Math 541a Assignment #2

1. a) Let $X$ be a non-negative random variable. Prove that the values of $t$ for which $M(t) = Ee^{tX}$ exists, that is, the set $I$ given by

$$I = \{ t : M(t) < \infty \}$$

is an interval, that is, a connected subset of $\mathbb{R}^1$, whose endpoints may or may not be included.

b) Prove that if $M(t)$ exists for any $t > 0$ then $EX^k < \infty$ for all $k = 1, 2, \ldots$.

c) Let $X$ be a random variable taking on both positive and negative values. Show that if the moment generating function of $X$ exists in an interval around zero, then $X$ has moments of all orders.

2. Let $X \sim G(p)$, the geometric distribution with parameter $p \in (0, 1)$,

$$P(X = k) = (1 - p)^{k-1}p \quad k = 1, 2, \ldots$$

a) Find the moment generating function of $X$, and, from that, its mean and variance.

b) Find the mean, variance, moment generating function of the sum $X = X_1 + \ldots + X_r$ of i.i.d. Geometric random variables with parameter $p \in (0, 1)$. Why would this distribution be called the ‘negative binomial’?

3. Show that if $X$ has a $\Gamma(\alpha, \beta)$ distribution then $X$ satisfies the Stein identity

$$E[Xf'(X)] = E[(X/\beta - \alpha)f(X)] \quad \text{for all } f \text{ for which these expectations exist},$$

and use the identity to find an expression for the moments, $EX^k, k = 1, 2, \ldots$ and $\text{Var}(X)$. 

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4. a) Prove that the all moments of the lognormal distribution exist, but that it does not posses a moment generating function at any \( t > 0 \).

b) Show that the lognormal distribution is not determined by its moment sequence. \( \text{Hint:} \) Let \( p_0(y) \) be the density of a standard lognormal distribution and consider

\[
p_a(y) = [1 + a \sin(2\pi \log y)]p_0(y) \quad \text{for } |a| \leq 1.
\]

5. Let \( U_1 < \cdots < U_n \) be the order statistics from an iid sample of \( n \) variables with the \( U[0, 1] \) distribution. Derive the density function of the range \( U_n - U_1 \) from the joint density of \( (U_1, U_n) \) by considering the transformation

\[
(u_1, u_n) \rightarrow (u_1, u_n - u_1).
\]

6. For vectors \( \mathbf{x}, \mathbf{y} \) in \( \mathbb{R}^n \) we say \( \mathbf{y} \geq \mathbf{x} \) if \( y_i \geq x_i, \ i = 1, \ldots, n \). For an event \( A \), let \( P_p(A) \) be the probability that \( \mathbf{X} = (X_1, \ldots, X_n) \in A \) where \( \mathbf{p} = (p_1, \ldots, p_n) \) and \( X_i \sim B(p_i) \) are independent Bernoulli variables. Say that an event \( A \subset \{0, 1\}^n \) is increasing if \( \mathbf{x} \in A \) implies that \( \mathbf{y} \in A \) for all \( \mathbf{y} \geq \mathbf{x} \).

a. For \( A \) increasing and \( \mathbf{p} \geq \mathbf{r} \), prove that

\[
P_p(A) \geq P_r(A).
\]

b. Let \( X_1, \ldots, X_n \) be independent Bernoulli random variables with respective success probabilities \( p_1, \ldots, p_n \), and let \( Y_1, \ldots, Y_n \) be independent Bernoulli random variables with respective success probabilities \( r_1, \ldots, r_n \). Show that if \( p_i \geq r_i, \ i = 1, \ldots, n \), then

\[
\sum_{i=1}^n X_i \geq_{st} \sum_{i=1}^n Y_i
\]

c. If \( X \sim B(n, p) \) and \( Y \sim B(n, r) \) with \( p \geq r \), prove that

\[
P(X \geq k) \geq P(Y \geq k) \quad \text{for all } k.
\]
7. Use Jensen’s inequality to prove the arithmetic/geometric mean inequality: If $a_1, \ldots, a_n$ are non-negative numbers, then

$$\left( \prod_{j=1}^{n} a_j \right)^{1/n} \leq \frac{1}{n} \sum_{j=1}^{n} a_j.$$ 

8. For $X$ a random variable with $E|X|^s < \infty$, $s > 0$ let

$$||X||_s = (E|X|^s)^{1/s}.$$

Hölder’s inequality states that for ‘conjugate exponents’ $1 < p, q < \infty$ satisfying

$$\frac{1}{p} + \frac{1}{q} = 1,$$

and $E|X|^p, E|Y|^q$ finite, that we always have

$$E|XY| \leq ||X||_p ||Y||_q.$$ 

First, prove the inequality by following the steps in parts a and b.

a. Prove that for $a, b > 0$

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Hint: Set $r = p - 1$ and consider the relationships between the areas under the curves $y = u^r$ and its inverse within a rectangle of length $a$ and height $b$ to derive

$$ab \leq \int_0^a u^r + \int_0^b u^{1/r}.$$ 

b. Apply the inequality in part a with $a = X/||X||_p$ and $b = Y/||Y||_q$.

c. Determine when the inequality is an equality.

d. Derive the Cauchy Schwarz inequality as a special case.
e. For $1 \leq s < t < \infty$ with $\|X\|_s < \infty$, prove Lyanpunov’s inequality: $\|X\|_s \leq \|X\|_t$.

9. We say $U$ and $V$ are positively correlated when $\text{Cov}(U, V) \geq 0$. Let $g$ and $f$ be any increasing functions, and $X$ any random variable such that $Ef^2(X)$ and $Eg^2(X)$ exist. Prove that

$$U = f(X) \quad \text{and} \quad V = g(X)$$

are positively correlated.