Math 541a Assignment #4

1. Let $X_1, \ldots, X_n$ be i.i.d. $U(0, \theta)$.
   a. Prove that $T_n(X) = ((n + 1)/n)X_{(n)}$ is UMVU for $\theta$.
   
   b. Calculate $\text{Var}_\theta(T_n)$ and compare its rate of decay to zero in $n$ with what is typically expected in the Cramer Rao bound for estimates based on $n$ i.i.d. observations.

2. Assuming differentiation under the integral is allowed, prove
   
   $$(I_X(\theta))_{ij} = -E_\theta \left( \frac{\partial^2 \log p(x, \theta)}{\partial \theta_i \partial \theta_j} \right).$$
   
3. Let $X_1, \ldots, X_n$ be i.i.d. $\Gamma(\alpha, \beta)$.
   a. If $\alpha$ is known, prove that $\hat{\beta} = \bar{X}_n/\alpha$ is UMVU for $\beta$ and achieves the Cramer Rao bound.
   
   b. Find the information matrix $I$ and its inverse when both parameters are unknown, and show that the presence of an unknown $\alpha$ makes the lower bound for $\beta$ increase.

4. a) Let $u_1$ and $u_2$ be non-zero vectors in $\mathbb{R}^n$, and define the projection of $u_1$ along $u_2$ to be
   
   $$P_{u_2}u_1 = \frac{u_2}{||u_2||}(u_1 \cdot \frac{u_2}{||u_2||}).$$
   
   Show that $u_2 \perp u_1 - P_{u_2}u_1$ and interpret this fact geometrically with a diagram.
   
   b. Show that
   
   $$||u_1 - P_{u_2}u_1||^2 = ||u_1||^2 - \frac{(u_1 \cdot u_2)^2}{||u_2||^2}.$$
c. Consider a smooth density \( p(x, \theta), \theta \in \Theta \subset \mathbb{R}^2 \). By inverting the information matrix \( I = (I_{ij})_{1 \leq i,j \leq 2} \), show that the Cramer Rao bound lower bound for estimating \( \theta_1 \) in the presence of \( \theta_2 \) is the reciprocal of

\[
I_{11} - \frac{I_{12}^2}{I_{22}}.
\]

d. For random variables \( V, W \) define a dot product

\[
V \cdot W = EVW \quad \text{so that} \quad ||V||^2 = EV^2,
\]

and orthogonality and projections can be defined as usual. Let \( U_1 = U(X, \theta_1), U_2 = U(X, \theta_2) \) be score functions associated with \( \theta_1, \theta_2 \) respectively. Show that the lower bound for estimating \( \theta_1 \) in the presence of \( \theta_2 \) is the reciprocal of

\[
||U_1 - P_{U_2} U_1||^2.
\]

5. Let \( (X_i, Y_i), i = 1, \ldots, n \) be i.i.d. from a bivariate normal distribution. When the covariance \( \rho \) is estimated by the sample correlation

\[
\hat{\rho} = \frac{\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{\left(\sum_{i=1}^{n}(X_i - \bar{X})^2\sum_{i=1}^{n}(Y_i - \bar{Y})^2\right)^{1/2}}
\]

it is known that

\[
\sqrt{n}(\hat{\rho} - \rho) \rightarrow_d \mathcal{N}(0, (1 - \rho^2)^2).
\]

Find a variance stabilizing transformation for \( \hat{\rho} \).