Agent Competition Double-Auction Mechanism

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This paper proposes an agent competition double-auction mechanism to simplify decision making and promote transactions for the customer-to-customer marketplaces. Under the proposed double-auction mechanism, bidding one’s true valuation (private information) is the best strategy for each individual buyer and seller even when shipping costs and sales taxes are different across various possible transactions. The proposed mechanism also achieves budget balance and asymptotic efficiency. Furthermore, these results not only hold for an environment where buyers and sellers exchange identical commodities, but also can be extended to an environment with multiple substitutable commodities.

Key words: mechanism design; double auction; strategyproof mechanism

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1. Introduction

Millions of transactions are arranged over eBay every day. Nevertheless, because transactions of desired commodities are carried out in many auctions, a potential customer needs to forecast the price movements to get the best deal. This problem is increased by the last-minute bidding phenomenon (Roth and Ockenfels 2002) as prices can change dramatically in the last minute of the auctions. The hassles involved can outweigh the savings and result in the loss of customers for online marketplaces (Porter 2001).

A double-auction mechanism can be used to simplify decision making and promote transactions for customer-to-customer marketplaces. The literature on double auctions typically assumes a collection of sellers each with one unit of the same object facing a collection of buyers each interested in buying at most one unit of the goods. Sellers differ in their opportunity costs and buyers in their reservation prices for the good. Furthermore, sellers and buyers in a consumer-to-consumer market can and do differ not just on the valuation for the good, but also on the cost of the transactions that would be executed. One example of such a transaction cost is shipping. In some cases this particular cost could be specific to the buyer-seller pair that transacts.

In this paper, we propose a direct-revelation mechanism\(^1\) for this two-sided exchange environment with pair-related transaction costs. Under this mechanism, the central auctioneer collects payments from buyers and pays the sellers. We show that this mechanism has the following properties.

- **Strategyproof**: that is, truthful bidding is a weakly dominant strategy for each buyer and each seller
- **Ex post individually rational**: so that each buyer’s and each seller’s utility are no less than their utility from nonparticipation for all possible outcomes; hence, potential buyers and sellers are willing to participate the auction
- **Ex post budget-balanced**: so that the auctioneer’s payoff is nonnegative for all possible outcomes; therefore, the auctioneer can hold the auction without an outside subsidy
- **Asymptotically efficient under quasi-linear preferences**: that is, the mechanism’s welfare loss compared to the maximum social welfare converges to 0 as the number of buyers and/or the number of sellers approach infinity; thus, as the auction becomes large, almost all feasible surplus will actually be realized

Note that to make an auction practical, the mechanism needs to be individually rational and budget-balanced. Moreover, strategyproofness is desired, because each agent may not know the number of agents in the exchange environment or the joint distribution of valuations.

Two examples of strategyproof, individually rational, and budget-balanced double-auction mechanisms are McAfee (1992) and Huang et al. (2002). McAfee (1992) describes a mechanism for a simple exchange environment, where buyers and sellers exchange single units of the same good, while Huang et al. (2002)
design a mechanism for a multiunit exchange environment. McAfee’s mechanism is also asymptotically efficient. However, these models assume no transaction costs.\(^2\)

The remainder of the paper is organized as follows. In §2, we present an exchange model with heterogeneous agents and propose a multistage mechanism design approach and the resulting truthful mechanism. In §3, we study the impact of transaction costs on efficiency and demonstrate the asymptotic efficiency property of the mechanism. In §4, we consider a more general multicommodity environment and prove that the proposed mechanism preserves the desired properties as long as the complementarity substitutability conditions (Shapley 1962) are satisfied.

2. Model and Mechanism

2.1. Model

Let \( I \) denote the group of buyers, and \( J \) the group of sellers, \( I \cap J = \emptyset \), hereafter both called agents. We also refer to a buyer as feminine and a seller as masculine. Consider the simple exchange environment in which each agent wants to buy (sell) one unit of the same item. We assume that when buyer \( i \) trades with seller \( j \), transaction cost \( d_{i,j} \) is incurred. The transaction costs are assumed to be common knowledge. Examples of where this common-knowledge assumption hold are when transaction costs are related to shipping costs.

We assume a private-value model, that is, each agent knows his or her own valuation of the item, but not others.\(^2\) We also assume quasi-linear utility, where an agent’s utility is the difference between his or her valuation of the item he or she sells (buys) and the amount of money he or she receives (pays). Social welfare is the sum of the auctioneer’s payoff and each individual agent’s utility.

The auctioneer and each agent may or may not know the number of agents in the exchange environment, the joint distribution of valuations, or any other relevant information. Although the exchange model is quite primitive, it captures two previously stated properties: information asymmetry and heterogeneity. In this paper, we use the term “bid” to denote both a buyer’s and a seller’s declaration, although some authors prefer to use the term “ask” to denote a seller’s declaration. Let \( f_i \) be the bid price of buyer \( i \), \( i \in I \), and \( g_j \) be the bid price of seller \( j \), \( j \in J \). If all the agents bid truthfully, the social welfare maximization problem can be formulated as follows:

Problem \( \mathcal{P} \):

\[
V(f, g, d) = \text{Maximize} \sum_i f_i x_i - \sum_j g_j y_j - \sum_{i,j} d_{i,j} z_{i,j}
\]

subject to

\[
\sum_j z_{i,j} = x_i \quad \text{for each } i \in I
\]

\[
\sum_i z_{i,j} = y_j \quad \text{for each } j \in J
\]

\[
x_i \in \{0, 1\} \quad \text{for each } i \in I
\]

\[
y_j \in \{0, 1\} \quad \text{for each } j \in J
\]

\[
z_{i,j} \in \{0, 1\} \quad \text{for each } i \in I \text{ and } j \in J,
\]

where \( x_i \) and \( y_j \) denote whether buyer \( i \), or seller \( j \), respectively, enters a transaction or not; \( z_{i,j} \) denotes whether buyer \( i \) transacts with seller \( j \) or not. Note that the transaction costs are shared among the agents, because the centralized auctioneer sets the clearing prices that account for the transaction costs.

Due to the network property of the formulation, the maximization problem is easy to solve; however, the challenge is to design a strategyproof, individually rational, and budget-balanced mechanism with high efficiency. To the best of our knowledge, no known strategyproof, individually rational, and budget-balanced mechanism is capable of conducting trades when no agent is a perfect substitute of another; that is, when there does not exist \( i_1 \in I, i_2 \in I, i_1 \neq i_2 \) such that \( d_{i_1,j} = d_{i_2,j} \) for all \( j \in J \), or \( j_1 \in J, j_2 \in J, j_1 \neq j_2 \) such that \( d_{i,j_1} = d_{i,j_2} \) for all \( i \in I \). Before proposing the truthful mechanism for this model, let us first review the existing approaches to truthful mechanism design.

2.2. Multistage Approach

There are two decisions an auction mechanism needs to make: the allocation decision and the pricing decision. The purpose of mechanism design is to construct a method for making these decisions to achieve desired properties. When designing a truthful mechanism, once the allocation rule is set, the payments can be determined; similarly, once the pricing rule is set, the allocation can be determined. Therefore, to identify a truthful mechanism, there are two straightforward approaches: pricing first or allocation first.

For the pricing-first approach, if we can calculate a threshold price for each buyer such that she wins an item if she outbids this threshold price and loses if she underbids, and vice versa for the seller, then the resulting mechanism is strategyproof. However, no known mechanism uses this approach, because it typically leads to a supply/demand volume mismatch even in the simple exchange environment. Moreover, as agents are heterogeneous and transaction costs
vary, it is not clear how a mechanism can determine the buying or selling prices for each individual agent and guarantee budget balance with high efficiency without information on the allocation decision and how much each transaction costs. For these reasons, if we make the pricing decision first, the resulting strategy-proof, individually rational, and budget-balanced mechanism is unlikely to have superior efficiency.

This is why all the existing strategy-proof, double-auction mechanisms for the two-sided exchange environments make the allocation decision before the pricing decision. To be strategy-proof and budget-balanced, given agents' bids, these mechanisms select a subset of trades that would be executed in an efficient allocation, which generally can be achieved by removing the least profitable trade(s) from the efficient allocation. Since before submitting the bids some agents are perfect substitutes for each other in these models, the bid prices in the removed trade(s) are used to set price for the trades that will be executed. This approach first appeared in McAfee (1992) and is called “trade reduction” in Babaioff and Walsh (2003). Using this approach, Babaioff and Walsh (2003) propose a mechanism for an exchange environment where each buyer tries to acquire a bundle of commodities and each seller provides a single unit of one commodity. They assume in the model that there is no transaction cost and prove their mechanism is strategy-proof, individually rational, and budget-balanced. However, their mechanism may lose a profitable trade for each bundle of interest, and the mechanism conducts no trade if each buyer wants a different bundle.

This trade-reduction approach can also be extended to the simple exchange environment with transaction costs. What we need to do is to partition the buyers into groups such that the buyers in the same group always incur identical transaction costs when trading with the sellers. Similarly, we partition the sellers. Then we can proceed with the trade-reduction approach and design a mechanism that removes one trade for each pair of buyer group–seller group. It can be shown readily that the resulting mechanism is strategy-proof and budget-balanced. However, one may find that most of the trades are removed and the mechanism suffers from extremely low efficiency. To remedy this problem, Babaioff et al. (2004) propose another trade-reduction mechanism that removes fewer trades by identifying the potential relationships between the groups. As a result, their strategy-proof, individually rational, and budget-balanced mechanism results in higher efficiency compared with the naive implementation. Nevertheless, their mechanism is still unable to conduct any transactions if each group contains only one agent. The efficiency of the trade-reduction mechanisms depends on not only the supply/demand information of the goods, but also the effectiveness of the partition.

To achieve higher efficiency, we propose a novel multistage mechanism design approach. Specifically, we first make the pricing decision for one side, say, the threshold price for each seller. We then take an efficient allocation among original buyers and winning sellers as the allocation decision. Finally, we decide the transaction prices for the winning buyers. The idea is to select a (small) group of sellers who are highly competitive during the first stage, such that when we take an efficient allocation among original buyers and these sellers, all of the sellers get transaction. Hopefully, the transaction prices for the winning buyers can be determined by competition among buyers in the final allocation, so that buyers have no incentive to speculate even though their bids can influence the sellers’ threshold prices and consequently the final allocation.

Compared with the trade-reduction approach, the multistage design has two major benefits. First, it offers great flexibility and can be applied to more general environments which no known mechanism under the trade-reduction approach is capable of doing. Second, it promotes transactions and provides mechanisms that dominate the mechanisms using the trade-reduction approach in both efficiency and agents’ payoffs. For detailed discussion, we refer the readers to Chu and Shen (2004a, b).

The critical decision in the multistage design is to determine the threshold prices in the first stage, where we face a trade-off. The lower the threshold prices are for sellers in the first stage, the more likely this approach will work. On the other hand, the lower the threshold prices are for sellers in the first stage, the smaller the realized allocation efficiency is. As we will show in the next section, by setting these threshold prices wisely, we can have a strategy-proof, individually rational, budget-balanced, and asymptotically efficient mechanism.

2.3. Mechanism

Before introducing the mechanism, we need to define the following notation:

Notation

- \( f_i \): The bid price of buyer \( i, i \in I \)
- \( g_j \): The bid price of seller \( j, j \in J \)
- \( d_{i,j} \): The transaction cost when buyer \( i \) trades with seller \( j \)
- \( V(I', J') \): The maximum feasible social welfare with respect to the bids of buyer set \( I' \) and seller set \( J' \)

\(^3\) The trade-reduction mechanism makes no trades when applied to the exchange environment studied in §3.2.
$V_{-k}(I', J')$ The maximum feasible social welfare with respect to the bids of agent set $I' \cup J' \setminus \{k\}$ ($k \in I' \cup J'$).

$V_{-k}(I', J')$ The maximum feasible social welfare with respect to the bids of agent set $I' \cup J' \setminus K$ ($K \subset I' \cup J'$).

$p_-(k)(I', J')$ The infimum (supremum) of bid prices of buyer (seller) $k$ satisfying $V(I', J') > V_{-k}(I', J')$.

$p_+(k)(I', J')$ The infimum (supremum) of bid prices of buyer (seller) $k$ satisfying $V_k(I', J') > V(I', J')$.

For simplicity of representation, we may drop parameter $(I', J')$ when the references to the buyer set and the seller set are obvious. Also, in the definition of $p_-(k)$ and $p_+(k)$, the infimum of an empty set is positive infinity, and the supremum of an empty set is negative infinity.

More comments on $p_-(k)$ and $p_+(k)$ are in order. Essentially, $p_-$ is the price at which an agent is able to trade in some efficient allocation, and $p_+$ is the price at which an agent is able to sell (buy) one more identical item in some efficient allocation. From a mathematical programming viewpoint, the shadow price of the constraint associated with a specific agent in the linear relaxation of formulation $\mathcal{P}$ can generally take any value in an interval: (minimal shadow price, maximum shadow price). When the minimal shadow price is positive, if an agent trades at $p_-$, his or her utility is the maximum shadow price, and if this agent trades at $p_+$, his or her utility is the minimal shadow price.

For a winning agent, $p_+$ is the agent’s Vickrey-Clarke-Groves (VCG) payment/revenue (Vickrey 1961, Clarke 1971, Groves 1973). If we employ an efficient allocation and complete the transactions at the VCG prices, the auctioneer will likely face a deficit. To avoid this, the auctioneer can retain some earnings from the agents by letting all or part of the agents trade at $p_+$, instead of $p_-$, is greater than or equal to $p_-$ for each buyer, and $p_+$ is less than or equal to $p_-$ for each seller. This is intuitive because $p_+$ is derived under an environment with more competition.

Now we present our double-auction mechanism. We let the $p_+(k)$s be the threshold prices for the pricing decision of one side of the market. We call the mechanism the agent competition double-auction mechanism. We provide two different versions, a seller competition mechanism and a buyer competition mechanism.

**Seller competition mechanism**
- Each agent submits one sealed bid.

**Buyer competition mechanism**
- For seller $j \in J$, if his bid $g_j$ is greater than $p_+(j)(I, J)$, he is eliminated from the auction. Let $\bar{J}$ denote the set of remaining sellers, $\{j \mid g_j \leq p_+(j)(I, J), j \in J\}$.

- The items are allocated among the remaining agents ($I$ and $\bar{J}$) in the most efficient way.

- The trading buyer $k$ pays $p_-(k)(I, \bar{J})$, and the trading seller $l$ receives $p_+(l)(I, J)$.

**Theorem 2.1.** The AC-DA mechanism is strategyproof, (ex post) individually rational, and (ex post) budget-balanced in the simple exchange environment with transaction costs.

The proof is deferred to §4 and the online appendix on the Management Science website (http://mansci.pubs.informs.org/eecompanion.html), where we provide a more general theorem.

Let us point out that the quasi-linear assumption about each agent’s utility is unnecessary with this theorem because each agent only tries to buy or sell a single unit and faces a take-it-or-leave-it situation. That is, if we calculate $p_-$ and $p_+$ according to formulation $\mathcal{P}$, the AC-DA mechanism is strategy-proof, individually rational, and budget-balanced. To the best of our knowledge, this is the first strategy-proof, (ex post) individually rational, and (ex post) budget-balanced double-auction mechanism applicable to the simple exchange environment with transaction costs.

3. **Efficiency Analysis**

3.1. **Impact of Transaction Costs**

If there are no transaction costs, the AC-DA mechanism either achieves an efficient allocation or misses at most one profitable trade in comparison to the

$\cdots$

4 For the seller competition mechanism, we may also eliminate seller $j$ if his bid is no less than $p_+(j)$, that is, we set $\bar{J} = \{j \mid g_j < p_+(j)(I, J), j \in J\}$. For the buyer competition mechanism, we may also eliminate buyer $i$ if her bid is no more than $p_-(i)$, that is, we set $\bar{J} = \{i \mid f_i > p_-(i)(I, J), i \in I\}$. The resulting mechanisms maintain the desired properties, and should be recognized as variants of the AC-DA mechanism.
efficient allocation. The AC-DA mechanism is asymptotically efficient as the number of trades approaches infinity.

Now we use the example in Figure 1 to illustrate how the AC-DA mechanism may behave under a simple exchange environment with transaction costs. In this example, we have two sellers and three buyers, and apply the seller competition mechanism. The transaction cost matrix is given in Figure 1. The agents have the incentive to truthfully bid their valuations, which are shown in the graph below the table. In this example, the \( p_i \)'s of the sellers are 88 and 90, respectively, and both sellers survive the elimination phase. Then in the efficient allocation, the mechanism executes two transactions, one between buyer 1 and seller 1, and another between buyer 2 and seller 2. The payments from buyers 1 and 2 are 92 and 94, respectively. The revenues for sellers 1 and 2 are 88 and 90, respectively.

Note that the seller competition mechanism achieves the efficient allocation in this example.

It is interesting to see how transaction costs affect the maximum efficiency of a mechanism. We will illustrate this effect using the example in Figure 2. This example is the same as the one in Figure 1 except that the transaction costs are about 70% higher. The new transaction cost matrix is given in Figure 2. In this example, \( p_i \)'s of the sellers are 85 and 86, respectively. The seller competition mechanism executes only one transaction (between buyer 2 at buying price 93 and seller 2 at selling price 86) instead of two in the previous example, while the efficient allocation allows two transactions.

Note that due to the high transaction costs, we essentially have two separate markets: the first market consists of buyer 1 and seller 1, whereas the second one has buyers 2 and 3, and seller 2. The first market is essentially an instance of the bilateral trade problem studied by Myerson and Satterthwaite (1983), who show that no incentive-compatible, budget-balanced mechanism is efficient. Thus, it is unlikely that there is any strategyproof, budget-balanced mechanism that always leads to a transaction between buyer 1 and seller 1. In this sense, the seller competition mechanism achieves the highest possible efficiency in this example.

The example illustrates that the transaction costs can prevent some possible transactions and separate the buyers and sellers into small markets, which results in a loss of efficiency. This effect can be aggravated if there is an infinite number of transaction costs.

To demonstrate this point, let us assume that \( \mathcal{F} \) is a family of continuous distributions with support contained in \([0, a]\) for some constant \(a\). Suppose distributions \(F\) and \(G\) are drawn from \( \mathcal{F} \) according to some stochastic process. Let each buyer’s valuation independently follow \(F\) and each seller’s valuation independently follow \(G\). Now, we assume that the agents are independently generated from some compact domain \(H\) according to some continuous distribution \(U\), and that the transaction cost, \(d_{ij}\), is the distance between buyer \(i\) and seller \(j\). That is, the transaction cost \(d\) depends on \(x\) and \(y\), the locations of \(i\) and \(j\), respectively, \((x, y) \in H\), and \(d(x, y)\) is a metric: \(d\) is symmetric; \(d\) satisfies the triangle inequality; and \(d(x, y) = 0\) if and only if \(x = y\). Let us also assume that \(d\) is continuous.

Then we have the following result.

**Theorem 3.1.** Given a finite buyer set, the efficiency achieved by the seller competition mechanism approaches zero almost surely as the number of sellers approaches infinity.

The proof is available in the appendix.

This counterintuitive result shows the importance of choosing the right mechanism to achieve higher efficiency. In fact, the buyer competition mechanism is asymptotically efficient in such a buyer’s market.
We discuss the asymptotic efficiency result further in the next subsection.

3.2. Asymptotic Efficiency

In this section, we will show that even though transaction costs hurt efficiency, the loss can be mitigated and becomes negligible in a large economy.

We adopt the same distribution and transaction costs assumptions as in the previous subsection. We have the following result.

Theorem 3.2. The AC-DA mechanism is asymptotically efficient if both the number of buyers \( m \) and the number of sellers \( n \) approach infinity and \( 0 < \lim (m/n) \leq \lim (m/n) < \infty \).\(^7\)

The proof appears in the appendix.

The asymptotic efficiency theorem shows that as long as the economy is large enough and balanced between buyers and sellers, both the buyer competition mechanism and the seller competition mechanism will achieve satisfactory efficiency. If the economy is unbalanced, say \( m/n \) approaches zero, then we should choose the buyer competition mechanism to achieve asymptotic efficiency. Similarly, if \( m/n \) approaches infinity, we should pick the seller competition mechanism to achieve asymptotic efficiency.

4. Substitutable Commodities

Up to now we have concentrated on a simple exchange environment to facilitate an understanding of the mechanism, because it is easy to think about the case in which each agent wants to buy (sell) one unit of the same item. In practice, there are many other factors that can make the auction more complicated. For instance, one buyer may find several different items to be perfect substitutes, while another buyer is only willing to accept one of them due to the brand name effect. Can the AC-DA mechanism be applied to this setting?

The answer is affirmative. The AC-DA mechanism maintains the desired properties if each agent submits a single bid and social welfare satisfies the following complementarity-substitutability conditions in Shapley (1962).

Property 1. If \( k \) and \( l \in I \) are two different buyers, then \( V + V_{[k,l]} \leq V_k + V_{-l} \).

Property 2. If \( k \) and \( l \in J \) are two different sellers, then \( V + V_{[k,l]} \leq V_k + V_{-l} \).

Property 3. If \( k \in I \) is a buyer and \( l \in J \) is a seller, then \( V + V_{[k,l]} \geq V_k + V_{-l} \).

Because we do not know how many agents are in the exchange environment nor their bid prices before the auction, these are properties of the underlying exchange environment. Property 1 means that buyers are substitutes for each other. Property 2 says that sellers are substitutes for each other. Property 3, the complementary condition, implies that the contribution to a system in which only new buyers are allowed to enter plus the contribution to a system in which only new sellers are allowed to enter is less than the contribution to a system in which both buyers and sellers are allowed to enter. To some extent, this condition suggests that the double-auction approach should achieve higher social welfare compared to separate one-sided auction approaches. Formally, we have the following result.

Theorem 4.1. The AC-DA mechanism is strategyproof, (ex post) individually rational, and (ex post) budget-balanced if the quasi-linear social welfare satisfies the complementarity-substitutability conditions.

As shown by Shapley (1962), the social welfare formulation \( \pi \) for the simple exchange environment satisfies the complementarity-substitutability conditions, and Theorem 2.1 in §2 becomes a special case of this theorem. The proof of this theorem is available in the online appendix.

5. Conclusion and Future Research Directions

In this paper, we propose a multistage double-auction mechanism design approach. By conducting transactions in an aggregate market, the resulting truthful mechanism can achieve higher social welfare and higher individual revenues. The mechanism can also be applied to any complementarity-substitutability exchange environment. To the best of our knowledge, this is the first strategyproof, budget-balanced double-auction mechanism applicable to an exchange environment with transaction costs.

We believe this work constitutes a useful basic framework for further development of general double auctions. In the near future, we plan to apply and possibly generalize the AC-DA mechanism to more general settings, especially for environments in which the social welfare does not satisfy the complementarity-substitutability conditions. We also plan to analyze in more detail the effects of the design approach on individual payoffs and social welfare. Also, because the literature focuses heavily on one-sided auctions, comparisons among different auctions, especially comparisons between one-sided auctions and double auctions, can be interesting future research directions.

An online supplement to this paper is available on the Management Science website (http://mansci.pubs.informs.org/e companion/html).
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Appendix. Asymptotic Efficiency Analysis*
Let \( \bar{T} \) be a family of continuous distributions with support contained in \([0, a]\) for some constant \( a \). Suppose distributions \( F \) and \( G \) are drawn from \( \bar{T} \) according to some stochastic process. Let each buyer’s valuation independently follow \( F \), and each seller’s valuation independently follow \( G \). We assume that the agents are independently generated from some compact domain \( H \) according to some continuous distribution \( U \), and that the transaction cost, \( d_{ij} \), is the distance between buyer \( i \) and seller \( j \). That is, the transaction cost \( d \) depends on \( x \) and \( y \), the locations of \( i \) and \( j \), respectively, \((x, y \in H)\), and \( d(x, y) = d \) is a metric: \( d \) is symmetric; \( d \) satisfies the triangle inequality; and \( d(x, y) = 0 \) if and only if \( x = y \). Let us also assume \( d \) is continuous.

Unlike the result for simple exchange environment, due to the transaction costs, we have the following negative result.

**Theorem A.1.** For a given finite buyer set, the social welfare achieved by seller competition mechanism approaches zero almost surely as the number of sellers approaches infinity.

**Proof.** Without loss of generality, assume that every buyer in the given buyer set has a positive probability of making a profitable transaction with a seller, otherwise we could remove this buyer, as she does not contribute to the system. Because \( U \) is continuous, the probability of two buyers being at the same location is zero. Let \( d_0 \) be the minimum of \( d(x, y) \), where \( x \) and \( y \) are locations of different buyers. Then \( d_0 \) is positive with probability 1. Set \( \epsilon = d_0/3 > 0 \), and define \( A(x, \epsilon) = \{ y \mid d(x, y) < \epsilon, y \in H \} \). Because \( d \) is continuous, the probability that some seller is from the area \( A \) is always positive. Due to the triangle inequality, we have a disjoint neighborhood \( A(x, \epsilon) \) for each buyer \( i \) at \( x \). Let us define \( r = \inf(y \mid G(y) > 0) \) and \( B = [r, r + \epsilon] \). We have \( \text{Prob}(Y \in B) > 0 \) for \( Y \) drawn from distribution \( G \). As the number of sellers approaches infinity, the number of sellers with a valuation belonging to \( B \) exceeds the number of buyers, \( m \), in every neighborhood \( A(x, \epsilon) \) with probability 1. We show under this condition that no seller survives the elimination phase. With no seller left in the remaining system, the social welfare achieved by the seller competition mechanism equals zero. It suffices to show that there is no efficient allocation in which seller \( j \), whose valuation is \( r \), can sell one more item with this condition. Note that the probability of some seller having a valuation lower than \( r \) is zero. Suppose there is such an efficient allocation where seller \( j \), with two items, makes transactions with both \( i_1 \) and \( i_2 \). Because the neighborhood \( A(x_{i_1}, \epsilon) \) and \( A(x_{i_2}, \epsilon) \) are disjoint, without loss of generality, we assume that seller \( j \) does not locate at \( A(x_{i_1}, \epsilon) \). There are only \( m \) buyers, so there must be some seller \( j' \) in \( A(x_{i_1}, \epsilon) \) who is not involved in the efficient transaction with a valuation belonging to \( B \). However, by the triangle inequality, one can improve the social welfare by making transactions between \( i_1 \) and \( j' \) instead of \( i_1 \) and \( j \). We have a contradiction; thus, no seller survives the elimination phase, and the social welfare achieved by the seller competition mechanism is zero almost surely as the number of buyers approaches infinity. \( \square \)

However, if both the number of buyers, \( m \), and the number of sellers, \( n \), go to infinity, we have the following asymptotic efficiency result:

**Theorem A.2.** The AC-DA mechanism is asymptotically efficient if both \( m \) and \( n \) approach infinity and \( m/n \) approaches some number \( p \) satisfying \( 0 < p < \infty \).

**Proof.** Let \( z \) be a number satisfying \( F(z) + pG(z) = 1 \). It is clear that \( z \) exists because both \( F \) and \( G \) are continuous and the left-hand side increases from 0 to 1 + \( p \). For any \( \epsilon > 0 \), since \( H \) is compact, there exists a finite \( \epsilon \)-partition \( A_1, A_2, \ldots, A_k \) of \( H \) (i.e., a partition such that \( A_j \) has radius less than \( \epsilon \)).

We prove the result by first calculating the limiting maximum social welfare per agent with complete information and then showing that the social welfare per agent achieved by the seller competition mechanism also approaches this limit almost surely.

Now, let us define \( r = \inf(y \mid G(y) \geq 0) \). We first consider those sellers who bid less than \( r - \epsilon \) in region \( A \). We have \( G(r - \epsilon) < G(z) \), thus \( pG(r - \epsilon) < 1 - F(z) \). As \( m/n \to p \), with probability 1, the number of sellers who bid less than \( r - \epsilon \) is less than the number of buyers who bid higher than \( z \). Thus, just by ranking the prices in each region and transacting according to the ranking, one can achieve social welfare per agent greater than

\[
\frac{[G(r - \epsilon)/G(z)] \int_{z}^{\infty} x dF(x) - p \int_{0}^{r-\epsilon} (y + \epsilon) dG(y)}{1 + p}
\]

as the transaction cost for each transaction is less than \( \epsilon \). Also note that \( G(r) = G(z) \) because \( G \) is left continuous, as \( \epsilon \to 0 \), it is almost sure that the social welfare achieved per agent is no less than

\[
\bar{C} \equiv \int_{z}^{\infty} x dF(x) - p \int_{0}^{r-\epsilon} y dG(y)
\]

as the number of agents approaches infinity. From the above arguments, with complete information, there is an allocation that achieves social welfare per agent no less than \( \bar{C} \) as the number of agents approaches infinity. We next show that \( \bar{C} \) is in fact also an upper bound on the social welfare per agent for any allocation. Thus, the maximum social welfare per agent with complete information approaches \( \bar{C} \) almost surely. To see this, we consider the case in which we ignore the transaction costs. The social welfare per agent with complete information is no greater than \( \bar{C} \) almost surely, which gives us an upper bound for the case with transaction costs.

Note that as \( n \to \infty \), it is almost sure that in each region \( A_j \) there exists some seller whose valuation is

*We only prove the results for the seller competition mechanism in the appendix, and it is understood that all theorems about the seller competition mechanism also hold, mutatis mutandis, for the buyer competition mechanism.*
between \( r - \epsilon \) and \( r \). Moreover, the percentage of these sellers whose valuations are between \( r - \epsilon \) and \( r \) approaches some positive number in each region. Due to the triangle inequality, a seller whose valuation is less than \( r - 2\epsilon \) is always more competitive than the seller whose valuation is between \( r - \epsilon \) and \( r \) in the same region. Thus, if in an efficient allocation, some seller with valuation between \( r - \epsilon \) and \( r \) completes a transaction, all sellers in the same region with valuations less than \( r - 2\epsilon \) must also get transactions. Moreover, each seller whose valuation is less than \( r - 2\epsilon \) survives in this case. By the calculation in the above paragraph, we know in an efficient allocation, the percentage of the sellers who bid between \( r - \epsilon \) and \( r \) and get a transaction can not go to zero, otherwise, the limiting maximum social welfare per agent with complete information is strictly less than \( \bar{C} \). That is, it is almost sure there is some seller with valuation between \( r - \epsilon \) and \( r \) who completes a transaction in each allocation. Thus, sellers whose valuation is less than \( r - 2\epsilon \) survive in the elimination phase. Because \( \epsilon \) can be arbitrary, by the same arguments as we used in the above paragraph, we know as \( n \rightarrow \infty \), the social welfare per agent achieved by the seller competition mechanism is no less than \( \bar{C} \). Note that \( \bar{C} \) is an upper bound on the social welfare per agent for any allocation, the social welfare per agent achieved by the seller competition mechanism approaches \( \bar{C} \) almost surely.

As both the maximum social welfare per agent with complete information and the social welfare per agent achieved by the seller competition mechanism approach \( \bar{C} \) almost surely, the ratio between the welfare achieved by the seller competition mechanism and the maximum social welfare with complete information converges to 1 as the number of agents approaches infinity. Thus, the seller competition mechanism achieves asymptotic efficiency. \( \square \)

**Theorem A.3.** The AC-DA mechanism is asymptotically efficient if \( m \rightarrow \infty \), \( n \rightarrow \infty \), and \( 0 < \liminf (m/n) \leq \limsup (m/n) < \infty \).

**Proof.** Given any \( \delta > 0 \), we use \( E \) to denote the following event: the efficiency achieved by the AC-DA mechanism is higher than \((1 - \delta) \times \) (maximum social welfare with complete information). We know from the proof of Theorem A.2 that for any \( \epsilon > 0 \) and \( 0 < p < \infty \), there is an open set containing \( p \), such that as long as \( m/n \) is in the set, \( \text{Prob}(E) > 1 - \epsilon \) if the number of agents is large enough. Also, because \( 0 < \lim (m/n) \leq \limsup (m/n) < \infty \), we have \( m/n \in \left[ \frac{1}{2} \lim (m/n), 2 \lim (m/n) \right] \). By Heine-Borel theorem, we can have finitely many these open sets covering the compact set \( \left[ \frac{1}{2} \lim (m/n), 2 \lim (m/n) \right] \). Thus, for any \( \epsilon, \delta > 0 \), \( \text{Prob}(E) > 1 - \epsilon \) when the number of agents approaches infinity. By letting \( \epsilon, \delta \rightarrow 0 \), the ratio between the welfare achieved by AC-DA mechanism and the maximum social welfare with complete information converges to 1 almost surely as the number of agents approaches infinity, that is, the AC-DA mechanism is asymptotically efficient. \( \square \)

**References**


