Heyting algebras generalize the well-known idea of Boolean algebras. In the 19th century, Luitzen Brouwer founded the mathematical philosophy of intuitionism, which believed that a statement could only be demonstrated by direct proof. Arend Heyting, a student of Brouwer’s, formalized this thinking into his namesake algebras. Our aim is to define Heyting algebras using only category-theoretic notions.

2. Definitions

A partially-ordered set (poset) is a set equipped with a reflexive, anti-symmetric, and transitive relation (≤). Posets formalize the notion of ordering or sequencing elements of a set.

Example 1. \((\mathbb{R}, \leq)\) is a partially-ordered set.

Some natural examples of posets have additional structure, namely meets and joins. A meet of two elements \(x\) and \(y\) of a poset is the unique element \(x \wedge y\) such that \(x \wedge y \leq x\), \(x \wedge y \leq y\), and for all \(z\) such that \(z \leq x\) and \(z \leq y\), \(z \leq x \wedge y\). A join of two elements \(x\) and \(y\) of a poset is the unique element \(x \vee y\) such that \(x \leq x \vee y\), \(y \leq x \vee y\), and for all \(z\) such that \(x \leq z\) and \(y \leq z\), \(x \vee y \leq z\).

If the meet and join is defined for all pairs of elements in a poset, we say that that poset has meets and has joins, respectively. A lattice is a poset that has joins and meets.

Example 2. \((\mathbb{N}, |)\) forms a lattice, with greatest-common-divisor and least-common-multiple as the meet and join operations, respectively.

Note that in the example, 1 divides into all elements, but there is no natural number that has all other natural numbers as divisor. We wish to exclude ourselves to lattices with both minimal elements (like 1) and maximal elements. A bounded lattice is a lattice with maximal and minimal elements.

Maximal elements are necessarily unique. Suppose we had two maximal elements \(M\) and \(M'\). Then \(M \leq M \vee M'\), but since \(M\) is maximal, \(M \vee M' \leq M\), so \(M = M \vee M'\), and by an identical argument, \(M \vee M' = M'\). Similarly, minimal elements are also unique.

Example 3. The power set of any finite set forms a bounded lattice with the \(\subseteq\) relation (union and intersection as join and meet, respectively).

Now we are finally ready to define Heyting algebras. A Heyting algebra is nothing more than a bounded lattice with another binary operation, yet it gives rise to many properties. Formally, a Heyting algebra is a bounded lattice (whose maximum and minimum elements are denoted 1 and 0, respectively) with a binary operation \(\Rightarrow\) such that \(a \wedge b \leq c\) if and only if \(a \leq (b \Rightarrow c)\)

In a Boolean algebra, we can set \(a \Rightarrow b := \neg a \lor b\) and it satisfies the above property. Hence, all Boolean algebras are Heyting algebras. The following are true in Heyting (and Boolean) algebras:

1. \(a \leq b\) iff \(a \Rightarrow b = 1\);
2. \(a = (1 \Rightarrow a)\);
3. \(a \Rightarrow (b \wedge c) = (a \Rightarrow b) \wedge (a \Rightarrow c)\);
4. \(b \wedge (b \Rightarrow c) \leq c\);
5. \(b \Rightarrow c = \lor\{a \mid a \wedge b \leq c\}\);

In fact, (4) and (5) form an equivalent definition of the \(\Rightarrow\) operation. The missing operation from a Heyting algebra is the complement operation. We can’t define this for all Heyting algebras, but the best we can do is define the
The regular elements of a Heyting algebra

Theorem 4. Elements in which double-negation and law of excluded middle hold. Just like the simple condition for MacLane’s coherence theorem and associativity, we have something similar for Heyting algebras. However, there are some elements in which double-negation and law of excluded middle hold. Theorem 4. The regular elements of a Heyting algebra \( H \), i.e. elements \( \{x \mid x = \neg\neg x\} \), constitute a Boolean algebra.

Most importantly, the original intent of Heyting was to show that this kind of algebra satisfied the axioms of intuitionistic calculus. In intuitionistic calculus, propositions follow from other propositions by modus ponens. Therefore, it is natural to use a partial ordering. If we let “implies” be the poset relation, the axioms become the following.

Theorem 5. Heyting algebras satisfy the following relations:

1. \( a \leq (b \Rightarrow a) \);
2. \( (a \Rightarrow (b \Rightarrow c)) \leq ((a \Rightarrow b) \Rightarrow (a \Rightarrow c)) \);
3. \( a \leq (b \Rightarrow (a \land b)) \);
4. \( (a \land b) \leq a, b \);
5. \( a, b \leq (a \lor b) \);
6. \( (a \Rightarrow c) \leq ((b \Rightarrow c) \Rightarrow ((a \lor b) \Rightarrow c)) \);
7. \( (a \Rightarrow b) \leq ((a \Rightarrow \neg b) \Rightarrow \neg a) \);
8. \( \neg a \leq (a \Rightarrow b) \);

3. Restating in category-theoretic terms

We already saw the category-theoretic definition of a poset in class. The following definitions build on viewing a poset as a category. A lattice is simply a poset category that has binary products and coproducts.

The corresponding notion of a binary product in a lattice category is the join. Recall that the join \( a \lor b \) has the property that any other element greater than both \( a \) and \( b \) is greater than \( a \lor b \). Since morphisms are unique if they exist, this is equivalent to the universal property of products. Similarly, the meet \( a \land b \) is a coproduct.

Furthermore, a bounded lattice is a lattice category with terminal and initial objects. The maximum element is greater than all other elements, but since morphisms are unique, this is equivalent to saying that it is a terminal object. Similarly, the minimum element is an initial object. Since equalizers and coequalizers are trivial in poset categories, bounded lattices are complete and cocomplete. A Heyting algebra \( H \) is a bounded lattice, in which for every element \( b \in H \), the functor \( - \land b : H \rightarrow H \) has right adjoint \( b \Rightarrow - : H \rightarrow H \).

To see that this agrees with our original definition of a Heyting algebra, we look at the hom-set definition of an adjunction. For all elements \( a, b, \) and \( c \), \( \text{Hom}(a \land b, c) \cong \text{Hom}(a, b \Rightarrow c) \), and since the size of a hom-set \( \text{Hom}(a, b) \) is 1 iff \( a \leq b \) and 0 otherwise in a poset, this is equivalent to saying that \( a \land b \leq c \) if and only if \( a \leq (b \Rightarrow c) \), which is our original definition. This means that any category satisfying this category-theoretic definition is a Heyting algebra. Since the morphisms in a poset are unique, any naturality square is necessarily commutative, so all Heyting algebras satisfy this definition.

4. Category of Heyting algebras

In order to turn the Heyting algebras into a category, we need to define morphisms. A morphism of Heyting algebras \( f : H_1 \rightarrow H_2 \) is one that preserves limits, colimits, and implication \( (f(0) = 0, f(a \lor b) = f(a) \lor f(b), \) and so on). The identity map \( f : H \rightarrow H, x \mapsto x \) is a morphism from a Heyting algebra to itself, and the composite of two such maps is also a morphism, so the Heyting algebras form a category.

5. References