}$; otherwise, set $p_+(i)(I,J)$ equal to the distance from the sink to her node in the residual graph of the optimal solution in which she trades. Remove buyer $i$ if her bid $f_i$ is less than $p_+(i)(I,J)$.

Calculate an efficient allocation for the remaining system consisting of the remaining buyers and original sellers, and conduct transactions according to this allocation. The transaction price for trading buyer $i$ is $p_+(i)(I,J)$, and the transaction price for a trading seller is the distance from the sink to his node in the residue graph in the remaining system.

Starting from an optimal solution to $\mathcal{P}$, whether a buyer trades in some optimal solution could be determined by a shortest path algorithm. Thus, if we formulate $\mathcal{P}$ as a maximum weight bipartite matching problem, the complexity of the mechanism is determined by the complexity of the matching algorithm. If there are many agents with the same transaction cost, they form a market as they do in the TRM. In this case, we can also formulate $\mathcal{P}$ as a convex minimal cost flow problem, and the complexity of the mechanism is the sum of the complexities of the convex minimum cost flow algorithm and sorting algorithm for each market. To implement the algorithm more quickly, instead of solving for the most efficient allocation in the remaining system, we can obtain it by removing the eliminated buyers and the corresponding transactions from the optimal solution to $\mathcal{P}$ via a shortest path algorithm. Then, we only need to solve the maximum weight bipartite matching problem or the convex minimum cost flow problem once for the Buyer Competition Mechanism. The Seller Competition Mechanism is implemented similarly and has similar complexity.

3.2.2. BC-LP mechanism

Chu and Shen (Working paper) propose a Buyer Competition (LP) mechanism, or BC-LP mechanism. This mechanism is strategy-proof, (ex post) individual-rational, and (ex post) weakly budget-balanced when applied in Environment $D$: the bilateral exchange environment with the single output restriction and transaction costs.

The key difference between the BC-LP mechanism and all the other mentioned truthful mechanisms is that the BC-LP mechanism focuses on the linear relaxation of the social welfare, $\tilde{\mathcal{W}}$.

As we did with the AC-DA mechanism, we can state the Buyer Competition (LP) mechanism as follows:

- Each agent submits one sealed bid.
- The mechanism applies the perturbation technique.
- For buyer $i \in I$, if her bid $f_i$ is no more than $\hat{p}_+(i)(I,J)$, she is eliminated from the auction. Let $\hat{I}$ denote the remaining buyer set, \{\(i \mid f_i > \hat{p}_+(i)(I,J), i \in I\}.
- The items are allocated among the remaining agents ($\hat{I}$ and $J$) according to the optimal solution to $\hat{\mathcal{V}}(\hat{I},J)$.
- The trading buyer $k$ pays $\hat{p}_+(k)(I,J)$, and the trading seller $l$ receives $\hat{p}_-(l)(I,J)$.

Even though the final allocation is achieved by solving a linear relaxation formulation $\hat{\mathcal{V}}(I,J)$, Chu and Shen (Working paper) show that the BC-LP mechanism does render a legitimate allocation with no agents trading partial bundles. In the BC-LP mechanism, $\hat{p}_+(i)$ and $\hat{\mathcal{V}}_+$ are closely related to the minimum shadow price of the constraint associated with buyer $i$, $x_i \leq 1$. We use $\hat{V}_i^+$ to denote this minimum shadow price and use $\hat{V}_i^-$ to denote the maximum shadow price of the constraint associated with buyer $i$. Similarly, let $\hat{V}_j^+$ and $\hat{V}_j^-$ denote the minimum and maximum shadow prices of the constraint associated with seller $j$, respectively. We connect the above shadow prices with $\hat{p}_+(i)$ and $\hat{p}_-(i)$ in the following propositions.

Proposition 1. For all $i \in I$, $f_i > \hat{p}_+(i)(I,J)$ if and only if $\hat{V}_i^+(I,J) > 0$.

Proposition 2. For all $i \in I$, if $\hat{V}_i^+(I,J) > 0$, then $\hat{p}_+(i)(I,J) = f_i - \hat{V}_i^+(I,J)$.

Proposition 3. In the remaining system $\hat{I} \cup J$, $\hat{p}_-(j)(\hat{I},J) = g_j + \hat{V}(\hat{I},J) - \hat{V}_j^-(\hat{I},J) = g_j + \hat{V}_j^-(\hat{I},J)$ for trading seller $j$.

These propositions enable us to implement the BC-LP mechanism in polynomial time:

- Collect one sealed bid from each agent.
- Generate an arbitrary lexicographic order.
- Solve linear program $\hat{\mathcal{V}}(I,J)$, and pick the optimal solution based on the lexicographic order.
For each buyer $i$, check whether this optimal solution changes if the constraint associated with buyer $i$, $x_i \leq 1$, is removed. If so, $\hat{p}_+(i)(I,J) = f_i - \hat{V}'_+(I,J)$, where $\hat{V}'_+(I,J)$ is the minimum shadow price of the above constraint in $\hat{V}(I,J)$; if not, buyer $i$ is eliminated.

- Solve linear program $\hat{V}(I,J)$, where $\hat{I}$ is the remaining buyer set, and pick the optimal solution based on the lexicographic order.

- For each trading seller $j$, calculate $\hat{p}_-(j)(\hat{I},J)$, which equals $g_j + \hat{V}'_-(I,J)$, where $\hat{V}'_-(I,J)$ is the maximum shadow price of the constraint associated with seller $j$, $y_j \leq 1$, in $\hat{V}(I,J)$.

- Conduct transactions according to the optimal solution to $\hat{V}(I,J)$. The transaction price for trading buyer $k$ is $\hat{p}_+(k)(I,J)$, and the transaction price for trading seller $l$ is $\hat{p}_-(l)(I,J)$.

To implement the BC-LP mechanism, we only need to solve two linear programs $\hat{V}(I,J)$ and $\hat{V}(I,J)$ and calculate the associated shadow prices.

When we implement the BC-LP mechanism under the simple exchange environment with transaction costs, we can have a faster algorithm by taking advantage of the structure of the exchange environment. Note that the linear relaxation formulation $\mathcal{P}(\tilde{I})$ has an integer-valued optimal solution and the same optimal objective value as $\mathcal{P}$. Therefore, $\hat{p}_+$ and $\hat{p}_-$ in the BC-LP mechanism are essentially equal to $p_+$ and $p_-$ in the AC-DA mechanism. So to implement the BC-LP mechanism in the simple exchange environment with transaction costs, the following procedure can be used:

- Collect one sealed bid from each agent.
- Apply the perturbation technique.
- Calculate the optimal solution to $\mathcal{P}$ based on the perturbation and construct the residual graph.
- Remove each buyer $i \in I$ if she does not trade in the optimal solution to $\mathcal{P}$. For each remaining buyer $i$, $\hat{p}_+(i)(I,J)$ is equal to the distance from the sink to her node in the residual graph. Remove buyer $i$ if her bid $f_i$ is less than $\hat{p}_+(i)(I,J)$.
- Calculate the most efficient allocation based on the perturbation for the remaining system consisting of the remaining buyers and original sellers.
- Conduct the transaction according to the most efficient allocation for the remaining system. The transaction price for trading buyer $i$ is $\hat{p}_+(i)(I,J)$, and the transaction price for a trading seller is the distance from the sink to his node in the residual graph in the remaining system.

As in the BC mechanism, instead of solving for the most efficient allocation in the remaining system, we can obtain it by removing the eliminated buyers and the corresponding transactions from the optimal solution to $\mathcal{P}$. Therefore, we only need to solve the maximum weight bipartite matching problem, or the convex minimum cost flow problem once we apply the BC-LP mechanism in the simple exchange environment with transaction costs.

3.2.3. Modified buyer competition mechanism

The modified buyer competition (MBC) mechanism is also proposed in Chu and Shen (Working paper) under the multi-stage design approach.

- Collect one sealed bid from each agent.
- Generate an arbitrary lexicographic order.
- Calculate the VCG payment $p_-$ for each agent.
- Calculate the optimal solution to $\mathcal{P}$ based on the lexicographic order.
- Remove all the buyers who are not involved in the optimal solution. Let $J$ denote the set of trading buyers in the optimal solution.
- Solve linear program $\hat{V}(J,J)$ and pick the optimal solution based on the lexicographic order.
- For each buyer $i$, check whether this optimal solution changes if the constraint associated with buyer $i$, $x_i \leq 1$, is removed. If so, $\hat{p}_+(i)(I,J) = f_i - \hat{V}'_+(I,J)$, where $\hat{V}'_+(I,J)$ is the minimum shadow price of the above constraint in $\hat{V}(I,J)$; if not, buyer $i$ is eliminated.
- Solve linear program $\hat{V}(I,J)$, where $\hat{I}$ is the remaining buyer set, and pick the optimal solution based on the lexicographic order.
- For each trading seller $j$, calculate $\hat{p}_-(j)(\hat{I},J)$ by solving $\hat{p}_-(j) = g_j + \hat{V}'_-(I,J)$, where $\hat{V}'_-(I,J)$ is the maximum shadow price of the constraint associated with seller $j$, $y_j \leq 1$, in $\hat{V}(I,J)$.
- Conduct transactions according to the optimal solution to $\hat{V}(I,J)$. The transaction price for trading buyer $k$ is the higher of her VCG price and $\hat{p}_+(k)(I,J)$, i.e. $\max(p_-(k)(I,J),\hat{p}_+(k)(I,J))$; and the transaction price for trading seller $l$ is the lower of his VCG price and $\hat{p}_-(l)(I,J)$, i.e. $\min(p_-(l)(I,J),\hat{p}_-(l)(I,J))$. 

\[\hat{p}_+(i)(I,J), \hat{p}_-(l)(I,J)\]
The MBC mechanism can be viewed as a BC-LP mechanism following a preliminary elimination phase based on the optimal solution to formulation $P$ and the VCG prices. If we apply the MBC mechanism in the simple exchange environment with transaction costs, then the MBC mechanism would be equivalent to the BC-LP mechanism because $p_-(k)(I,J) \leq p_2(k)(i,j)$ for buyer $k$ and $p_-(l)(I,J) \geq p_-(l)(i,j)$ for seller $l$.

4. Mechanism comparison: Implementation, efficiency, and payoffs

After we reviewed the existing mechanisms in Section 3, a nature question is how easy it is to implement each mechanism for each exchange environment of interest, and which mechanism renders the highest efficiency. In this section, we first compare the implementation and applicability of all the mechanisms, and then investigate the efficiencies of the mechanisms that are designed for Environment $B$ and Environment $C$, respectively. Our investigation shows that the multi-stage approach offers mechanisms with simpler implementation and higher applicability. The multi-stage approach offers mechanisms that are applicable in Environment $D$, where no known mechanism under the trade reduction approach is applicable. Furthermore, in terms of both efficiency and individual payoffs, the mechanisms under the multi-stage design approach dominate the corresponding mechanisms under the trade reduction approach.

4.1. Implementation and applicability

We have introduced the following mechanisms in this paper: the TRM, BC and SC (both variants of AC-DA), KSM-TR, MBC and BC-LP mechanisms. Similarly to the BC-LP mechanism, we can define a Seller Competition (LP) mechanism, or SC-LP mechanism in short, in which we first remove sellers who bid lower than their $p$, then conduct the efficient allocation in the remaining system. Note that the SC-LP mechanism maintains all the properties under the simple exchange environment with transaction costs, but may reach an allocation with agents trading partial bundles under the bilateral exchange environment with the single output restriction. Therefore, we should only apply the SC-LP mechanism under the simple exchange environment, or the simple exchange environment with transaction costs. Similar to the MBC mechanism, we can define a Modified Seller Competition mechanism, or MSC mechanism in short. We should only apply the MSC mechanism under the simple exchange environment, or the simple exchange environment with transaction costs. Therefore, the MSC mechanism is equivalent to the SC-LP mechanism and we will omit the detail discussions on the MSC mechanism.

We first compare the implementation complexity under Environment $A$, the simple exchange environment. All the above mechanisms are applicable to this environment. We can find the efficient allocation by ranking all the bid prices, and these mechanisms either allocate resource efficiently or remove one least profitable transaction. Therefore, the implementation complexity of the implementation is determined by the sorting algorithm adopted.

Now, we compare the implementation complexity under Environment $B$, the simple exchange environment with transaction costs. All the mechanisms except the KSM-TR mechanism are applicable to this environment. In Sections 3.1.2 and 3.2.1, we have seen that the TRM, BC and SC mechanisms can be implemented by solving a maximum weight bipartite matching problem or a convex minimum cost flow problem. In Sections 3.2.2 and 3.2.3, we have seen that the MBC mechanism is equivalent to the BC-LP mechanism under this environment, and both mechanisms can be implemented by solving a maximum weight bipartite matching problem or a convex minimum cost flow problem. All the transaction prices are then determined by the distance labels via a shortest path algorithm. Each mechanism can be implemented by iteratively removing buyers from the optimal solution to $P$ via a shortest path algorithm.

The BC-LP/SC-LP mechanism has a slight advantage in terms of implementation. Unlike the BC or SC mechanism, the implementation of the BC-LP mechanism needs not to check whether buyer $i$ trades in any optimal solution to $P$ since the lexicographic order determines a unique optimal solution. Unlike the TRM, the implementation of the BC-LP mechanism does not require constructing the reduced residual graph. It also requires less effort to obtain the final allocation because it removes less trades than the TRM does, as we will see in Theorem 2.

Now, we compare the implementation complexity under Environment $C$, the bilateral exchange environment with the single output restriction. The KSM-TR, MBC, and BC-LP mechanisms are
applicable to this environment. Both the KSM-TR and MBC mechanisms require solving the integer formulation \( Q \), while the BC-LP mechanism only need to solve its linear relaxation \( \tilde{Q} \). Therefore, the BC-LP mechanism has a great advantage in terms of implementation.

Finally, we compare the implementation complexity under Environment \( D \), the bilateral exchange environment with the single output restriction and transaction costs. Only the MBC and BC-LP mechanisms are applicable to this environment while no known mechanism under the trade reduction approach is applicable. The MBC mechanism requires solving the integer formulation \( Q \), while the BC-LP mechanism only need to solve its linear relaxation \( Q \). Once again, the BC-LP mechanism has a great advantage in terms of implementation.

Table 1 summarizes the finding of this section. We use “Network Program” to denote that a mechanism can be implemented under the specified environment via a maximum weight bipartite matching algorithm or a convex minimum cost flow algorithm. Similarly, “Sorting algorithm”, “Linear Program”, and “Integer Program” denote that a mechanism can be implemented under the specified environment via a sorting algorithm, a linear program, and an integer program, respectively. We see that the BC-LP mechanism has the lowest implementation complexity while it is applicable to all the environments of interest.

4.2. Efficiency and payoff comparison

Now, we investigate the efficiencies and the payoffs of the mechanisms that are applicable to Environment \( B \) and Environment \( C \), the simple exchange environment with transaction costs, and the bilateral exchange environment with single output restriction, respectively.

4.2.1. Simple exchange environment with transaction costs

We have the following mechanisms that are applicable in the simple exchange environment with transaction costs: the TRM, BC and SC, BC-LP and SC-LP, and MBC and MSC mechanisms.

Since under this environment the MBC mechanism is equivalent to the BC-LP mechanism and the MSC mechanism is equivalent to the SC-LP mechanism, we will only compare the efficiency and agent payoffs of the TRM, BC and SC, BC-LP and SC-LP mechanisms. We say that mechanism \( Y \) dominates mechanism \( X \) if, for any instance, mechanism \( Y \) is at least as efficient as mechanism \( X \) and each agent’s payoff under mechanism \( Y \) is at least as high as that under mechanism \( X \).

We have the following theorem (whose proof is in Appendix along with other proofs).

**Theorem 1.** The BC mechanism dominates the BC-LP mechanism.

There are instances where the BC mechanism strictly outperforms the BC-LP mechanism in efficiency. Consider the following example with two buyers, two sellers, and no transaction costs. The bid price for both buyers is 1, and the bid price for both sellers is 0. In the BC mechanism, \( p_+ \) for both buyers is 1, both buyers survive the elimination stage, and the final allocation makes two transactions at price 1. This allocation is the efficient allocation for the system. In the BC-LP mechanism, according to the perturbation technique, we take the buyers’ bid prices as \( 1 + \epsilon_1 \) and \( 1 + \epsilon_2 \), respectively, and the buyers’ \( p_+ \)s are \( 1 + \epsilon_2 \) and \( 1 + \epsilon_1 \) respectively. One of the buyers is eliminated, and the final allocation makes one transaction where the trading buyer pays 1 and the trading seller receives 0 since \( \epsilon \ll 1 \). We see that in this example, the BC mechanism achieves a higher efficiency than the BC-LP mechanism.

### Table 1

<table>
<thead>
<tr>
<th></th>
<th>Environment ( A )</th>
<th>Environment ( B )</th>
<th>Environment ( C )</th>
<th>Environment ( D )</th>
</tr>
</thead>
<tbody>
<tr>
<td>TRM</td>
<td>Sorting algorithm</td>
<td>Network program</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>BC and SC</td>
<td>Sorting algorithm</td>
<td>Network program</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>KSM-TR</td>
<td>Sorting algorithm</td>
<td>–</td>
<td>Integer program</td>
<td>–</td>
</tr>
<tr>
<td>MBC</td>
<td>Sorting algorithm</td>
<td>Network program</td>
<td>Integer program</td>
<td>Integer program</td>
</tr>
<tr>
<td>SC-LP</td>
<td>Sorting algorithm</td>
<td>Network program</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>BC-LP</td>
<td>Sorting algorithm</td>
<td>Network program</td>
<td>Linear program</td>
<td>Linear program</td>
</tr>
</tbody>
</table>

“–” denotes that a mechanism is not applicable to the specified environment.
Theorem 2. The BC-LP mechanism dominates the TRM.

There are instances where the BC-LP mechanism achieves efficient allocation while TRM does not. Consider the following example with one buyer, two sellers, and no transaction costs. The bid price for the buyer is 1, and the bid price for both sellers is 0. In the BC-LP mechanism, \( p_+ \) for the buyer is 0, the buyer survives the elimination stage, and the final allocation makes one transaction at price 0. This allocation is the efficient allocation for the system. In the TRM, this transaction is removed and no transaction takes place. We see that in this example, the BC-LP mechanism achieves a higher efficiency than the TRM.

Since the TRM is symmetric between buyers and sellers, by symmetry between the BC-LP and SC-LP mechanisms, we have:

Theorem 3. The SC-LP mechanism dominates the TRM.

Furthermore, due to the symmetry between the BC and SC mechanisms, we have:

Theorem 4. The SC mechanism dominates the SC-LP mechanism.

Since the dominance relationship is transitive, by Theorems 1 and 2, and Theorems 3 and 4, we have:

Corollary 1. The BC mechanism dominates the TRM.

Corollary 2. The SC mechanism dominates the TRM.

4.2.2. Bilateral exchange environment with the single output restriction

In this section, we compare the mechanisms that are designed for the bilateral exchange environment with the single output restriction. We have three mechanisms that are applicable to this environment: the KSM-TR, BC-LP, and MBC mechanisms. Since the KSM-TR mechanism can only be applied to environments without transaction costs, we assume no transaction costs and focus on Environment C instead of Environment D in this section.

The BC-LP mechanism can be implemented polynomially, while both the KSM-TR and MBC mechanisms need to be implemented by computing the VCG prices and the optimal solution to formulation \( \mathcal{Q} \), which is NP-hard. We are especially interested in comparing the efficiency and the payoffs of the KSM-TR and MBC mechanisms. We have the following theorem.

Theorem 5. The MBC mechanism dominates the KSM-TR mechanism.

There are instances where the MBC mechanism achieves efficient allocation while KSM-TR mechanism does not. Consider the following example with one buyer, two sellers, and no transaction costs. The bid price for the buyer is 1, and the bid price for both sellers is 0. The VCG price for the buyer is 0, while the VCG price for both sellers is 0. In the MBC mechanism, \( \tilde{p}_+ \) for the buyer is 0, and the buyer trades in the final allocation at price 0. The final allocation is the efficient allocation for the system. In the KSM-TR mechanism, this trade is removed, and no transaction takes place. We see that in this example, the MBC mechanism achieves a higher efficiency than the KSM-TR mechanism.

Table 2 summaries the efficiency and payoff comparisons from Sections 4.2.1 and 4.2.2.

To study the efficiencies of the BC-LP and MBC mechanism, we conduct computational tests. We assume that there are three types of commodities and investigate the performances of these mechanisms when there are different numbers of bundle types, different numbers of buyers per bundle type, and different variances in the valuation distribution. Parameters and notation used in the computational tests are summarized in Table 3.

In the computational test, the number of bundle types is set to be either 5 or 10, and the bundle acquired is an integer triple \((i, j, k)\), where \(i\), \(j\), and \(k\) each corresponds to the demand for one commodity, and each is independently drawn from 0 to 10. For each bundle type, the number of buyers is either 5 or 10, while the number of sellers for each commodity equals the expected total demand of each commodity. The valuations of the agents are independent random variables. The valuation of a seller
Table 3
Parameter settings

<table>
<thead>
<tr>
<th>Variable name</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of bundle types</td>
<td>$5(M_s), 10(M_L)$</td>
</tr>
<tr>
<td>Number of buyers for</td>
<td>$5(N_s), 10(N_L)$</td>
</tr>
<tr>
<td>each bundle type</td>
<td></td>
</tr>
<tr>
<td>Standard deviation of</td>
<td>$10(\sigma_s), 20(\sigma_L)$</td>
</tr>
<tr>
<td>seller’s valuation $\sigma$</td>
<td></td>
</tr>
<tr>
<td>Bundle formation $(i,j,k)$</td>
<td>Integers independent uniform $(0,10)$</td>
</tr>
<tr>
<td>Number of sellers</td>
<td>The size of the expected total demand</td>
</tr>
<tr>
<td>Valuation of sellers</td>
<td>Independent normal $(100,\sigma^2)$</td>
</tr>
<tr>
<td>Valuation of buyers</td>
<td>$(i+j+k) \ast 100, (i+j+k)\sigma^2$</td>
</tr>
</tbody>
</table>

is normally distributed with mean 100 and variance $\sigma^2$, where $\sigma$ is either 10 or 20. The valuation of a buyer is normally distributed with mean $(i+j+k)100$ and variance $(i+j+k)\sigma^2$, where $(i,j,k)$ is the bundle she wants.

In Fig. 3, we plot the percentages of average efficiency for the KSM-TR, BC-LP, and MBC mechanisms under all eight scenarios from $(M_s, N_s, \sigma_s)$ to $(M_L, N_L, \sigma_L)$. MBC achieves the highest efficiency in all scenarios. BC-LP is also highly desirable because it only involves solving linear programming problems and still achieves over 95% efficiency in all scenarios. The computational results show that the efficiency achieved by KSM-TR is substantially lower than the efficiency by the MBC and BC-LP mechanisms.

5. Conclusion

In this paper, we compare two truthful double auction design approaches: the Trade Reduction Approach and the Multi-Stage Design Approach. The multi-stage design approach offers mechanisms applicable to more complicated exchange environments. For example, both the BC-LP and MBC mechanisms are applicable to the bilateral exchange environment with the single output restriction and transaction costs and maintaining good auction properties, while no known mechanism under the trade reduction approach is applicable.

The mechanisms under both approaches have similar implementation complexity, while the mechanisms under the multi-stage design approach dominate the mechanisms under the trade reduction approach in both efficiency and individual payoffs in each type of exchange environment of interest.

We also want to emphasize the BC-LP mechanism’s contribution in terms of implementation as this mechanism can be implemented by solving two linear programs and calculating the associated shadow prices. From the computational results, we know that the BC-LP mechanism successfully captures most of the possible efficiency.

Appendix

Proof of Theorem 1. We first compare the efficiencies of the BC and BC-LP mechanisms. Let $I_{BC}$ denote the remaining buyer set under the BC mechanism, and $\tilde{I}_{BC-LP}$ denote the remaining buyer set under the BC-LP mechanism. We will first show that $\tilde{I}_{BC} \supseteq \tilde{I}_{BC-LP}$ and that the BC mechanism is at least as efficient as the BC-LP mechanism.

Under the simple exchange environment, $p_{+}(k)(I,J) = \tilde{p}_{+}(k)(I,J)$ for each agent $k \in I \cup J$. The BC mechanism only eliminates each buyer who bids lower than her $p_{+}$, that is, $\tilde{I}_{BC} = \{i | f_i \geq p_{+}(i)(I,J), i \in I\}$. Because the BC-LP mechanism applies the perturbation technique, these threshold prices may differ additions and subtractions of the perturbation factors. Therefore, the BC-LP mechanism also eliminates each buyer who bids lower than her $p_{+}$, and may eliminate buyers whose bid prices equal their $p_{+}$s. Thus, the remaining buyer set $\tilde{I}_{BC}$ under the BC mechanism is at least as large as $\tilde{I}_{BC-LP}$ under the BC-LP mechanism. Since both mechanisms implement the most efficient allocation of the remaining system, the BC mechanism is at least as efficient as the BC-LP mechanism.

Now, we compare the buyers’ payoffs. For each non-trading buyer, the payoff is zero. For trading buyer $i$, the payoff is $f_i - p_{+}(i)(I,J)$ under the BC mechanism. Due to the strategy-proofness of the BC mechanism, each buyer who bids higher than her $p_{+}$ is involved in a transaction. Thus, under the BC
mechanism, buyer $i$’s payoff is zero if $f_i \leq p_+(i)(I,J)$ and $f_i - p_+(i)(I,J)$ if $f_i > p_+(i)(I,J)$. Due to the strategy-proofness of the BC-LP mechanism, each remaining buyer is involved in a transaction. Thus, under the BC-LP mechanism, buyer $i$’s payoff is zero if $f_i \leq p_+(i)(I,J)$ and $f_i - p_+(i)(I,J)$ if $f_i > p_+(i)(I,J)$. Under the simple exchange environment, $p_+(k)(I,J) = p_+(k)(I,J)$ for each agent $k \in I \cup J$, and each buyer receives the same payoff under both the BC and BC-LP mechanisms.

Now, we consider the sellers’ payoffs. We will show that each seller’s payoff under the BC mechanism is at least as high as it under the BC-LP mechanism. Since the BC mechanism is (ex post) individual-rational, it suffices to show that for each seller with a positive payoff under the BC-LP mechanism, his payoff under the BC mechanism is at least as high as his payoff under the BC-LP mechanism.

Under the BC-LP mechanism, the payoff for a non-trading seller is zero. Consider trading seller $j$ with a positive payoff. This payoff is $\hat{p}_-(j)(I_{BC-LP},J) - g_j = V(\hat{I}_{BC-LP},J) - V(\hat{I}_{BC-LP},J \setminus \{j\})$. Shapley (1962) shows that the simple exchange environment with transaction costs satisfies the complementarity-substitutability conditions, which guarantee that $V(\hat{I}_{BC-LP},J) - V(\hat{I}_{BC-LP},J \setminus \{j\}) \leq V(\hat{I}_{BC},J) - V(\hat{I}_{BC},J \setminus \{j\})$ if $\hat{I}_{BC} \supseteq \hat{I}_{BC-LP}$. Since seller $j$ has a positive payoff under the BC-LP mechanism, $V(\hat{I}_{BC-LP},J) - V(\hat{I}_{BC-LP},J \setminus \{j\}) > 0$; therefore, $V(\hat{I}_{BC},J) - V(\hat{I}_{BC},J \setminus \{j\}) > 0$, that is, seller $j$ must trade under the BC mechanism, and seller $j$’s payoff under the BC mechanism equals $p_-(j)(\hat{I}_{BC},J) - g_j = V(\hat{I}_{BC},J) - V(\hat{I}_{BC},J \setminus \{j\})$. This is no less than his payoff under the BC-LP mechanism.

**Proof of Theorem 2.** We first compare the efficiencies of the BC-LP and TRM mechanisms. Let $\hat{I}_{BC-LP}$ denote the remaining buyer set under the BC-LP mechanism, and $\hat{I}_{TRM}$ denote the trading buyer set under the TRM. We will first show that $\hat{I}_{BC-LP} \supseteq \hat{I}_{TRM}$ and that the BC-LP mechanism is at least as efficient as the TRM.

To prove $\hat{I}_{BC-LP} \supseteq \hat{I}_{TRM}$, it suffices to show that all the trading buyers under the TRM survive the elimination under the BC-LP mechanism. Note that both mechanisms apply the perturbation technique, so we know that any cycle length in the (reduced) residual graph is non-zero.

Under the TRM, a buyer $i$ is removed from the transactions if she is the first buyer starting from the sink in the minimal positive cycle in the reduced residual graph (RRG) for her commercial relationship component (CRC). Let $s$ denote the last seller starting from the sink in the minimal positive cycle in the RRG for this CRC. Now, consider any buyer $b \neq i$ in the CRC, by the definition of RRG and CRC, there exists a path from $b$ to $s$. Consider the cycle in the RRG consisting of the edge from the sink to $b$, the path from $b$ to $s$, and the edge from $s$ to the sink. Since perturbation technique is applied, the cycle has a non-negative length. Furthermore, since it belongs to the residual graph related to the efficient allocation in the original system, the cycle length must be positive. Because no two cycles have the same length under perturbation technique, the cycle “sink–$b$–$s$–sink” has a length greater than the length of the minimal positive cycle “sink–$i$–$s$–sink”.

Assume now we have one more buyer who is identical to buyer $b$. The residual graph of the original allocation would have one more edge from $b$ to sink with length/cost equal to the negative of the bid price of $b$. Consider the cycle “sink–$i$–$b$–$s$–sink”, which is the difference of the cycles “sink–$i$–$s$–sink” and “sink–$b$–$s$–sink”. This cycle has a negative length, and the maximum social welfare would improve if we had one more buyer who was identical to buyer $b$. This shows that $V_b(I,J) > V(I,J)$. Since both $V_b(I,J)$ and $V(I,J)$ are continuous functions of buyer $b$’s bid price $f_b$, $f_b > p_+(b)(I,J)$ by the definition of $p_+(b)$, and buyer $b$ survives under the BC-LP mechanism. Thus, all trading buyers under the TRM survive the elimination stage under the BC-LP mechanism, that is, $\hat{I}_{BC-LP} \supseteq \hat{I}_{TRM}$. Since the BC-LP mechanism implements the most efficient allocation of the remaining system, it is at least as efficient as the TRM.

Now, we compare the buyers’ payoffs. The strategy-proofness of the BC-LP mechanism guarantees that all the remaining buyers in $\hat{I}_{BC-LP}$ are involved in transactions under this mechanism. Thus, $\hat{I}_{BC-LP} \supseteq \hat{I}_{TRM}$ shows that if a buyer trades under the TRM, she trades under the BC-LP mechanism. Note that both mechanisms are deterministic mechanisms, that is, there is some critical price such that buyer $i$ trades at this critical price if her bid price is higher, and she loses the transaction if her bid price is lower. Thus, the critical price for each buyer under the BC-LP mechanism is no more than the critical price under the TRM. The payoff of a trading buyer is the difference between her bid price and the critical price. Thus, each buyer’s
payoff under the BC-LP mechanism is no less than her payoff under the TRM.

Now, we consider the sellers’ payoffs. Under the TRM, if we removes all non-trading buyers in the exchange system, the corresponding residual graph has no negative cycle. Thus, the final allocation under the TRM is an efficient allocation among the trading buyer set $\hat{I}_{\text{TRM}}$ and original seller set $J$. We will show that if a seller trades under the TRM, he trades under the BC-LP mechanism. Shapley (1962) shows that the simple exchange environment with transaction costs satisfies the complementarity–substitutability conditions, which guarantee that $V(\hat{I}_{\text{BC-LP}}, J) - V(\hat{I}_{\text{BC-LP}}, J \setminus \{j\}) \geq V(\hat{I}_{\text{TRM}}, J) - V(\hat{I}_{\text{TRM}}, J \setminus \{j\})$ if $\hat{I}_{\text{BC-LP}} \supseteq \hat{I}_{\text{TRM}}$. Since perturbation technique is applied, each trading seller $j$ under the TRM must have $V(\hat{I}_{\text{TRM}}, J) - V(\hat{I}_{\text{TRM}}, J \setminus \{j\}) > 0$; thus, $V(\hat{I}_{\text{BC-LP}}, J) - V(\hat{I}_{\text{BC-LP}}, J \setminus \{j\}) > 0$ and seller $j$ trades under the BC-LP mechanism. Therefore, if a seller trades under the TRM, he trades under the BC-LP mechanism. Note both mechanisms are deterministic mechanisms, that is, there is some critical price such that seller $j$ trades at this critical price if his bid price is lower, and he loses the transaction if his bid price is higher. Thus, for each seller, the critical price under the BC-LP mechanism is no less than the critical price under the TRM. The payoff of a trading buyer is the difference between the critical price and the bid price. Thus, each seller’s payoff under the BC-LP mechanism is no less than his payoff under the TRM. □

**Proof of Theorem 5.** We first compare the efficiencies of the MBC and KSM-TR mechanism. Both mechanisms apply the same perturbation technique/lexicographic order and calculate the optimal solution to $\mathcal{P}$. Let $\hat{I}$ denote the set of trading buyers in the optimal solution to $\mathcal{P}$.

Each trading buyer $i$ in the KSM-TR mechanism, that is, each buyer in $\hat{I}$ who does not rank last in her market, survives the elimination stage of the MBC mechanism. To see this, suppose the right-hand side of the constraint associated with buyer $i$, $x_i \leq 1$, changes from 1 to 2, i.e., suppose we have one more buyer who is identical to buyer $i$, we can then improve the optimal solution to $\hat{V}(I,J)$ by using this additional buyer to replace the buyer who ranks last in the market. Thus, $\hat{V}(I,J) > \hat{V}(I,J)$, and each trading buyer in the KSM-TR mechanism survives the elimination stage of the MBC mechanism. That is, the trading buyer set of the KSM-TR mechanism is a subset of the remaining buyer set $\hat{I}$ in the MBC mechanism. Since the MBC mechanism implements the efficient allocation of the remaining system consisting of $\hat{I}$ and $J$, the MBC mechanism is at least as efficient as the KSM-TR mechanism.

Now, we compare the buyers’ payoffs. The strategy-proofness of the MBC mechanism guarantees that each buyer in $\hat{I}$ is involved in the transactions under the MBC mechanism. Thus, if a buyer trades under the KSM-TR mechanism, she trades under the MBC mechanism. Note both mechanisms are deterministic mechanisms, where there is some critical price such that buyer $i$ trades at this critical price if her bid price is higher, and she loses the transaction if her bid price is lower. Thus, the critical price for each buyer under the MBC mechanism is no more than the critical price under the KSM-TR mechanism. The payoff of a trading buyer is the difference between her bid price and the critical price. Thus, each buyer’s payoff under the MBC mechanism is no less than her payoff under the KSM-TR mechanism.

Now, we consider the sellers’ payoffs. Since the trading buyer set under the KSM-TR mechanism is a subset of the trading buyer set under the MBC mechanism, the total transaction quantity under the KSM-TR mechanism is no more than the total transaction quantity under the MBC mechanism for each commodity. For each commodity, we can decide the trading sellers by ranking their bid prices and breaking the tie based on the perturbation technique/lexicographic order; thus, the trading seller set under the KSM-TR mechanism is a subset of the trading seller set under the MBC mechanism. Note both mechanisms are deterministic mechanisms, where there is some critical price such that seller $j$ trades at this critical price if his bid price is lower, and he loses the transaction if his bid price is higher. Thus, the critical price for each seller under the MBC mechanism is no less than the critical price under the KSM-TR mechanism. The payoff of a trading seller is the difference between the critical price and his bid price. Thus, each seller’s payoff under the MBC mechanism is no less than his payoff under the KSM-TR mechanism. □

**References**