Method of Moments. Let $X, X_1, \ldots, X_n$ be independent from distribution $p(x, \theta), \theta \in \Theta \subset \mathbb{R}^d$. Then by the weak law of large numbers, when $E_\theta |X|^k < \infty$, for $r = 1, \ldots, k$, as $n \to \infty$,

$$\hat{m}_r = \frac{1}{n} \sum_{i=1}^n X_i^r \to_p m_r(\theta) \quad \text{where} \quad m_r(\theta) = E_\theta X^r.$$

If we can express the quantity that we want to estimate, $q(\theta) \in \mathbb{R}^p$, as a function $g : \mathbb{R}^k \to \mathbb{R}^p$ of the first $k$ moments

$$q(\theta) = g(m(\theta)) \quad \text{where} \quad m(\theta) = (m_1(\theta), \ldots, m_k(\theta))^T,$$

then it is natural to consider the estimator

$$\hat{q}(\theta) = g(\hat{m}) \quad \text{where} \quad \hat{m} = (\hat{m}_1, \ldots, \hat{m}_k)^T.$$

If $g$ is continuous at $m(\theta) = (m_1(\theta), \ldots, m_k(\theta))$ at the true $\theta$ then the estimate is consistent; that is, since

$$\hat{m} \to_p m(\theta), \quad \text{we have} \quad \hat{q}(\theta) = g(\hat{m}) \to_p g(m(\theta)) = q(\theta).$$

To derive the asymptotic distribution, we apply the delta method. Start with the multivariate central limit theorem for i.i.d. variables, which gives, suppressing the dependence of $\Sigma$ on $\theta$,

$$\sqrt{n}(\hat{m} - m(\theta)) \to_d N_k(0, \Sigma)$$

where

$$\{\Sigma\}_{i,j} = m_{i+j}(\theta) - m_i(\theta)m_j(\theta)$$

and so

$$\sqrt{n}(g(\hat{m}) - g(m(\theta))) \to_d N_p(0, g'\Sigma g'^T).$$

We note that $g(m) \in \mathbb{R}^p$ and $g'$ denotes the $\mathbb{R}^{p \times k}$ matrix evaluated at $m(\theta)$.

Taking the $\Gamma(\alpha, \beta)$ for example, we have $EX^k = \beta^k k(\alpha)$ and so in particular $EX = \alpha\beta, \text{Var}(X) = \alpha\beta^2$, and for estimating the parameters of the distribution $(\alpha, \beta)$ themselves we use

$$(\alpha, \beta)^T = g(m_1, m_2) = \left(\frac{m_1^2}{m_2 - m_1^2}, \frac{m_2 - m_1^2}{m_1}\right)^T.$$
Hence we have $k = 2$ and

$$
\Sigma = \begin{bmatrix}
m_2 - m_1^2 & m_3 - m_2 m_1 \\
m_3 - m_2 m_1 & m_4 - m_2^2
\end{bmatrix} = \begin{bmatrix}
\alpha \beta^2 & 2\alpha(\alpha + 1)\beta^3 \\
2\alpha(\alpha + 1)\beta^3 & 2\alpha(\alpha + 1)(2\alpha + 3)\beta^4
\end{bmatrix}.
$$

Then

$$
g'(m_1, m_2) = \begin{bmatrix}
\frac{2m_1 m_2}{(m_2 - m_1)^2} - \frac{m_2^2}{m_1^2} \\
-\frac{m_2^2 - m_2}{m_1^2} \frac{1}{m_1}
\end{bmatrix}
$$

so substituting,

$$
g'\left(\alpha \beta, \alpha(\alpha + 1)\beta^2\right) = \begin{bmatrix}
2(\alpha + 1)\beta^{-1} & -\beta^{-2} \\
-(2\alpha + 1)\alpha^{-1} & (\alpha \beta)^{-1}
\end{bmatrix}
$$

gives the asymptotic covariance matrix of the limiting multivariate normal as

$$
g'\Sigma g'^T = \begin{bmatrix}
2\alpha(\alpha + 1) & -2(\alpha + 1)\beta \\
-2(\alpha + 1)\beta & (2\alpha + 3)\alpha^{-1}\beta^2
\end{bmatrix}
$$