A Power-of-Two Ordering Policy for One-Warehouse Multiretailer Systems with Stochastic Demand

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We study a two-echelon supply chain with one warehouse and \( N \) (nonidentical) retailers facing stochastic demand. An easy-to-implement inventory policy, the so-called power-of-two (POT) policy, is proposed to manage inventory for the system. To maintain a certain service level, safety stocks are kept at the warehouse and each retailer outlet to buffer random demand. Our analysis highlights the important role of the warehouse safety stock level, which, in addition to the length of the warehouse order interval, significantly affects the lengths of the retailers’ order intervals. By combining the length of the warehouse order interval with the warehouse safety stock level, we introduce a plane partition method and develop a polynomial time algorithm to find a POT policy for arbitrary target service levels. The long-run average cost of the proposed POT policy is guaranteed to be no more than 1.26 times the optimal POT policy cost. We also show that our proposed policy can be computed in \( O(N^3) \).

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1. Introduction

We consider a distribution system with one warehouse and multiple retailers. The warehouse orders from an outside supplier with unlimited supply and replenishes the retailers’ inventories. Random demand occurs at the retailers only, while unsatisfied demand is back-ordered. To maintain appropriate service levels, safety stock is maintained at the warehouse and the retailers. The warehouse and each retailer follow a periodic order inventory policy with certain service level requirements. The planning horizon is infinite, and the objective is to minimize the long-run average system-wide cost subject to the service level requirements.

Multiechelon stochastic inventory models have received much attention from the research community. In terms of exact results, many papers focus on the continuous review policy. For example, Axsäter (2000) provides exact results for compound Poisson demand and nonidentical retailers when the warehouse and the retailers use the \((R, nQ)\) policy. Cheung and Hausman (2000) analyze exact results for the central warehouse serving nonidentical retailers. Chen and Zheng (1997) provide exact results when the central warehouse uses a continuous review reorder point policy, while the reorder point is calculated according to echelon stock inventory. For periodic review models, Liljenberg (1996) and Cachon (2001) provide exact valuation methods for identical retailers under a batch ordering policy.

It is recognized that the structure of an optimal policy for most multiechelon stochastic inventory systems is either unknown or extremely complex; thus, it is difficult to implement them in practice. As a result, much research has focused on heuristic policies that are easier to implement. For excellent reviews of this literature, we refer the readers to Axsäter (1993) and Federgruen (1993). Recent developments include Lee and Billington (1993), Tempelmeier (1993), Hausman and Erkip (1994), Chen and Samroengraja (2000), and Levi et al. (2008).

However, because these policies are only heuristic, a natural question to ask is how much is the gap between the heuristic solution and the optimal one. The literature provides only some numerical evidence (e.g., Federgruen and Zipkin 1984a, b; Chen and Zheng 1994, 1997). But no matter how extensively the numerical experiments have been conducted, there are no guarantees with respect to the heuristic’s performance on other problem instances. Thus, it is important to design algorithms that can provide good worst-case bound for any problem instance.

For a single-echelon continuous-review inventory system with stationary stochastic demand, Zheng (1992) shows...
that the maximum relative error is bounded by 0.125 when using the deterministic EOQ formula as a heuristic solution in the \((Q, r)\) policy. Axsäter (1996) gives a slightly better bound of 0.118. Levi et al. (2007) analyze stochastic inventory problems with correlated and nonstationary demands. They provide a 2-approximation algorithm for the periodic-review stochastic inventory control problem and a 3-approximation algorithm for the stochastic lot-sizing problem. Levi et al. (2006) extend the results in Levi et al. (2007) to multiechelon supply chains. Their approach is guaranteed to produce a policy with total expected cost no more than twice the expected cost of an optimal policy for the multiechelon system.

Other than Levi et al. (2006), we have not seen much work for the stochastic multiechelon inventory systems. In contrast, for the deterministic multiechelon inventory systems, it is well known that a class of easily implementable policies, the so-called power-of-two (POT) policies, performs surprisingly well (e.g., Maxwell and Muckstadt 1985, Roundy 1985). Under a POT policy, each item is replenished at constant reorder intervals that are POT multiples of some fixed or variable base planning period. An optimal POT policy can be found very easily, and its cost is guaranteed to be within 2% of optimality. Recent works that analyze easy-to-implement policies with worst-case bounds include Chen (2000), Chan et al. (2002), and Levi et al. (2005).

In this paper, we apply the POT policy to one-warehouse multiretailer (OWMR) models with stochastic demand. Because most companies determine the safety stock levels according to their target service levels, we minimize the total order and holding costs for the given service levels instead of assuming a linear out-of-stock penalty cost and explicitly considering this backlogging cost in the objective function. We develop a polynomial time algorithm to find a close-to-optimal POT policy for arbitrary target service levels. Our approach highlights the important role of the warehouse safety stock level, which, in addition to the length of the warehouse order interval, significantly affects the lengths of the retailers’ order intervals. To derive the close-to-optimal policy, we employ a plane partition method using information related to the warehouse’s order frequency and safety stock level. The long-run average cost of the proposed POT policy is guaranteed to be no more than \(\sqrt{2} \approx 1.26\) times the optimal periodic order policy cost under which the warehouse and retailers place synchronized orders periodically.

The rest of this paper is organized as follows. Section 2 describes the POT inventory model. Section 3 discusses the OWMR model without the POT constraints. Based on this relaxation model, we discuss how to construct a close-to-optimal POT policy in §4. We conclude in §5. An electronic companion to this paper is available as part of the online version that can be found at http://or.journal.informs.org/.

2. Model

We use term facility to refer to either the warehouse or a retailer. Facility 0 is the warehouse and retailers are facilities 1 through \(N\). Let the holding cost rate and the fixed setup cost at retailer \(n\), \(n = 1, \ldots, N\), be \(h'_{n}, K_{n}\), and at the warehouse be \(h_{0}, K_{0}\). Let the average demand at retailer \(n\) be \(\lambda_{n}\) and the variance \(\sigma_{n}^{2}\) per unit time. We define the echelon holding cost rate \(h_{n} \equiv h'_{n} - h_{0}\), which can also take negative values (salvage values). Order lead time is \(L_{0}\) at the warehouse and \(L_{n}\) for each retailer \(n = 1, \ldots, N\).

A periodic ordering policy is one in which each facility \(n\) places an order once every \(T_{n} > 0\) units of time, beginning at \(t = 0\). Periodic ordering policies enable straightforward coordinations among the warehouse and the retailers and are widely used in practice. A power-of-two (POT) policy is a periodic ordering policy such that the ratio \(T_{n}/T_{0}\) is an integer power of two. In this model, the retailers’ orders are synchronized, i.e., all \(N\) retailers order in the same periods. In this paper, we will focus on the synchronized ordering policy. Lee et al. (1997) and Cachon (1999) study the scheduled ordering policies in which the retailers’ orders are balanced, i.e., the same number of retailers’ order at each period. They show that the scheduled ordering policies might lead to higher or lower total supply chain cost because the synchronized ordering policy might have lower demand uncertainty. Designing efficient and effective algorithms for the scheduled ordering policy is left as an important future research direction.

In a random demand environment, safety stocks are needed at the warehouse and each retailer site to ensure a certain service level. Type I service level measures the probability of satisfying the demand using on-hand inventory during a single order cycle. Type II service level is also known as fill rate, which measures proportion of demand satisfied. In this paper, we focus on Type I service level, and the same idea can be extended to the systems with Type II service requirement.

Capturing demand uncertainty via Normal approximations is a common practice in most retail and distribution chains. In this paper, we focus on systems in which the Normal approximation for demands is appropriate. In this case, the standard deviation \(\sigma\) characterizes the demand uncertainty and the amount of safety stocks required can be calculated as \(z\sigma\). For example, if we specify a 90% service level, \(z = 1.28\) for Type I service level (Nahmias 2001). This square-root formula can be used for any Normal demand distribution (Eppen 1979), or it can be used as a good pooling approximation for demand distributions that are not normalized. For example, for single-parameter demand distributions (e.g., Poisson) or additive two-parameter demand distributions that have a constant variance-to-mean ratio, it is possible to construct an appropriate safety stock pooling function numerically by plotting the total amount of stock required to achieve the given customer service level (as a function of the average lead time
demand) then subtracting the average lead time demand. More details can be found in Caggiano et al. (2002).

When retailer demands are independent and the normal approximation is appropriate, the warehouse demand can also be represented by Normal distribution, and the variance of warehouse demand during an order cycle is the summation of the variances imposed from each retailer.

A periodic ordering policy for the system can be characterized by an \((N + 1)-\)tuple, \(\bar{T} = (T_0, T_1, \ldots, T_N)\), and the corresponding average cost can be written as

\[ c(\bar{T}) \equiv K_0/T_0 + h_0 s_0 + \sum_{1 \leq n \leq N} (c_n(T_0, T_n) + h'_n s_n), \]

where \(s_n\) is the safety stock level and \(c_n(T_0, T_n)\) includes the setup costs and cycle inventory costs associated with retailer \(n\). When either \(T_n/T_0\) or \(T_n/T_n\) is an integer, \(c_n(T_0, T_n) = K_n/T_n + (\lambda_n/2)(h_n T_n + h_0(T_0 \vee T_n))\), where \(x \vee y\) denotes the larger of \(x\) and \(y\) (Roundy 1985); on the other hand, if neither \(T_n/T_0\) nor \(T_n/T_n\) is an integer, \(K_n/T_n + (\lambda_n/2)(h_n T_n + h_0(T_0 \vee T_n))\) provides a lower bound for \(c_n(T_0, T_n)\). The safety stock level \(s_n\) at retailer \(n\) can be written as \(\theta_n \sqrt{(L_n + T_n)\sigma_n^2} = \theta_n \sigma_n \sqrt{L_n + T_n}\), where \(\theta_n\) depends on the specified Type 1 service level at retailer \(n\). To calculate the safety stock level at the warehouse, let us first examine how each retailer contributes to the demand variance at the warehouse.

**Case 1.** \(T_n \geq T_0\): Retailer \(n\) orders less frequently than the warehouse does. When \(T_n/T_0\) is an integer, each time retailer \(n\) orders, the order introduces variance \(L_0\sigma_n^2\) to the warehouse. When \(T_n/T_0\) is not an integer, the order introduces variance no less than \(L_0\sigma_n^2\).

**Case 2.** \(T_n < T_0\): Retailer \(n\) orders more frequently than the warehouse does. When \(T_n/T_0\) is an integer, after each warehouse’s order, retailer \(n\) would order \((T_n/T_0 - 1)\) more times until the warehouse’s next order. In this case, retailer \(n\)’s orders introduce variance \((L_0 + T_n - T_n)\sigma_n^2\) to the warehouse. When \(T_0/T_n\) is not an integer, retailer \(n\) would order \([T_0/T_n - 1]\) more times after each warehouse’s order \((\lceil x \rceil\) is the smallest integer greater than or equal to \(x\)). The variance introduced is no less than \((L_n + T_0 - T_n)\sigma_n^2\) to the warehouse.

The maximum safety stock level for the warehouse occurs when all the retailers order at the same time, and the corresponding safety stock level \(s_0\) is

\[ \theta_0 \sqrt{L_0 \sum_n \sigma_n^2 + \sum_n (0 \vee (T_0 - T_n))\sigma_n^2}, \]

where \(\theta_0\) depends on the specified service level at the warehouse. We assume that the warehouse adopts a stationary safety stock policy and keeps this safety stock level all the time. This assumption enables us to use a single parameter to characterize the level of warehouse safety stock and investigate the interplay between the warehouse safety stock level and the length of the warehouse order interval. An important future research is how to represent and characterize a close-to-optimal policy when the warehouse safety stock level is not stationary.

The optimal periodic order policy problem can then be formulated as follows:

\[ \min c(\bar{T}) \equiv K_0/T_0 + h_0 s_0 + \sum_{1 \leq n \leq N} (c_n(T_0, T_n) + h'_n s_n) \]

subject to \(T_n > 0\) for \(n = 0, 1, \ldots, N\), \(T_n/T_0\) is an integer,

where

\[ c_n(T_0, T_n) := K_n/T_n + \frac{\lambda_n}{2}(h_n T_n + h_0(T_0 \vee T_n)), \]

\[ s_n := \theta_n \sigma_n \sqrt{L_n + T_n}, \]

and

\[ s_0 := \theta_0 \sqrt{L_0 \sum_n \sigma_n^2 + \sum_n (0 \vee (T_0 - T_n))\sigma_n^2}. \]

The objective function minimizes the total order and holding costs, while the safety stock levels are controlled by the target service levels. Each time retailer \(n\) orders, the inventory position is raised to the order-up-to level, which equals the safety stock level \(s_n\) plus the expected demand \(\lambda_n T_n\). Each time the warehouse orders, the new inventory position equals the safety stock level \(s_0\) plus the expected demand during the following cycle, which might vary because some retailers will not order from time to time.

### 3. Relaxation of the OWMR Model

Similar to the spirits of Roundy (1985), we will construct a POT policy based on the optimal solution to the relaxation problem. The major difference here is that for each facility, we have an extra term for the safety stock cost. The optimal order cycles at all facilities are linked together through the warehouse safety stock level. Like the length of the warehouse order interval, the warehouse safety stock information needs to be considered in deciding which retailers should order more frequently than the warehouse and which retailers should order less frequently.

We minimize the relaxation problem by dropping the integer ratio constraints that specify \(T_n/T_0\) to be an integer. Let \(\breve{G}\) denote the retailer set with \(T_n > T_0\), \(\breve{C}\) the retailer set with \(T_n = T_0\), and \(\breve{D}\) the retailer set with \(T_n < T_0\). We solve this relaxation problem by decomposing it into smaller subproblems: (i) we first determine all the possible set partitions of \(\breve{G}\), \(\breve{C}\), and \(\breve{D}\); then (ii) we solve the optimal solution for the relaxation problem for each possible partition. Unlike Roundy (1985), in which we can determine the partition straightforwardly based on the warehouse order interval alone, a novel plane partition method is proposed to determine only \(O(N^2)\) instead of an exponential number of the possible set partitions. This plane partition method incorporates both the order interval information and the safety stock information. Then, for each possible partition,
we show that at most two feasible solutions satisfy the first-order condition, and we can find the optimal solution via a binary search or a golden search on the warehouse safety stock level.

3.1. Determining \( q_* \), \( c_* \), and \( \mathcal{L} \)

Notice that if \( \sigma_n = 0 \), our problem reduces to the original problem in Roudy (1985), and we can follow his approach to decide whether retailer \( n \) belongs to \( q_* \), \( c_* \), or \( \mathcal{L} \). In the following, we assume that \( \sigma_n > 0 \) for \( n = 1, \ldots, N \).

Let \( f_n(T) = K_n/T + (\lambda_n/2)h_nT + h_n\theta_n\sigma_n/\sqrt{T_n + x} \). When \( T_n \geq T_0 \), \( f_n(T) = c_n(T_0, T_n) + h_n\sigma_n \) characterizes the cost at retailer \( n \) because the term \( h_nT_n + h_0(T_0 \vee T_n) \) is just \( h_nT_n \). By Corollary 1 in the online appendix, the first-order condition of \( f_n(T) \) has a unique solution, which minimizes \( f_n(T) \). Denote this minimum solution \( T_n^* \).

Define \( g_n(T) = K_n/T + (\lambda_n/2)h_nT + h_n\theta_n\sigma_n/\sqrt{T_n + x} \). When \( T_n \leq T_0 \), \( g_n(T) = c_n(T_0, T_n) + h_n\sigma_n \) characterizes the cost at retailer \( n \). Because \( h_n \equiv h_n^* - h_0 \) can be negative, \( g_n(T) \) has at most two solutions, and \( g_n(T) \) has at most one local minimum by Lemma EC.1 in the online appendix. If \( g_n(T) \) has one local minimum, denote it as \( T_n^* \); if \( g_n(T) \) has no local minimum, \( g_n(T) \) is a decreasing function and we define \( T_n^* = \infty \). \( f_n(T) \) is negative in \( (0, T_n^*) \) and positive in \( (T_n^*, \infty) \), while \( g_n(T) \) is negative on \( (0, T_n^*) \), \( g_n(T_n^*) = 0 \) if \( T_n^* < \infty \).

**Lemma 1.** \( T_n > T_n^* \) for all \( n \).

**Proof of Lemma 1.** Because \( g_n'(T) = f_n'(T) - (\lambda_n/2)h_0 < f_n'(T) \) and \( f_n'(T) < 0 \) on \( (0, T_n^*) \), \( g_n'(T_n^*) < 0 \). Because either \( T_n = \infty \) or \( g_n'(T_n) = 0 \), we conclude that \( T_n > T_n^* \).

To determine whether \( n \) belongs to \( q_* \), \( c_* \), or \( \mathcal{L} \), that is, whether \( T_n \) is greater, equal to, or less than \( T_0 \), we first need to know how \( T_n \) influences the total cost. Notice that four terms in the cost function contain \( T_n \): \( K_n/T_n \), \( (h_nT_n + h_0(T_0 \vee T_n)) \), \( h_n\theta_n\sigma_n/\sqrt{T_n + x} \), and \( h_0\sigma_n \), where \( s_0 = \theta_0\sqrt{L_0} \sum \sigma_n^2 + \sum (0 \vee (T_0 - T_n))\sigma_n^2 \).

We observe that:

- If \( T_n \geq T_0 \), the warehouse stock level \( s_0 \) is independent of \( T_n \), and the total cost is a function of \( T_n \) that varies according to \( f_n(T_n) \), that is, \( \partial \mathcal{C}_n/[\partial T_n] = \partial f_n(T_n)/\partial T_n \).
- If \( T_n \leq T_0 \), \( s_0 \) is a decreasing function of the order interval \( T_n \). We may explicitly express the relationship by writing \( s_0(T_n) \) instead of \( s_0 \). The total cost varies according to \( g_n(T_n) + h_0\sigma_n(T_0) \), that is, \( \partial \mathcal{C}_n/[\partial T_n] = (\partial g_n(T_n) + h_0\sigma_n(T_0))/\partial T_n \).

**Proposition 1.** \( n \in q_* \) if and only if \( T_n < T_n^* \).

**Proof of Proposition 1.** Let \( T_n^* \) be the optimal \( T_n \) for given \( T_0 \).

\[ \Rightarrow: \] Suppose that \( T_n \geq T_n^* \) and \( f_n(T_n) \) is a strictly increasing function on \( (T_0, \infty) \). Because the total cost changes according to \( f_n(T_n) \), we can decrease the total cost by setting \( T_n = T_0 \) if \( T_n > T_0 \). This, \( T_n \leq T_0 \) and \( n \in q_* \).

\[ \Leftarrow: \] Suppose that \( T_n > T_n^* \); we show that \( T_n = T_n^* \). Recall that when \( T_n \leq T_0 \), the total cost varies according to \( g_n(T_n) + h_0\sigma_n(T_0) \). The derivative of the total cost is \( (g_n(T_n) + h_0\sigma_n(T_0)) = f_n'(T_n) - (\lambda_n/2)h_0 - h_0(\theta_n\sigma_n^2/(2s_0)) \). Because \( f_n'(T_n) \leq 0 \) on \( (0, T_n^*) \) and \( T_n < T_n^* \), \( f_n'(T_n) \leq 0 \) for \( T_n \in (0, T_0) \), thus \( (g_n(T_n) + h_0\sigma_n(T_0)) < 0 \) and \( (g_n(T_n) + h_0\sigma_n(T_0)) \) is a strictly decreasing function on \( (T_0, \infty) \). If \( T_n < T_0 \), we can decrease the total cost by setting \( T_n = T_0 \). Therefore, \( T_n^* > T_0 \), and the total cost changes according to \( f_n(T_n) \). \( T_n^* = T_n^* > T_0 \) minimizes \( f_n(T_n) \), and \( n \in q_* \). □

**Proposition 2.** \( n \in c_* \) if \( T_n^* \leq T_0 \leq T_n \).

**Proof of Proposition 2.** Because \( T_n^* \leq T_0 \), \( n \in \mathcal{L} \cup c_* \) by Proposition 1. \( T_n \in (0, T_0) \) minimizes the total cost, which is equivalent to minimizing \( (g_n(T_n) + h_0\sigma_n(T_0)) \) on \( (0, T_0) \). When \( T_n \leq T_0 \leq T_n^* \), \( g_n(T_n) > 0 \), while \( s_0 \) is also a decreasing function. \( T_n^* = T_0 \) minimizes the cost, and \( n \in c_* \). □

If the demand is deterministic, our definitions of \( T_n \) and \( T_n^* \) are consistent with the corresponding definitions in Roudy (1985). In this case, \( n \in q_* \) if and only if \( T_n < T_n^* \); \( n \in c_* \) if and only if \( T_n^* \leq T_0 \leq T_n \); and \( n \in \mathcal{L} \) if and only if \( T_n < T_0 \). However, with stochastic demand, if \( T_n > T_n^* \), \( n \) might belong to \( q_* \) or \( \mathcal{L} \) depending on the values of \( T_n \) and \( s_0 \), the level of warehouse safety stock. When the warehouse safety stock level is low, that is, the warehouse mainly serves as a cross-docking point, the retailers and the warehouse operate at the same pace (i.e., \( n \in \mathcal{L} \)). When the warehouse safety stock level is high and the warehouse serves as a major storage location, it is profitable for the retailers to order more frequently than the warehouse to take advantage of the risk-pooling effect (i.e., \( n \in q_* \)). We can determine whether retailer \( n \) is optimal \( n \) depending on the values of \( T_n \) and \( s_0 \), the level of warehouse safety stock. When the warehouse safety stock level is low, that is, the warehouse mainly serves as a cross-docking point, the retailers and the warehouse operate at the same pace (i.e., \( n \in \mathcal{L} \)). When the warehouse safety stock level is high and the warehouse serves as a major storage location, it is profitable for the retailers to order more frequently than the warehouse to take advantage of the risk-pooling effect (i.e., \( n \in q_* \)).
To facilitate the comparison, we rearrange the terms and define
\( \tilde{g}_n(T_n) \equiv -h_n s_0(T_n) + h_0 s_0(T_n) + g_n(T_n) \). It suffices to compare \( g_n(T_n) \) and \( \tilde{g}_n(T_n) \) on \((0, T_0)\). Notice that \( g_n(T_n) \) depends only on \( T_n \) and not on \( C_{-n} \), while \( \tilde{g}_n(T_n) \) depends not only on \( T_n \) but also on \( T_n \) and \( C_{-n} \). This separation enables us to investigate the critical value of \( C_{-n} \) at which retailer \( n \) shifts from \( \mathcal{S} \) to \( \mathcal{E} \).

In Figure 2, we draw \( g_n(T_n) \) and \( \tilde{g}_n(T_n) \) as functions of \( T_n \) on \((0, T_0)\). Notice that \( g_n(T_n) \) approaches positive infinity when \( T_n \) goes to zero, while \( \tilde{g}_n(T_n) \leq g_n(T_n) \) for all \( T_n \in (0, T_0) \). Thus, \( \tilde{g}_n(T_n) \) is either below, intersecting, or tangent with \( g_n(T_n) \) on \((0, T_0)\).

- If \( \tilde{g}_n(T_n) \) is below \( g_n(T_n) \) on \((0, T_0)\), then \( g_n(T_n) > \tilde{g}_n(T_n) \) is equivalent to \( g_n(T_n) + h_0 s_0(T_n) > g_n(T_n) + h_0 s_0(T_n) \). This means that \( T_n \) minimizes \( g_n(\cdot) + h_0 s_0(\cdot) \) on \((0, T_0)\). Furthermore, \( T_n^* \in (0, T_0) \) minimizes the total cost, which is equivalent to minimizing \( (g_n(T_n) + h_0 s_0(T_n)) \) on \((0, T_0)\). Therefore, \( T_n^* = T_n \), and \( n \in \mathcal{E} \).

Figure 2. \( g_n(T_n) \) and \( \tilde{g}_n(T_n) \) when \( \tilde{g}_n(T_n) \) is tangent with \( g_n(T_n) \).
on \((0, \tau_0]\). Because \(T^*_n\) minimizes \(g_n(T_n) + h_0 s_0(T_0)\) on \((0, \tau_0]\), one possible value of \(T^*_n\) is \(T_n^* = T_0\) and \(n \in \mathbb{Z}^+\); then \(s_0 = \theta_0 \sqrt{C_n - n} \); another possible value is \(T^*_n = \bar{T}_n\) (or other tangent points) and \(n \notin \mathbb{Z}\); then \(s_0 = \theta_0 \sqrt{C_n^* + (T_0 - \bar{T}_n) \sigma_n^2}\) 

\(\text{(or } s_0 \geq \theta_0 \sqrt{C_n^* + (T_0 - \bar{T}_n) \sigma_n^2}\).

\(C_n > C_n^*\): \(\bar{g}_n(T_n)\) intersects with \(g_n(T_n)\) on \(T_n \in (0, \tau_0)\). Because \(\bar{T}_n^* < T_0\), and \(n \notin \mathbb{Z}\). Furthermore, by Lemma 5 in the appendix, the optimal \(T_n^*\) is no more than \(\bar{T}_n\); then \(s_0 = \theta_0 \sqrt{C_n^* + (T_0 - \bar{T}_n) \sigma_n^2}\).

Notice that the optimal warehouse safety stock level \(s_0^*\) is either smaller than or equal to \(\theta_0 \sqrt{C_n^* - n}\) when \(n \in \mathbb{Z}^+\), or larger than or equal to \(\theta_0 \sqrt{C_n^* + (T_0 - \bar{T}_n) \sigma_n^2}\) when \(n \notin \mathbb{Z}\), and \(s_0^*\) will never pick a value between \(\theta_0 \sqrt{C_n^* - n}\) and \(\theta_0 \sqrt{C_n^* + (T_0 - \bar{T}_n) \sigma_n^2}\). Thus, if we can define a function \(r_n(T_n)\) whose value is always between \(\theta_0 \sqrt{C_n^* - n}\) and \(\theta_0 \sqrt{C_n^* + (T_0 - \bar{T}_n) \sigma_n^2}\) for each retailer \(n\), then for retailer \(n\), if \(T_0 > \tau_n\), \(n \in \mathbb{Z}^+\) if and only if \(s_0^* \leq r_n(T_n)\) and \(n \notin \mathbb{Z}\) if and only if \(s_0^* > r_n(T_n)\). This way, we can use \(T_0\) and \(s_0\) to find out whether \(n \in \mathbb{Z}^+\) or \(n \notin \mathbb{Z}\) without calculating \(C_n^*\), whose value varies with \(n\).

Next, we show that such function \(r_n(T_n)\) does exist. In Figure 3, we draw a line through \((T_0, g_n(T_0))\) such that this line is tangent with \(g_n(T_n)\) and lies below \(g_n(T_0)\) on \((0, \tau_0)\). We define \(r_n(T_n)\) such that \(h_0(\theta_0^2 \sigma_n^2 / (2r_n(T_n)))\) is the slope of the tangent line.

**Lemma 2.** \(\theta_0 \sqrt{C_n^* - n} \leq r_n(T_n) < \theta_0 \sqrt{C_n^* + (T_0 - \bar{T}_n) \sigma_n^2}\).

**Proof of Lemma 2.** We fix \(C_n = C_n^*\), \(-h_0 s_0(T_n)\) is a convex function of \(T_n\); the slope (derivative with regard to \(T_n\)) is 

\((\theta_0^2 \sigma_n^2 / (2s_0(T_n)))\), which is an increasing function of \(T_n\).

Because \(\bar{g}_n(T_n)\) is convex and below \(g_n(T_n)\) on \((0, \tau_0)\) at \(C_n = C_n^*\), the slope at \(T_n\), \(h_0(\theta_0^2 \sigma_n^2 / (2s_0(T_n))) = \theta_0(\theta_0^2 \sigma_n^2 / (2\theta_0 \sqrt{C_n^*}))\), is no less than the slope of the tangent line \(h_0(\theta_0^2 \sigma_n^2 / (2r_n(T_n)))\). That is, \(\theta_0 \sqrt{C_n^*} \leq r_n(T_n)\). Meanwhile, the slope at \(T_n\) \(h_0(\theta_0^2 \sigma_n^2 / (2s_0(T_n))) = h_0(\theta_0(\theta_0^2 \sigma_n^2 / (2\theta_0 \sqrt{C_n^*} + (T_0 - \bar{T}_n) \sigma_n^2)))\), is less than the slope of the tangent line. That is,

\(r_n(T_n) < \theta_0 \sqrt{C_n^* + (T_0 - \bar{T}_n) \sigma_n^2}\).

Thus, \(\theta_0 \sqrt{C_n^* - n} \leq r_n(T_n) < \theta_0 \sqrt{C_n^* + (T_0 - \bar{T}_n) \sigma_n^2}\).

Figure 3 shows the graphic proof and how to represent \(r_n(T_n)\) as a function of \(T_0\). □

Therefore, if \(T_0 > \tau_n\), we can claim that under scenario (IIb), if \(s_0 \leq r_n(T_n), n \in \mathbb{Z}^+\), and if \(s_0 > r_n(T_n), n \notin \mathbb{Z}\).

This statement is also true for scenario (IIa), where \(\bar{g}_n(T_n)\) is tangent with \(g_n(T_n)\) at \(T_0\) when \(C_n = C_n^*\). Under this scenario, \(r_n = \theta_0 \sqrt{C_n^* - n}\). When \(n \in \mathbb{Z}^+\), we have \(s_0 > \theta_0 \sqrt{C_n^* - n}\), which is larger than \(r_n = \theta_0 \sqrt{C_n^* - n}\). Therefore, if \(s_0 \leq r_n(T_n), n \in \mathbb{Z}^+\), and if \(s_0 > r_n(T_n), n \notin \mathbb{Z}\).

Furthermore, this statement is also true for scenario (I), where \(\bar{g}_n(T_n)\) intersects with \(g_n(T_n)\) when we decrease \(C_n^*\) to zero. Under this scenario, \(n \notin \mathbb{Z}\), \(T_n^*\) minimizes \(g_n(T_n) + h_0 s_0(T_0)\) on \((0, \tau_0)\). Because \(\bar{g}_n(T_n) = -h_0 s_0(T_0) + g_n(T_0)\), \(T_n^*\) maximizes \(\bar{g}_n(T_n) - g_n(T_0)\) on \((0, \tau_0)\). \(\bar{g}_n(T_n)\) lies above \(g_n(T_n)\) as \(g_n(T_n)\) intersects with \(g_n(T_0)\). Because the tangent line is either below or tangent with \(g_n(T_n), (T_n^*, \bar{g}_n(T_n^*))\) lies above the tangent line. In the following, we show that the optimal warehouse safety stock level \(s_0^*\) is greater than \(r_n(T_n)\). Now we prove \(s_0^* > r_n(T_0)\) by contradiction. Suppose that \(s_0^*\) is no more than \(r_n(T_n)\); then \(h_0(\theta_0^2 \sigma_n^2 / (2s_0^*))\), the slope of \(\bar{g}_n(T_n)\) at \(T_n^*\), is no less than the slope of the tangent line, \(h_0(\theta_0^2 \sigma_n^2 / (2r_n(T_n)))\). Because \(s_0^*\) is concave, the slope of \(\bar{g}_n(T_n)\) is no less than the slope of the tangent line on \([T_n^*, T_n]\). Therefore, \(\bar{g}_n(T_n)\) is above the tangent line on \([T_n^*, T_n]\). This cannot happen because both \(\bar{g}_n(T_n)\) and the tangent line pass through \((T_0, g_n(T_0))\). Therefore, the optimal warehouse safety stock level \(s_0^* > r_n(T_n)\), and we can still use \(r_n(T_n)\) to determine whether \(n \in \mathbb{Z}^+\) or \(n \notin \mathbb{Z}\).

Finally, we can claim that if \(T_0 > \tau_n\), \(n \in \mathbb{Z}^+\) if and only if \(s_0 \leq r_n(T_0)\), and if \(s_0 > r_n(T_0)\) under all possible scenarios.

The curve \(s_0 = r_n(T_n)\) and the line \(T_0 = \tau_n\) for retailer \(n\) divide the place into three areas. A total of \(N\) curves and \(N\) lines divide the plane into many small areas. Each area corresponds to one possible partition of \(\mathbb{Z}, \mathbb{Z}^+, \text{and } \mathbb{Z}^+\). The next theorem shows that the number of intersections is bounded and at most \(O(N^2)\) areas exist.

**Theorem 1.** There exists a constant \(k\), such that for any two retailers \(i\) and \(j\), \(r_i(T_0)\) and \(r_j(T_0)\) do not intersect for more than \(k\) times.

The proof is based on the fact that \(r_i(T_0)\) and \(T_0\) satisfy some polynomial equation of bounded degree. The detailed proof of the theorem is provided in the online appendix.
By Theorem 1, a curve \( r_i(T_0) \) intersects no more than \( k \) times with another curve \( r_j(T_0) \), and a curve for a particular retailer may have at most \( O(N) \) intersections. Thus, there will be at most \( O(N^2) \) intersections, and we can find the \( T_0 \) value for each intersection because each curve \( r_j(T_0) \) can be represented by a polynomial by Lemma EC.6 in the online appendix. By ranking these \( O(N^2) \) values of \( \tau_n \)'s and \( \tau_n \)'s, we can divide the \( T_0 \) axis into \( O(N^2) \) segments. Starting from the leftmost interval, which corresponds to all \( n \in \mathcal{S} \), as we increase the value of \( T_0 \) and move from left to right, we may introduce new partitions as \( T_0 \) crosses \( \tau_n \), \( \tau_n \), or an intersection point.

The following example illustrates how we can enumerate all the possible partitions efficiently. To keep the example simple, we assume that there are only two retailers, and \( r_1 \) and \( r_2 \) intersects only once at \( T_{12} \) as shown in Figure 4. \( \tau_1, \tau_2, \tau_1, \tau_2, \) and \( T_{12} \) divide the \( T_0 \) axis into six intervals. We start at \( T_0 = 0 \) and continuously increase the value of \( T_0 \) until it passes \( T_{12} \), and we discuss the possible partition changes during this process.

- \((0, \tau_1)\): the only possible partition is \((\emptyset, \emptyset, \emptyset) = \{(1, 2), \emptyset, \emptyset\}\).
- \((\tau_1, \tau_1)\): Retailer 1 is moved from \( \emptyset \) to \( \emptyset \), and we get a new partition \((\emptyset, \emptyset, \emptyset) = \{(2), (1), \emptyset\}\).
- \((\tau_1, \tau_1)\): A new partition, \((\emptyset, \emptyset, \{1\})\), is added for the case \( s_0 > r_1(T_0) \), in which \( 1 \in \mathcal{S} \) and the rest of retailers remains unchanged. Now we have two possible partitions, \((\emptyset, \emptyset, \{1\}) \) and \((\emptyset, \{1\}, \emptyset)\).
- \((\tau_2, \tau_2)\): Retailer 2 is removed from \( \emptyset \) to \( \emptyset \) for all the partitions in the previous interval. Thus, we have two possible partitions, \((\emptyset, \{1\}, \emptyset)\) and \((\emptyset, \{1, 2\}, \emptyset)\).
- \((\tau_2, T_{12})\): A new partition, \((\emptyset, \emptyset, \{1\})\), is added for the case \( s_0 > r_2(T_0) \) in which \( 2 \in \mathcal{S} \) and the rest of retailers remains unchanged. Now we have three possible partitions, \((\emptyset, \emptyset, \{1, 2\})\), \((\emptyset, \{1\}, \emptyset)\), and \((\emptyset, \{1, 2\}, \emptyset)\).

- \((T_{12}, \infty)\): A new partition \((\emptyset, \{1\}, \{2\})\) is added and an old partition \((\emptyset, \{2\}, \{1\})\) is removed. We have three possible partitions, \((\emptyset, \emptyset, \{1, 2\})\), \((\emptyset, \{1\}, \{2\})\), and \((\emptyset, \{1, 2\}, \emptyset)\).

In the above example, notice that for each of the intervals, we have at most three possible partitions because there are at most two \( r_j(T_0) \) functions separating the partitions. We list the partitions corresponding to large \( s_0 \) \((s_0 > r_j(T_0))\) first, then the partitions corresponding to small \( s_0 \) \((s_0 < r_j(T_0))\). Note that no sorting is needed because when \( T_0 \) crosses \( \tau_n \), the newly introduced partition always have large \( s_0 \).

In general, the partition enumeration can be carried out in \( O(N^2 \log N) \), and the complexity is determined by sorting the \( \tau_n \)'s, \( \tau_n \)'s, and the \( (O(N^2)) \) intersection positions.

### 3.2. Solution with Known \( \mathcal{S}, \mathcal{E}, \) and \( \mathcal{L} \)

In Roundy (1985), once the partition is determined, the optimal order cycle for each facility is characterized uniquely by its first-order condition. Here, the first-order condition is linked together by the warehouse safety stock level. We should note that at most two feasible solutions satisfy the first-order condition and illustrate how to find both, and thus the optimal solution, efficiently via a binary search or a golden search on the warehouse safety stock level.

Recall that the average cost of relaxed policy is \( c(\bar{T}) = K_0/T_0 + h_0 s_0 + \sum_{n \in \mathcal{S}} (c_n(T_n, T_0) + h_n s_n) \).

For given \( \mathcal{S}, \mathcal{E}, \) and \( \mathcal{L} \), we regroup the terms as

\[
K_0/T_0 + h_0 s_0 + \sum_{n \in \mathcal{S}} (c_n(T_n, T_0) + h_n s_n) + \sum_{n \in \mathcal{E}} (c_n(T_n, T_0) + h_n s_n)
\]

where

- \( \sum_{n \in \mathcal{S}} (c_n(T_n, T_0) + h_n s_n) = \sum_{n \in \mathcal{S}} (K_n/T_0 + (\lambda_n/2) \cdot h_n s_n + h_n s_n) = \sum_{n \in \mathcal{S}} (K_n/T_0 + (\lambda_n/2) \cdot h_n s_n + h_n s_n) \)
- \( \sum_{n \in \mathcal{E}} (c_n(T_n, T_0) + h_n s_n) = \sum_{n \in \mathcal{E}} (K_n/T_0 + (\lambda_n/2) \cdot h_n s_n + h_n s_n) \)
- \( \sum_{n \in \mathcal{E}} (c_n(T_n, T_0) + h_n s_n) = \sum_{n \in \mathcal{E}} (K_n/T_0 + (\lambda_n/2) \cdot h_n s_n + h_n s_n) \)
- \( n \in \mathcal{S} \) can solve the optimal \( T_0 \) by minimizing \( K_n/T_0 + (\lambda_n/2) h_n s_n + h_n s_n \), as the first-order condition characterizes the optimal solution. The solution is exactly \( \tau_n \), which we defined previously. However, we cannot use the same approach to solve \( T_n \) with \( n \in \mathcal{L} \) because \( s_0 \) is also a function of these \( T_n \)s. The first-order condition with respect to \( T_n \) is

\[
-\frac{K_n}{T_n^2} + \frac{\lambda_n}{2} h_n s_n + h_n s_n + \frac{1}{2\sqrt{T_n + T_0}} - h_0 \frac{\theta_n^2 \sigma_n^2}{2s_0} = 0.
\]

The first-order condition of \( T_0 \) gives us

\[
\sum_{n \in \mathcal{S}} \frac{\lambda_n}{T_n^2} h_n s_n + \sum_{n \in \mathcal{S}} \left( -\frac{K_n}{T_n^2} + \frac{\lambda_n}{2} h_n s_n + h_n s_n + \frac{1}{2\sqrt{T_n + T_0}} \right) - \frac{\theta_n^2 \sigma_n^2}{2s_0} = 0.
\]
We show that Equations (1) and (2) have at most two solutions. By (1), $T_n$ can be written as a function of the warehouse stock level $s_0$ by Corollary EC.2 in the online appendix. Furthermore, by Lemma EC.4 in the online appendix, $T_0(s_0)$ is a strictly convex function of $s_0$. By (2), $T_0$ can be written as a function of $s_0$ by Corollary EC.3, and $T_0(s_0)$ is a strictly concave function of $s_0$ by Lemma EC.5 in the online appendix. Thus, if we know the optimal $s_0$, we can determine the optimal $T_0$ and $T_n$.

Notice that if we know the values of $T_0$ and $T_n$, we can calculate $s_0$ by

$$\theta_0 \sqrt{L_0 \sum \sigma_n^2 + \left( \sum_{n \in \mathcal{L}} \sigma_n^2 \right)T_0 - \sum \sigma_n^2 T_n}.$$ 

That is, the optimal solution $(s_0, T_0, T_n)$ must also satisfy the equation that defines $s_0$. Define $\Psi(s_0) = \theta_0^2 (L_0 \sum \sigma_n^2 + (\sum_{n \in \mathcal{L}} \sigma_n^2)T_0(s_0) - \sum_{n \in \mathcal{L}} \sigma_n^2 T_n(s_0))$. $\Psi(s_0)$ is a strictly concave function of $s_0$. At optimality, we must have $\Psi(s_0) = s_0^2$ or $\sqrt{\Psi(s_0)} = s_0$, according to the definition of $s_0$. Because $(\sqrt{\Psi(s_0)} - s_0)^2 = (1/4)(2\Psi'(s_0) - (\Psi(s_0))^2)/\Psi^{3/2} < 0$ for $\Psi''(s_0) < 0$, $\sqrt{\Psi(s_0)} = s_0$ is also a strictly concave function, and $\sqrt{\Psi(s_0)} - s_0 = 0$ has at most two solutions. Via a binary search or a golden search, we can find these potential optimal $s_0$ values and determine the corresponding $T_0$ and $T_n$ in $(\mathcal{N})$. By comparing the two solutions, we can select the solution with lower total cost. Thus, given $\mathcal{S}, \mathcal{E}$, and $\mathcal{L}$, we can solve the minimization problem efficiently. Because there are only $O(N^3)$ possible partitions of $\mathcal{S}, \mathcal{E}$, and $\mathcal{L}$, we can solve the relaxation problem in $O(N^3)$.


In this section, we will use the optimal solution of the relaxation problem to build a close-to-optimal POT policy. Based on $T^*\_n^\_j$, the optimal solution of the relaxation problem, what we will do next is to construct an ordering policy satisfying $T_n = 2^{n\_i} T_L$ (for $n \in \mathcal{Z}$) with good performance.

Recall that for the arbitrage periodic ordering policy $\overline{T} = (T_1, T_2, \ldots, T_n)$, $K_0/T_0 + h_0 s_0 + \sum_{n \in \mathcal{N}} (K_n/T_n + (\lambda_n/2)(h_0 T_n + h_0(T_0 \vee T_n))) + h_0^2 s_0$ is a lower bound of the policy cost, where $s_0 = \theta_0 \sqrt{L_0 \sum \sigma_n^2 + \sum_{n \in \mathcal{L}} (0 \vee (T_0 - T_n)) \sigma_n^2}$. Therefore, the relaxation formulation provides a lower bound for the optimal periodic ordering policy cost.

We investigate two scenarios: (i) the base interval length $T_L$ is given and fixed, and find a power-of-two policy satisfying $T_n = 2^{n\_i} T_L$ (for $n \in \mathcal{Z}$); (ii) we have the flexibility to choose the base interval length $T_L$ and the corresponding POT policy.

4.1. A Close-to-Optimal Power-of-Two Ordering Policy with Given $T_L$

For arbitrary target service levels, we can construct an ordering policy satisfying $T_n = 2^{n\_i} T_L$ (for $n \in \mathcal{Z}$), which is no more than $\Phi \equiv (\sqrt{5} + 1)/2 \approx 1.618$ times the cost of the relaxation problem by the following procedure:

(i) Let $T_n^*$ denote the optimal solution of the relaxation problem. Define $q_n = \lfloor \log_2(T_n^*/T_L) \rfloor$, that is, $q_n$ is the integer part of $\log_2(T_n^*/T_L)$, and define $r_n = \log_2(T_n^*/T_L) - q_n$.

(ii) For $n \in \mathcal{S} \cup \mathcal{E}$, if $r_n < \log_2(\Phi)$, set $T_n = 2^{n\_i} T_L$, and set $T_n = 2^{n\_i + 1} T_L$ otherwise.

(iii) For $n \in \mathcal{S}$, if $r_n \leq 0.5$, set $T_n = 2^{n\_i} T_L$, and set $T_n = 2^{n\_i + 1} T_L$ otherwise.

(iv) For $n \in \mathcal{L}$, we consider two cases: $q_n < q_0$ and $q_n = q_0$.

- $q_n < q_0$: set $T_n = 2^{n\_i} T_L$ if $r_n < 0.5$, and set $T_n = 2^{n\_i + 1} T_L$ otherwise.
- $q_n = q_0$: if $r_0 < \log_2(\Phi)$, set $T_n = 2^{n\_i} T_L = T_0$; if $r_0 \geq \log_2(\Phi)$, set $T_n = 2^{n\_i} T_L = T_0$ if $r_n \leq 1 - \log_2(\Phi)$, and set $T_n = 2^{n\_i + 1} T_L$ otherwise.

Following the above procedure, if $n \in \mathcal{S}$, $T_n \geq T_0$; if $n \in \mathcal{E}$, $T_n \geq T_0$; and if $n \in \mathcal{L}$, $T_n \geq T_0$. Now we show that each term in the objective function is positive, and the cost ratio between the constructed ordering policy and the optimal solution for the relaxation problem is no more than $\Phi$.

The objective function has the following terms: $K_n/T_n$ for all $n$, $(\lambda_n/2) h_0 T_n$ for $n \in \mathcal{L}$, $(\lambda_n/2) h_0 T_n$ for $n \in \mathcal{S} \cup \mathcal{E}$, $h_0 \theta_0 \sqrt{L_0 \sum \sigma_n^2 + \sum_{n \in \mathcal{L}} (T_0 - T_n) \sigma_n^2}$.

To prove that each term in the objective function is positive, it suffices to show $h_n > 0$ for $n \in \mathcal{L}$. Now suppose this is not true, and $h_n \leq 0$; then both $g_n(T_n)$ and $h_n s_n(T_n)$ are decreasing functions on $(0, T_0)$; thus, $n \in \mathcal{E} \cup \mathcal{S}$. This is a contradiction. Therefore, $h_n > 0$ for $n \in \mathcal{L}$.

Next, we show that the cost ratio between the constructed ordering policy and the optimal solution for the relaxation problem is no more than $\Phi$. The proof of this result relies on the following two lemmas.

Lemma 3. $\max\{T_n^*/T_n^\_j, T_n^\_n/T_n^\_j\} < \Phi$ for all $n$. 

Proof of Lemma 3. We discuss the following three cases:

- For $n \in \mathcal{S} \cup \mathcal{E}$, $T_n^*/T_n^\_j = 2^{n\_j}$ if $r_n < \log_2(\Phi)$, and $T_n^*/T_n^\_j = 2^{n\_j - 1}$ if $r_n \geq \log_2(\Phi)$, because $0 \leq r_n < 1$, $1/2 \leq T_n^*/T_n^\_j < \Phi$, $\max\{T_n^*/T_n^\_j, T_n^\_n/T_n^\_j\} < \Phi$ for $n \in \mathcal{S} \cup \mathcal{E}$.

- For $n \in \mathcal{S}$, $T_n^*/T_n^\_j = 2^{n\_j}$ if $r_n < 0.5$, and $T_n^*/T_n^\_j = 2^{n\_j - 1}$ if $r_n \geq 0.5$, $\max\{T_n^*/T_n^\_j, T_n^\_n/T_n^\_j\} \leq \sqrt{2}/\Phi$ for $n \in \mathcal{S}$.

- For $n \in \mathcal{L}$, when $q_n < q_0$, same as the case $n \in \mathcal{S}$, $\max\{T_n^*/T_n^\_j, T_n^\_n/T_n^\_j\} < \Phi$; when $q_n = q_0$, if $r_0 < \log_2(\Phi)$, we have $T_n^*/T_n^\_j = 2^{n\_j}$, because $n \in \mathcal{L}$ and $T_n^\_j < T_n^* \leq 1$ and $T_n^*/T_n^\_j < \Phi$; if $r_0 \geq \log_2(\Phi)$, we have $T_n^*/T_n^\_j = 2^{n\_j}$, if $r_0 < 1 - \log_2(\Phi)$, and $T_n^*/T_n^\_j = 2^{n\_j - 1}$ if $r_n > 1 - \log_2(\Phi)$, and $1/\Phi < T_n^*/T_n^\_j \leq 2/\Phi$. Thus, $\max\{T_n^*/T_n^\_j, T_n^\_n/T_n^\_j\} \leq \sqrt{2}/\Phi$ for $n \in \mathcal{L}$.

Therefore, $\max\{T_n^*/T_n^\_j, T_n^\_n/T_n^\_j\} < \Phi$ for all $n$. 

Lemma 4. $\max_{n \in \mathcal{L}} \{T_n - T_0\}/(T_n^* - T_0^*) \leq \Phi^2$. 

Proof of Lemma 4. Notice that $T_n \leq T_0$ for $n \in \mathcal{L}$. If $T_n = T_0$, $(T_0 - T_n)/(T_n^* - T_0^*) < \Phi^2$. Now we focus on the
case with \( T_n < T_0 \). Because \( T_0/T_n \) is an integer power of two, \( T_n \leq T_0/2 \). Notice that \( T_0/T_n < 2/\Phi \). According to \( r_0 \), we have the following two scenarios:

- \( r_0 < \log_2(\Phi) \) in this case, \( T_0 = T_0^*/2^n < T_0^* \), while \( T_0^* < \Phi T_n \), \( (T_0 - T_n)/(T_0^* - T_n) < (T_0 - T_n)/(T_0 - \Phi T_n) \).

Because \( T_n < T_0/2 \), \( (T_0 - T_n)/(T_0 - \Phi T_n) \leq (2 - 1)/(2 - \Phi) = \Phi^2 \). Thus, \( (T_0 - T_n)/(T_0^* - T_n) \leq \Phi^2 \).

- \( r_0 \geq \log_2(\Phi) \) in this case, \( T_0 = 2^{n-1} T_n \), we further divide this case into two subcases:

  - If \( T_0^* < T_n \), \( q_0 = q_0^* \), and \( r_n \leq 1 - \log_2(\Phi) \) because \( T_n = 2^{n} T_n \). Thus, \( T_n^* \) is no more than \( 2T_0/\Phi \). Because \( T_0 = T_0^*/2^n \) and \( T_n = T_n^*/2^n \), \( T_0 - T_n \) and \( (T_0^* - T_n^*) \) are less than \( (T_0 - T_n)/(\Phi/2) - 2T_0/\Phi \) and \( (2 - 1)/(2 - \Phi) = \Phi^2 \). Thus, \( (T_0 - T_n)/(T_0^* - T_n^*) \leq \Phi^2 \).

  - If \( T_0^* = T_0 \) or \( T_0^*/T_n > 1/4 \), then \( T_n^* < T_0/4 \), \( (T_0 - T_n)/(T_0^* - T_n^*) < (T_0 - T_n)/(\Phi/2) - T_0 \Phi T_n \).

Because \( T_n < T_0/4 \), \( (T_0 - T_n)/(\Phi/2) - T_0 \Phi T_n \) is no more than \( (4 - 1)/(2 - \Phi) = \Phi^2 \). Thus, \( (T_0 - T_n)/(T_0^* - T_n^*) \leq \Phi^2 \).

Therefore, \( \max_{n \in \mathcal{F}} ((T_0 - T_n)/(T_0^* - T_n^*)) \leq \Phi^2 \) for all \( n \).

\( \square \)

**Theorem 2.** The cost ratio between the constructed ordering policy and the optimal solution for the relaxation problem is no more than \( \Phi \).

**Proof of Theorem 2.** Lemma 3 says that \( T_0^*/T_n < \Phi \) for all \( n \). Thus, the cost ratios between \( K_n/T_n \) and \( K_n^*/T_n^* \) are less than \( \Phi \). Similarly, the ratios between \( (\lambda_n/2)h_i T_n^* \) and \( (\lambda_n/2)h_i T_n^* \) are less than \( \Phi \). Furthermore, the ratios between \( h_i^*\theta_n^* \sigma_n^* \sqrt{L_n + T_n^*} \) and \( h_i^* \theta_n^* \sigma_n^* \sqrt{L_n + T_n^*} \) are less than \( \sqrt{\Phi} \) for all \( n \) in \( \mathcal{G} \cup \mathcal{E} \).

So far, we have compared all the terms except the warehouse stock cost. According to Lemma 4, it is easy to show that the ratios between

\[
h_0 \theta_n \sqrt{L_n \sum_{n \in \mathcal{F}} \sigma_n^2 + \sum_{n \in \mathcal{E}} (T_0 - T_n) \sigma_n^2}
\]

and

\[
h_0 \theta_n \sqrt{L_n \sum_{n \in \mathcal{F}} \sigma_n^2 + \sum_{n \in \mathcal{E}} (T_0^* - T_n^*) \sigma_n^2}
\]

are no more than \( \sqrt{\Phi^2} = \Phi \).

We have shown that each term in the objective function of the POT policy is no more than \( \Phi \approx 1.618 \) times the corresponding term in the optimal solution for the relaxation problem. Because the relaxation problem provides a lower bound of the optimal periodic ordering policy cost, we claim that the cost of the resulting POT policy is no more than 1.618 times the optimal periodic ordering policy cost. \( \square \)

### 4.2. A Close-to-Optimal Power-of-Two Ordering Policy with Unrestricted \( T_L \)

For arbitrary target service levels, if the base period \( T_L \) is not fixed in advance, we can construct a POT policy whose cost is no more than \( \sqrt{2} \approx 1.260 \) times the cost of the relaxation problem by the following procedure:

(i) Let \( T_n^* \) denote the optimal solution of the relaxation problem. Define \( T_n^* = T_n^*/\sqrt{2} \), \( q_n = \lfloor \log_2(T_n^*/T_L) \rfloor \), that is, \( q_n \) is the integer part of \( \log_2(T_n^*/T_L) \), and define \( r_n = \log_2(T_n^*/T_L - q_n) \).

(ii) For \( n \in \{ 0 \} \cup \mathcal{E} \), set \( T_0 = T_L \).

(iii) For \( n \in \mathcal{E} \cup \mathcal{F} \),

\( \circ K_n/T_n^* \geq (\lambda_n/2)h_i^* T_n^* \) if \( r_n \leq 1/3 \), set \( T_n = 2^{1/3} T_L \) and set \( T_n^* = 2^{2/3} T_L \).

\( \circ K_n/T_n^* < (\lambda_n/2)h_i^* T_n^* \) if \( r_n \leq 2/3 \), set \( T_n = 2^{1/3} T_L \) and set \( T_n^* = 2^{2/3} T_L \).

Following the above procedure, if \( n \in \mathcal{E} \), \( T_n > T_0 \); if \( n \in \mathcal{E} \), \( T_n = T_0 \); and if \( n \in \mathcal{F} \), \( T_n < T_0 \). By regrouping the terms, the objective function has the following six combined terms:

1. \( K_0/T_0^* \), \( K_n/T_n^* \) for \( n \in \mathcal{E} \);
2. \( (\lambda_n/2)h_i T_0^* \) for \( n \in \mathcal{E} \);
3. \( (\lambda_n/2)h_i T_0^* \) for \( n \in \mathcal{E} \);
4. \( (\lambda_n/2)h_i T_0^* \) for \( n \in \mathcal{E} \);
5. \( h_i^* \theta_n^* \sigma_n^* \sqrt{L_n + T_n^*} \approx \Phi^2 \) for all \( n \) in \( \mathcal{G} \cup \mathcal{E} \).

Now for each combined term in the objective function, we show that the cost ratio between the constructed ordering policy and the optimal solution for the relaxation problem is no more than \( \sqrt{2} \).

**Term 1.** \( K_0/T_0^* \) and \( K_n/T_n^* \) for \( n \in \mathcal{E} \):

\( T_0 = T_0^*/\sqrt{2} \) and \( T_n = T_n^*/\sqrt{2} \) for \( n \in \mathcal{E} \). The cost ratios are \( \sqrt{2} \).

**Term 2.** \( (\lambda_n/2)h_i T_0^* \) for \( n \in \mathcal{E} \) and \( (\lambda_n/2)h_i T_0^* \) for \( n \in \mathcal{E} \):

\( T_0 = T_0^*/\sqrt{2} \) and \( T_n = T_n^*/\sqrt{2} \) for \( n \in \mathcal{E} \). The cost ratios are \( 1/\sqrt{2} < \sqrt{2} \).

**Term 3.** \( (\lambda_n/2)h_i T_0^* \) for \( n \in \mathcal{E} \):

We have two scenarios:

(i) \( (\lambda_n/2)h_i T_0^* > \Phi \): if \( r_n \leq 1/3, \lambda_n^* < T_n^*/T_0 \leq \sqrt{2/3} \), then the cost ratio is smaller than \( \sqrt{2/3} < \sqrt{2} \).

(ii) \( (\lambda_n/2)h_i T_0^* < \Phi \): if \( r_n > 2/3, \lambda_n^* < T_n^*/T_0 \leq \sqrt{2/3} < \sqrt{2} \).

**Term 4.** \( (\lambda_n/2)h_i T_0^* \) for \( n \in \mathcal{E} \):

We have two scenarios:

(i) \( (\lambda_n/2)h_i T_0^* > \Phi \): if \( r_n \leq 1/3, \lambda_n^* < T_n^*/T_0 \leq \sqrt{2/3} \), then the cost ratio is smaller than \( \sqrt{2/3} < \sqrt{2} \).

(ii) \( (\lambda_n/2)h_i T_0^* < \Phi \): if \( r_n > 2/3, \lambda_n^* < T_n^*/T_0 \leq \sqrt{2/3} < \sqrt{2} \).
if } r_n > 2/3, 1 \leq T_n/T_n^* \leq \sqrt{2}, \text{ then the cost ratio is smaller than } \sqrt{2}.

\text{Lemma 5. } h_1^* \theta_0 \sigma_n \sqrt{T_n + T_n^*},

T_n/T_n^* \leq 2^{2/3}, \ (T_n + T_n^*)/(T_n^* + T_n^*) \leq 2^{1/3}, \text{ and the cost ratios are no more than } \sqrt{2^{2/3}} = \sqrt{2}.

\text{Term 6. } h_0 \theta_0 \sqrt{T_n \sum \sigma_n^2 + \sum_{n \in J}(T_n - T_n^*) \sigma_n^2}.

We first show } (T_0 - T_n)/(T_0 - T_n^*) \leq 2^{1/3}. \text{ It suffices to prove the case where } T_n > T_0. \text{ Notice that } T_0/T_n \text{ is an integer power of two; thus, } T_n \leq T_0/2. \text{ Also note that } T_n^* = \sqrt{2} T_0, \ T_n^* \leq 2^{1/3} T_3; \text{ therefore, } (T_0 - T_n)/(T_0 - T_n^*) \leq (2 - 1)/(2 \sqrt{2} - 2^{1/3}) < 2^{2/3}. \text{ So, the ratio of the warehouse safety stock cost is less than } \sqrt{2^{2/3}} = \sqrt{2}.

We have shown that each combined term in the objective function of the POT policy is no more than } \sqrt{2} \approx 1.26 \text{ times the corresponding term in the optimal solution for the relaxation problem. Because the relaxation problem provides a lower bound of the optimal periodic ordering policy cost, the cost of the resulting POT policy guarantees to be no more than 1.260 times the optimal periodic ordering policy cost.

5. Conclusions

In this paper, we propose a power-of-two (POT) policy to manage inventory in one-warehouse multiretailer models with stochastic demand. For any given target service levels, we develop a polynomial time algorithm that runs in time } O(N^3) \text{ to find a POT policy whose cost is guaranteed to be no more than } \sqrt{2} \approx 1.26 \text{ times the optimal periodic ordering policy cost.}

In the future, we plan to investigate the possibility of generalizing the results in this paper to more complex stochastic inventory systems. Interesting directions include the balanced ordering policies, the ordering policies with additional restrictions/constraints, and the extensions to multi-item inventory systems.

6. Electronic Companion

An electronic companion to this paper is available as part of the online version that can be found at http://or.journal.informs.org/.

Appendix. Proof of Lemma 5

\text{Lemma 5. } \text{If } C_{-n} > C_{-n}^*, \text{ the optimal } T_n^* \text{ that minimizes } g_n(T_n) + h_0 \theta_0 \sqrt{C_{-n} + (T_0 - T_n) \sigma_n^2} \text{ on } (0, T_0) \text{ is no more than } \tilde{T}_n.

\text{Proof of Lemma 5. } \text{Recall that } \tilde{T}_n \text{ is the rightmost tangent point on } (0, T_0). \text{ Therefore, } \tilde{T}_n \text{ minimizes } g_n(T_n) - \tilde{g}_n(T_n) \text{ on } (0, T_0). \text{ By definition, } \tilde{g}_n(T_n) = -h_0 S_0(T_n) + h_0 S_0(T_0) + g_n(T_0), \text{ and } \tilde{T}_n \text{ minimizes } (g_n(T_n) + h_0 \theta_0 \sqrt{C_{-n} + (T_0 - T_n) \sigma_n^2}) \text{ on } (0, T_0). \text{ For } \tilde{T}_n < T_n < T_0, \text{ we have}

\text{That is, for } \tilde{T}_n < T_n < T_0,

\[ g_n(T_n) - g_n(\tilde{T}_n) > h_0 \theta_0 \sqrt{C_{-n} + (T_0 - \tilde{T}_n) \sigma_n^2} - h_0 \theta_0 \sqrt{C_{-n} + (T_0 - T_n) \sigma_n^2}. \]

For } \tilde{T}_n \leq T_n < T_0, \text{ we also have}

\[ \sqrt{C_{-n} + (T_0 - \tilde{T}_n) \sigma_n^2} - \sqrt{C_{-n} + (T_0 - T_n) \sigma_n^2} > \sqrt{C_{-n} + (T_0 - \tilde{T}_n) \sigma_n^2} - \sqrt{C_{-n} + (T_0 - T_n) \sigma_n^2}, \]

because } C_{-n} > C_{-n}^*. \text{ Thus, for } \tilde{T}_n < T_n < T_0,

\text{or}

\[ g_n(\tilde{T}_n) + h_0 \theta_0 \sqrt{C_{-n} + (T_0 - \tilde{T}_n) \sigma_n^2} < g_n(T_n) + h_0 \theta_0 \sqrt{C_{-n} + (T_0 - T_n) \sigma_n^2}. \]

\text{That is, the optimal solution } T_n^* \text{ is no more than } \tilde{T}_n. \square

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References


Chu and Shen: Power-of-Two Ordering Policy with Stochastic Demand


