1. Behrens - Fisher Problem. Let \(X_1, \ldots, X_n\) be iid \(N(\mu_x, \sigma^2_x)\), and \(Y_1, \ldots, Y_n\) be iid \(N(\mu_y, \sigma^2_y)\).

(a) What is the Generalized Likelihood Ratio (GLR) test statistic for

\[ H_0 : \mu_x = \mu_y \]
\[ H_1 : \mu_x \neq \mu_y. \]

(b) Can the GLR statistic be used for this hypotheses test? Explain.

(c) Derive (directly) the asymptotic distribution of the GLR test statistic. Can the GLR test statistic be used asymptotically?

2. We say \(X = (X_1, \ldots, X_b)\) has a multinomial distribution \(\mathcal{M}(n, \theta)\), \(\theta = (\theta_1, \ldots, \theta_b)\) when

\[
P(X_1 = n_1, \ldots, X_b = n_b) = \begin{cases} \binom{n}{m_1, \ldots, m_b} \Pi_{i=1}^b \theta_i^{n_i} & \text{if } \sum_{i=1}^b n_i = n, \\ 0 & \text{otherwise.} \end{cases} \tag{1}
\]

In some models, the probabilities \(\theta\) are themselves functions of a (possibly vector) parameter. For instance, in a \(b = 4\) four cell linkage model of Fisher,

\[ \theta_1 = \frac{2 + p}{4}, \quad \theta_2 = \theta_3 = \frac{1 - p}{4}, \quad \text{and} \quad \theta_4 = \frac{p}{4}. \]

As another example, for testing independence of discrete variables \(U\) and \(V\) in a \(a \times c\) (contingency) table, if \(\theta_{ij}\) is the probability of cell \(i, j\), then

\[ \theta_{ij} = p_i q_j \]

where \(p_i\) and \(q_j\) are the probability of (row) \(U = i\), and (column) \(V = j\), respectively. For this reason, we consider the framework where \(p \in \Theta \subset \mathbb{R}^k\), and we test

\[ H_0 : \theta = \theta(p) \]
\[ H_1 : \theta \neq \theta(p). \]
a) Show the asymptotic equivalence of the GLR procedure and Pearson’s Chi-squared test, which rejects $H_0$ when

$$
\chi^2 = \sum_{i=1}^b \frac{(X_i - n\hat{\theta}_i(\hat{p}))^2}{n\hat{\theta}_i(\hat{p})}
$$

is large, where $\hat{p}$ is the mle of $p$.

b) What is the asymptotic ($n \to \infty$) distribution of $\chi^2$?

c) Give details for using Person’s Chi-squared statistic for testing for independence between two factors in a $a \times c$ table.

3. For $i = 1, \ldots, k$ let $X_{i1}, \ldots, X_{ik}$ be independent Poisson random variables with distribution $P(\theta_i)$, respectively. Find the GLR test, and its asymptotic distribution, for testing $H_0: \theta_1 = \cdots = \theta_k$, against the alternative that there exists some unequal $\theta_i$’s.

4. Show that if $Z \sim \mathcal{N}_n(\delta, I)$ and $P$ a symmetric projection matrix of rank $r$, then

$$
Z'PZ \sim \chi^2_r(\delta'P\delta).
$$