1. (125 points, 25 points each part) Consider the following linear state/observation equations in $\mathbb{R}$, where $a, c, h$ and $g$ are nonzero constants, $c, g$ strictly positive, and $v_t, w_t$ are independent standard Brownian motions, independent of $x_0$:

$$
\begin{align*}
\, dx_t &= a x_t dt + c dv_t \\
\, dy_t &= h x_t dt + g dw_t.
\end{align*}
$$

Suppose that $x_0$ is normal, with $E x_0 = 0$ and $\text{Var}(x_0) = p_0$.

a) Calculate the conditional covariance at time $t$, $p(t)$, by solving the associated Ricatti equation.

b) Write out the filter equation for $d\hat{x}_t$.

c) What happens under the change of scale (i) $a, c$ replaced by $\alpha a, \alpha c$ for $\alpha \neq 0$? Explain. What happens under the change of scale $h, g$ replaced by $\alpha h, \alpha g$ for $\alpha \neq 0$? Explain.

d) Determine the behavior of the system as $t \to \infty$.

e) Determine the behavior of the system as $t \to \infty$, when $a = 0$.

2. (105 points, 35 points each part) Consider the scalar linear/quadratic
regulator problem on the interval \([0, T]\),

\[
\begin{align*}
\mathrm{d}x_t &= (ax_t + bu_t)\mathrm{d}t + cdv_t \\
\mathrm{d}y_t &= hx_t\mathrm{d}t + gdw_t \\
J(u) &= E\left(\int_0^T (qx_t^2 + ru_t^2)\mathrm{d}t + f x_T^2\right)
\end{align*}
\]

\(a, b, c, h, g, q, r, f\) nonzero constants, \(c, g, q, r, f\) strictly positive, and \(v_t, w_t\) are independent standard Brownian motions, independent of \(x_0\), a normal variable, with \(Ex_0 = 0\) and \(\text{Var}(x_0) = p_0\).

a) Find the optimal (feedback) control for this system.

b) Suppose now that the cost function becomes

\[
J(u) = E\left(\int_0^T (qx_t^2 + ru_t^2)\mathrm{d}t + f_1 x_T^2 + f_2 x_T^2\right),
\]

for \(f_1, f_2\) fixed positive constants. Write out the necessary equations to specify the optimal (feedback) control (you do not need to solve).

c) The (standard) linear/quadratic regulator is dual to the (standard) Kalman filter. What filtering problem is the dual to the regulator problem with cost function as in b)?)

3. (120 points, 20 points each part) a) Let \(U, X\) be vectors space with inner product \(<\cdot, \cdot>_U, <\cdot, \cdot>_X\), and norms \(||u||_U^2 = <u, u>_U, ||x||_X^2 = <x, x>_X\) respectively. (We will henceforth drop the subscripts, as the space in which we are working will be clear from context.) Let \(L\) be a linear map from \(U\) to \(X\), and \(b \in X\). Now consider the following optimization problem:

Minimize \(J(u) = ||Lu + b||^2 + ||u||^2\) over \(u \in U\).

Find the \(u^0 \in U\) corresponding to the minimum. Hint: Expand out norms as inner products, combine quadratic terms, and “complete the square.”
b) Consider the problem of finding the optimal open loop control for the (cannonical) deterministic linear system with quadratic cost:

\[ \dot{x} = A(t)x_t + B(t)u_t, \quad x_0 \in \mathbb{R}^n \text{ fixed, } t \in [0, T] \text{ and } \]

\[ J(u) = \int_0^T (x_s'Qx_s + u_s'Ru_s)ds + x_T'Fx_T. \]

As usual, we take the state to lie in \( \mathbb{R}^n \) and the control in \( \mathbb{R}^m \). As functions in \( t \), we will consider fixed (predetermined) controls as functions \( u \in \mathcal{U} = L^2(\mathbb{R}^m) \), the set of all measurable \( \mathbb{R}^m \) valued functions with \( \int_0^T u_t'u_t dt < \infty \). For such a fixed (predetermined) function \( u \in \mathcal{U} \), use the fundamental matrix to write the solution \( x \), as an element of \( \mathcal{X} = L^2(\mathbb{R}^n) \), as

\[ x = Lu + b, \]

for some linear operator \( L : \mathcal{U} \to \mathcal{X} \) and \( b \in \mathcal{X} \).

c) Define inner products, inducing norms, on \( \mathcal{X} \) and \( \mathcal{U} \) such that

\[ J(u) = ||x||^2 + ||u||^2. \]

d) Find a formula for the optimal open loop control \( u^0 \) by using the framework of part a). You should explain how certain operators on the \( L^2 \) spaces which appear (involving adjoints and inverses) should be calculated in principle (e.g, by a certain integration, or the finding the solution to the certain differential equation ), but you need not be more specific.

e) Now consider finding the optimal open loop control for the stochastic linear regulator

\[ dx_t = (A(t)x_t + B(t)u_t)dt + C(t)dv_t \]

\[ J(u) = E \left( \int_0^T (x_s'Qx_s + u_s'Ru_s)ds + x_T'Fx_T \right). \]

where \( v_t \) is a standard Brownian motion. Use the fundamental matrix to write the solution in the form

\[ x = Lu + b + v \]
where \( v \) is a random element of \( \mathcal{X} \). Write out \( J \) in a form in which the techniques above can be applied, and find an expression for the optimal open loop control \( u^0 \) for the stochastic system.

f) Are the open loop controls the same for both problems?

g) Extra Credit: Prove directly (not using uniqueness!) that when \( u^0 \) is unique, the \( u^0 \) found in e) is the same as the optimal open loop control as given in, by Davis. In particular, that

\[
u^0(t) = R^{-1}B\Lambda(t)x^*_t
\]

where

\[
dx^*_t = Ax^*_t dt + Bu^0_t dt,
\]

and \( \Lambda \) satisfies the Ricatti equation

\[
\dot{\Lambda} + A'\Lambda + \Lambda A + Q - \Lambda'BR^{-1}B'\Lambda = 0
\]

\[
\Lambda(T) = F.
\]