**Technical Appendix to Accompany**

“*A Note on Optimal Procurement Contracts with Limited Direct Cost Inflation*”

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**Prof of Lemma 1**

From (3) in the text:

\[ u(\beta) \geq u(\hat{\beta} \mid \beta) = t(\hat{\beta}) - [\hat{\beta} - e(\hat{\beta})] - \psi(e(\hat{\beta} \mid \beta)) = u(\hat{\beta}) + \psi(e(\hat{\beta})) - \psi(e(\hat{\beta} \mid \beta)). \]

Therefore:

\[ u(\hat{\beta}) - u(\beta) \leq \psi(e(\hat{\beta} \mid \beta)) - \psi(e(\hat{\beta})). \]  

(A1)

Since (A1) holds for all \( \beta \) and \( \hat{\beta} \), it follows that:

\[ \psi(e(\beta)) - \psi(e(\beta \mid \hat{\beta})) \leq u(\hat{\beta}) - u(\beta). \]  

(A2)

Because \( \psi(\cdot) \) is a continuous function of \( e \), (A2) implies \( \lim_{\beta \to \hat{\beta}} |u(\beta) - u(\hat{\beta})| = 0 \), provided \( e(\beta) \) is piece-wise continuous. Since \( u(\beta) \) is a continuous function on a compact set, constraint (2) in the text must bind at some \( \tilde{\beta} \in [\underline{\beta}, \bar{\beta}] \) in any solution to [BP]. For this innate cost realization, \( u(\tilde{\beta}) = 0 \). Therefore, the following problem identifies an upper bound on the cost saving that can be achieved relative to a cost reimbursement contract (in which the procurement cost is precisely the agent’s realized innate cost) on \( \beta \in [\underline{\beta}, \bar{\beta}] \):

\[
\begin{align*}
\text{Minimize} & \quad t(\beta) \\
\text{subject to, for all } & \quad \beta \in [\underline{\beta}, \bar{\beta}]: \\
& \quad u(\beta) \equiv t(\beta) - [\beta - e(\beta)] - \psi(e(\beta)) \geq 0, \\
& \quad u(\beta) \geq u(\tilde{\beta} \mid \beta) = t(\tilde{\beta}) - [\tilde{\beta} - e(\tilde{\beta})] - \psi(e(\tilde{\beta} \mid \beta)), \\
& \quad \text{and } u(\tilde{\beta}) = 0.
\end{align*}
\]

LT’s analysis is readily employed to prove that when \( \alpha \leq 1 \), the solution to this problem is the LT contract, under which no cost inflation occurs.

For \( \beta \in [\tilde{\beta}, \bar{\beta}] \), incentive compatibility requires \( u'(\beta) = -\psi'(e(\beta)) \) almost everywhere.

Since \( u(\bar{\beta}) \geq 0 \) and \( u(\tilde{\beta}) = 0 \), it follows that:

\[ 0 \leq u(\bar{\beta}) - u(\tilde{\beta}) = \int_{\tilde{\beta}}^{\bar{\beta}} u'(\beta) d\beta = -\int_{\tilde{\beta}}^{\bar{\beta}} \psi'(e(\beta)) d\beta \leq -\int_{\tilde{\beta}}^{\bar{\beta}} \frac{1}{2k} e(\beta) d\beta. \]  

(A3)
The last inequality in (A3) holds because $\psi'(e(\beta)) = \frac{1}{2k} e(\beta)$ when $e(\beta) \geq 0$ and $\psi'(e(\beta)) = \frac{\alpha}{2k} e(\beta)$ when $e(\beta) \leq 0$, since $\alpha \leq 1$. (A3) implies $\int_{\beta} e(\beta) d\beta \leq 0$.

For any $\beta \in [\bar{\beta}, \overline{\beta}]$, the reduction in the buyer’s procurement cost relative to the cost reimbursement contract is $\beta - t(\beta)$. Since $u(\beta) = t(\beta) - [\beta - e(\beta)] - \psi(e(\beta))$ from (2) in the text, this reduction in procurement cost is:

$$\beta - t(\beta) = e(\beta) - \psi(e(\beta)) - u(\beta) \leq e(\beta). \quad (A4)$$

The inequality in (A4) holds because $\psi(e(\beta))$ and $\mu(\beta)$ are both non-negative for all innate cost realizations. (A4) implies:

$$\int_{\beta} [\beta - t(\beta)] dF(\beta) \leq \int_{\beta} e(\beta) d\beta \leq 0. \quad (A5)$$

(A5) implies that the expected reduction in procurement cost relative to the cost reimbursement contract on $[\bar{\beta}, \overline{\beta}]$ is at most zero. Therefore, because the LT contract minimizes expected procurement costs when $\bar{\beta} = \overline{\beta}$, the optimal value of $\bar{\beta}$ is $\overline{\beta}$, and so the LT contract is the optimal contract when $\alpha \in [0, 1]$.

Finally, notice that:

$$\int_{\beta} t'(\beta) dF(\beta) = \frac{1}{2k} \int_{\beta} \left[ 2\beta - \bar{\beta} + \frac{1}{2k} [\bar{\beta} - \beta]^2 \right] d\beta = \frac{1}{2k} \left\{ \bar{\beta}^2 - \beta^2 - 2k \bar{\beta} + \frac{1}{6k} [\bar{\beta} - \beta]^3 \right\}$$

$$= \frac{1}{2k} \left\{ 4k \beta + 4k^2 - 2k [\beta + 2k] + \frac{8k^3}{6k} \right\} = \frac{1}{2k} \left\{ 2k \beta + \frac{4 k^2}{3} \right\} = \beta + \frac{2k}{3}.$$

**Proof of Lemma 2.**

To demonstrate that the discontinuous contract is individually rational, notice first that for all $\beta \in [\underline{\beta}, \overline{\beta} + l)$, the supplier can undertake the same effort and receive the same payment less $[2k - l]^2/[4k]$, as under the LT contract. It is readily verified that the LT contract ensures the supplier a utility of at least $[2k - l]^2/[4k]$ for all $\beta \in [\underline{\beta}, \overline{\beta} + l)$. Therefore, the discontinuous contract ensures non-negative utility for the supplier for all $\beta \in [\underline{\beta}, \overline{\beta} + l)$.  


For $\beta \in [\underline{\beta} + l, \overline{\beta}]$, the maximum amount of cost inflation the supplier will undertake under the discontinuous contract is the difference between the highest final cost admitted under the discontinuous contract and the smallest innate cost in $[\underline{\beta} + l, \overline{\beta}]$. This difference is:

$$\frac{\sqrt{\alpha}}{1 + \sqrt{\alpha}} [\beta + l] + \frac{1}{1 + \sqrt{\alpha}} \overline{\beta} - [\beta + l] = \frac{1}{1 + \sqrt{\alpha}} [\overline{\beta} - (\beta + l)] = \frac{1}{1 + \sqrt{\alpha}} [2k - l].$$

The personal cost the supplier experiences from undertaking this level of cost inflation is:

$$\frac{\alpha}{4k[1 + \sqrt{\alpha}]} [2k - l]^2 = \bar{\tau}^d - \bar{c}^d.$$  \hfill (A6)

The equality in (A6) follows from properties (iii) and (iv) of the discontinuous contract.

The maximum amount of cost-decreasing effort the supplier can supply under the discontinuous contract is the difference between the highest innate cost in $[\underline{\beta} + l, \overline{\beta}]$ and the smallest final cost the supplier can realize under the contract. This difference is:

$$\overline{\beta} - \left[ \frac{\sqrt{\alpha} [\underline{\beta} + l + \overline{\beta}]}{1 + \sqrt{\alpha}} \right] = \frac{\sqrt{\alpha}}{1 + \sqrt{\alpha}} [\overline{\beta} - (\beta + l)] = \frac{\sqrt{\alpha}}{1 + \sqrt{\alpha}} [2k - l].$$

The personal cost the supplier incurs from delivering this effort is:

$$\frac{\alpha}{4k[1 + \sqrt{\alpha}]} [2k - l]^2 = \bar{\tau}^d - \bar{c}^d.$$  \hfill (A7)

The equality in (A7) follows from properties (iii) and (iv) of the discontinuous contract. (A6) and (A7) imply that the supplier can always secure a utility of at least $t^d(\beta) - c^d(\beta) - [\bar{\tau}^d - \bar{c}^d] = 0$ for all $\beta \in [\underline{\beta} + l, \overline{\beta}]$ under the discontinuous contract. Therefore, the contract is individually rational.

To demonstrate that the discontinuous contract is incentive compatible, it is sufficient to show that the supplier will never claim to have innate cost $\underline{\beta} + l$ when $\beta \in [\underline{\beta}, \overline{\beta} + l)$ or claim to have an innate cost below $\underline{\beta} + l$ when $\beta \in [\overline{\beta} + l, \overline{\beta}]$. This is the case because: (i) the required final cost and associated payment do not vary with $\beta$ for all $\beta \in [\underline{\beta} + l, \overline{\beta}]$; and (ii) the LT contract is incentive compatible and the discontinuous contract coincides with the LT contract for all $\beta \in [\underline{\beta}, \overline{\beta} + l)$ except that the payment is systematically reduced by $[2k - l]^2 / [4k]$. 
First suppose the supplier claims to have innate cost $\beta + l$ when $\beta \in [\underline{\beta}, \overline{\beta} + l)$. The claim implies the supplier must realize final cost $\frac{\sqrt{\alpha}[\beta + l] + \overline{\beta}}{1 + \sqrt{\alpha}}$. To secure this final cost, the supplier with innate cost $\beta$ must undertake cost inflation $\beta + l - \beta + \frac{2k-l}{1 + \sqrt{\alpha}}$. This level of cost inflation entails personal disutility that exceeds $\frac{\alpha[2k-l]^2}{4k[1 + \sqrt{\alpha}]^2}$. Therefore, the supplier’s utility from this action is less than:

$$\bar{r}^d - \bar{c}^d - \frac{\alpha[2k-l]^2}{4k[1 + \sqrt{\alpha}]^2} = 0.$$  \hspace{1cm} (A8)

The equality in (A8) follows from properties (iii) and (iv) of the discontinuous contract. Consequently, the supplier would receive negative utility from the exaggeration in question, and so will not undertake the exaggeration.

Now suppose the supplier’s innate cost is $\beta + l$. If the supplier understates his innate cost marginally and so claims to have innate cost $(\beta + l)^- \equiv \lim_{\beta \uparrow \beta + l} \beta$, he must deliver effort:

$$e^d((\beta + l)^- | \beta + l) = \beta + l - c((\beta + l)^-) = \beta + l - [2(\beta + l) - \overline{\beta}] = 2k - l.$$  \hspace{1cm} (A9)

The second equality in (A9) follows from Lemma 1. The agent’s utility from this understatement of $\beta$ is:

$$u((\beta + l)^- | \beta + l) = t^d((\beta + l)^-) - c^d((\beta + l)^-) - \frac{1}{4k} [e^d((\beta + l)^- | \beta + l)]^2$$

$$= \frac{1}{2k} [\bar{\beta} - (\beta + l)]^2 - \frac{1}{4k} [2k-l]^2 - \frac{1}{4k} [2k-l]^2 = 0.$$  \hspace{1cm} (A10)

The second equality in (A10) follows from Lemma 1 and property (i) of the discontinuous contract. Because the LT contract is incentive compatible, $u(\beta | \beta^+) \leq u((\beta + l)^- | \beta + l) = 0$ for all $\beta \in [\underline{\beta}, \beta + l)$. Therefore, the supplier with innate cost $\beta + l$ will not understate this cost. Furthermore, because $e^d(\beta | \beta)$ is increasing in $\bar{\beta}$ for all $\beta \in [\underline{\beta}, \beta + l)$ and $\bar{\beta} \in [\beta + l, \overline{\beta}]$, $u(\beta | \bar{\beta}) < u(\beta | \beta + l) \leq 0$ for any $\beta \in [\underline{\beta}, \beta + l)$, for all $\bar{\beta} \in (\beta + l, \overline{\beta}]$. Therefore, the discontinuous contract is incentive compatible.
The identified expressions for effort follow directly from Lemma 1 and from the fact that \( e^d(\beta) = \beta - c^d(\beta) \). The identified expressions for rent holds because \( u^d(\beta) = t^d(\beta) - c^d(\beta) - \psi(e^d) \), where, recall, \( \bar{t}^d - \bar{c}^d = \frac{\alpha[2k-1]^2}{4k[1+\sqrt{\alpha}]^2} \), \( e^d(\beta) = \frac{r}{1+\sqrt{\alpha}} \) for \( \beta \in [\bar{\beta} + l, \bar{\beta}] \), and \( \psi(e) = e^2 /[4k] \) for \( e \geq 0 \) and \( \psi(e) = \alpha e^2 /[4k] \) for \( e < 0 \).

**Proof of Proposition 1.**

From Lemma 1 and property (i) of the discontinuous contract, the buyer’s expected payment under the discontinuous contract on the interval \( [\bar{\beta}, \bar{\beta} + l] \) is:

\[
\int_{\bar{\beta}}^{\bar{\beta} + l} \left( 2\beta - \bar{\beta} + \frac{1}{2k}[\beta - \bar{\beta}]^2 - \frac{1}{4k}(2k-l)^2 \right) \frac{d\beta}{2k} \\
= \frac{1}{2k} \left\{ \beta + l \right\}^2 - \bar{\beta}^2 - \bar{\beta} l - \frac{1}{6k}\left\{ \beta - \bar{\beta} \right\}^3 \bar{\beta} + l - \frac{1}{8k^2}[2k-l]^2 \\
= \frac{1}{2k} \left\{ 2\beta l + l^2 - (\beta + 2k) l - \frac{1}{6k}\left\{ (2k-l)^3 - (2k)^3 \right\} \right\} - \frac{1}{8k^2}[4k^2 - 4kl + l^2] \\
= \frac{1}{2k} \left\{ \beta l + l^2 - 2kl - kl + l^2 - \frac{l^3}{4k} - \frac{1}{6k}\left\{ -12k^2 l + 6kl^2 - l^3 \right\} \right\} \\
= \frac{1}{2k} \left\{ \beta l + l^2 - kl - \frac{l^3}{12k} \right\} = \frac{l}{2k} \left[ \beta + l - k - \frac{l^2}{12k} \right]. \tag{A11}
\]

From property (iii) of the discontinuous contract, the buyer’s payment under the discontinuous contract for all \( \beta \in [\bar{\beta} + l, \bar{\beta}] \) is \( \frac{\sqrt{\alpha}[\beta + l] + \bar{\beta}}{1+\sqrt{\alpha}} + \frac{\alpha [2k-l]^2}{4k[1+\sqrt{\alpha}]^2}. \) Therefore, the buyer’s expected payment under the discontinuous contract over this portion of the support is:

\[
\left[ \frac{\sqrt{\alpha}[\beta + l] + \bar{\beta}}{1+\sqrt{\alpha}} + \frac{\alpha[2k-l]^2}{4k[1+\sqrt{\alpha}]^2} \right] \left[ \frac{\beta - (\beta + l)}{2k} \right] \\
= \left[ \frac{\sqrt{\alpha}[\beta + l] + \bar{\beta}}{1+\sqrt{\alpha}} + \frac{\alpha[2k-l]^2}{4k[1+\sqrt{\alpha}]^2} \right] \left[ \frac{2k-l}{2k} \right]. \tag{A12}
\]
(A11) and (A12) imply that the buyer’s expected payment under the discontinuous contract is:

\[
\frac{l}{2k} \left[ \beta + l - k - \frac{l^2}{12k} \right] + \left[ \sqrt{\alpha} \frac{\beta + l}{1 + \sqrt{\alpha}} + \frac{\alpha [2k-l]^2}{4k[1+\sqrt{\alpha}]^2} \right] \left[ \frac{2k-l}{2k} \right].
\] (A13)

Taking derivatives of the terms in (A13) with respect to \( l \) provides:

\[
\frac{\partial}{\partial l} \left( \frac{l}{2k} \left[ \beta + l - k - \frac{l^2}{12k} \right] \right) = \frac{1}{2k} \left[ l - \frac{l^2}{6k} + \beta + l - k - \frac{l^2}{12k} \right]
\]

\[
= \frac{1}{2k} \left[ 2l + \beta - k - \frac{l^2}{4k} \right] = \frac{\beta + 2k}{2k} - \frac{1}{8k^2}[2k-l][6k-l];
\] (A14)

\[
\frac{\partial}{\partial l} \left( \sqrt{\alpha} \frac{\beta + l}{1 + \sqrt{\alpha}} + \frac{\alpha [2k-l]^2}{4k[1+\sqrt{\alpha}]^2} \right) \left[ \frac{2k-l}{2k} \right]
\]

\[
= -\frac{1}{2k} \left( \sqrt{\alpha} \frac{\beta + l}{1 + \sqrt{\alpha}} + \frac{\beta + 2k}{1 + \sqrt{\alpha}} + \frac{\alpha [2k-l]^2}{4k[1+\sqrt{\alpha}]^2} \right) + \left[ \frac{\sqrt{\alpha}}{1 + \sqrt{\alpha}} - \frac{\alpha [2k-l]}{2k[1+\sqrt{\alpha}]^2} \right] \left[ \frac{2k-l}{2k} \right]
\]

\[
= -\frac{1}{2k} \left( \beta + 2k + \frac{\sqrt{\alpha} [l-2k]}{1 + \sqrt{\alpha}} + \frac{3\alpha [2k-l]^2}{4k[1+\sqrt{\alpha}]^2} \right) + \frac{\sqrt{\alpha}}{1 + \sqrt{\alpha}} \left[ \frac{2k-l}{2k} \right]
\]

\[
= -\frac{\beta + 2k}{2k} - \frac{1}{2k} \left( \frac{\sqrt{\alpha} [l-2k]}{1 + \sqrt{\alpha}} + \frac{3\alpha [2k-l]^2}{4k[1+\sqrt{\alpha}]^2} \right) + \frac{\sqrt{\alpha}}{1 + \sqrt{\alpha}} \left[ \frac{2k-l}{2k} \right]
\]

\[
= -\frac{\beta + 2k}{2k} + \frac{1}{8k^2} \left( -\frac{\sqrt{\alpha} [l-2k]}{1 + \sqrt{\alpha}} - \frac{3\alpha [2k-l]^2}{4k[1+\sqrt{\alpha}]^2} + \frac{\sqrt{\alpha}}{1 + \sqrt{\alpha}} \frac{4k[2k-l]}{[1+\sqrt{\alpha}]^2} \right)
\]

\[
= -\frac{\beta + 2k}{2k} + \frac{1}{8k^2} [2k-l] \left( \frac{\sqrt{\alpha}}{1 + \sqrt{\alpha}} - \frac{3\alpha}{[1 + \sqrt{\alpha}]^2} [2k-l] + \frac{\sqrt{\alpha}}{1 + \sqrt{\alpha}} 4k \right)
\]

\[
= -\frac{\beta + 2k}{2k} + \frac{1}{8k^2} [2k-l] \left( \frac{4\sqrt{\alpha} + 4\alpha}{[1 + \sqrt{\alpha}]^2} 2k - \frac{3\alpha}{[1 + \sqrt{\alpha}]^2} 2k + \frac{3\alpha}{[1 + \sqrt{\alpha}]^2} l \right)
\]
\[
\begin{align*}
    &\quad = -\frac{\beta + 2k}{2k} + \frac{1}{8k^2} [2k-l] \left( \frac{4\sqrt{\alpha + \alpha}}{[1 + \sqrt{\alpha}]^2} 2k + \frac{3\alpha}{[1 + \sqrt{\alpha}]^2} l \right) . \\
    \text{The sum of the derivatives in (A14) and (A15) is:} &\quad \frac{\beta + 2k}{2k} - \frac{1}{8k^2} [2k-l][6k-l] - \frac{\beta + 2k}{2k} + \frac{1}{8k^2} [2k-l] \left( \frac{4\sqrt{\alpha + \alpha}}{[1 + \sqrt{\alpha}]^2} 2k + \frac{3\alpha l}{[1 + \sqrt{\alpha}]^2} \right) \\
    &\quad = \frac{1}{8k^2} [2k-l] \left( \frac{4\sqrt{\alpha + \alpha}}{[1 + \sqrt{\alpha}]^2} [2k] + \frac{3\alpha l}{[1 + \sqrt{\alpha}]^2} + l - 6k \right) \\
    &\quad = \frac{1}{8k^2} [2k-l] \left( -3 + 2\sqrt{\alpha + 2\alpha} \right) [2k] + l \left( \frac{1+2\sqrt{\alpha + 4\alpha}}{[1 + \sqrt{\alpha}]^2} \right) \\
    &\quad = \frac{1+2\sqrt{\alpha + 4\alpha}}{8k^2 (1 + \sqrt{\alpha})^2} [2k-l] \left( l - 2k \left[ \frac{3+2\sqrt{\alpha + 2\alpha}}{1+2\sqrt{\alpha + 4\alpha}} \right] \right) .
\end{align*}
\]

From (A16), the buyer’s expected payment is minimized when \( l = 2k \left[ \frac{3+2\sqrt{\alpha + 2\alpha}}{1+2\sqrt{\alpha + 4\alpha}} \right] \). From (A13), the buyer’s expected payment when \( l = 2k \) is \( \beta + 2k - k - \frac{(2k)^2}{12k} = \beta + \frac{2k}{3} \).

Therefore, letting \( \gamma = \frac{3+2\sqrt{\alpha + 2\alpha}}{1+2\sqrt{\alpha + 4\alpha}} \) and \( x = 2kl \), (A16) implies that the buyer’s expected payment under the discontinuous contract is:

\[
\begin{align*}
    \beta + \frac{2}{3} k - \int_{\frac{2k}{l}}^{2k} \frac{4\sqrt{\alpha + \alpha} + 1}{8k^2 [1 + \sqrt{\alpha}]^2} [2k-l][l-2k\gamma] \, dl \\
    &\quad = \beta + \frac{2}{3} k + \left[ \frac{1+2\sqrt{\alpha + 4\alpha}}{[1 + \sqrt{\alpha}]^2} \right] k \int_{\gamma}^{1} [l-x][\gamma-x] \, dx .
\end{align*}
\]

\[
\int_{\gamma}^{1} [l-x][\gamma-x] \, dx = \int_{\gamma}^{1} [x^2 - (1+\gamma)x + \gamma] \, dx \\
    = \left( \frac{1}{3} x^3 - \frac{1}{2} \gamma x^2 + \gamma x \right) \bigg|_{\gamma}^{1} = \frac{1}{3} \gamma^3 - \frac{1}{2} \gamma^2 + \frac{1}{2} \gamma - \frac{1}{2} \gamma + \frac{1}{2} \gamma^3 + \gamma - \gamma^2 \\
    = -\frac{1}{6} + \frac{1}{6} \gamma^3 + \frac{1}{2} \gamma - \frac{1}{2} \gamma^2 = \frac{1}{6} [\gamma^3 - 3\gamma^2 + 3\gamma - 1] = -\frac{1}{6} [1 - \gamma] .
\]
From (A17) and (A18), the buyer’s expected payment under the discontinuous contract is:

\[
\beta + \frac{2}{3}k - \left[ \frac{1+2\sqrt{\alpha}+4\alpha}{(1+\sqrt{\alpha})^2} \right] \frac{k}{6} \left( \frac{2\alpha-2}{1+2\sqrt{\alpha}+4\alpha} \right)^3
\]

\[
= \beta + \frac{2}{3}k - \left[ \frac{2^3(\alpha-1)^3}{6(1+\sqrt{\alpha})^2(1+2\sqrt{\alpha}+4\alpha)^2} \right] k
\]

\[
= \beta + \frac{2}{3}k - G(\alpha), \quad \text{(A19)}
\]

where \( G(\alpha) \equiv \frac{4k}{3} \left[ \frac{(\sqrt{\alpha}-1)^3(1+\sqrt{\alpha})}{(1+2\sqrt{\alpha}+4\alpha)^2} \right]. \quad \text{(A20)}

It is apparent from (A20) that \( G(1) = 0 \). Furthermore:

\[
G'(\alpha) = \frac{s}{[1+2\sqrt{\alpha}+4\alpha][\sqrt{\alpha}-1]^3 \alpha^{-1/2} [3(1+\sqrt{\alpha})+\sqrt{\alpha}-1] / 2}
\]

\[
- 2 [1+\sqrt{\alpha}][\sqrt{\alpha}-1][4+\alpha^{-1/2}]
\]

\[
= \frac{s}{[1+2\sqrt{\alpha}+4\alpha][1+2\sqrt{\alpha}] \alpha^{-1/2} - 2 [1+\sqrt{\alpha}][\sqrt{\alpha}-1][4+\alpha^{-1/2}] > 0}
\]

\[
\Leftrightarrow [1+2\sqrt{\alpha}+4\alpha][1+2\sqrt{\alpha}] > 2 [\alpha-1][4\sqrt{\alpha}+1] \Leftrightarrow 3 + 12\sqrt{\alpha} + 6\alpha > 0. \quad \text{(A21)}
\]

Because the last inequality in (A21) holds for all \( \alpha \geq 0 \), \( G(\alpha) \) is a strictly increasing function of \( \alpha \). Finally, from (A20):

\[
\lim_{\alpha \to \infty} G(\alpha) = \frac{4k}{3} \lim_{\alpha \to \infty} \left\{ \frac{[\sqrt{\alpha}-1][1+\sqrt{\alpha}]}{[1+2\sqrt{\alpha}+4\alpha]^2} \right\} = \frac{4k}{3} \lim_{\alpha \to \infty} \left\{ \frac{\alpha^2}{16\alpha^2} \right\} = \frac{k}{12}.
\]