The Regularity of Solutions of the Primitive Equations of the
Ocean in Space Dimension Three

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1. Introduction.

In this Note, we prove the global existence and uniqueness of solutions of the primitive equations of the ocean in a bounded domain with Dirichlet boundary conditions. These equations are the fundamental physical model in geophysical dynamics and oceanography [P,WP]. The mathematical formulation of the primitive equations was initiated by J.L. Lions, Temam, and Wang in [LTW1,LTW2,LTW3]. They proved the global existence of weak solutions and they studied asymptotic and numerical properties of the equations. The local existence of strong solutions for initial data in \(H^3\) was proven in [TZ,GMR]. Recently, Cao and Titi proved in [CT] the global existence of strong solutions for the primitive equations in the case of Neumann boundary conditions on the bottom and the top (see also [K]).

In this paper, we settle the case of (physical) Dirichlet boundary conditions. We provide the sketch of the proof for the case of Dirichlet boundary conditions on the bottom and the sides, and the Neumann boundary condition on the top for flat topography. For further details, for other physical boundary conditions, and for the non-flat boundary topography, cf. [KZ].

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2. The regularity of the primitive equations. We address the existence and the uniqueness of strong solutions for the primitive equations of the ocean

\[
\frac{\partial u_k}{\partial t} - \nu \Delta u_k + \sum_{j=1}^{3} \partial_j (u_j u_k) + \partial_k p = f_k, \quad k = 1, 2
\]

(PE\(_k\))

with the divergence free condition \(\sum_{k=1}^{2} \partial_k u_k = 0\). Denote \(v = (u_1, u_2)\) and \(u = (u_1, u_2, u_3)\).

The equations are derived from the 3D Navier-Stokes system under the hydrostatic approximation assumption. The differences with the 3D NSE are the lack of an evolution equation for \(u_3\) and the fact that \(p\) is independent of \(x_3\). The initial condition is \(v(\cdot, 0) = v_0\), where \(v_0 = (u_{01}, u_{02}): \Omega \to \mathbb{R}^2\) satisfies \(\text{div}_2 \int_{-h}^0 v_0 \, dx_3 = 0\). The equations are set in a bounded domain \(\Omega = \Omega_2 \times (-h, 0)\), where \(h\) is a positive constant and \(\Omega_2 \subseteq \mathbb{R}^2\) is a smooth bounded domain.

The boundary conditions are the following. On the top we have \(\partial v/\partial x_3 = 0\) and \(u_3 = 0\), for \((x_1, x_2, x_3) \in \Gamma_t = \Omega_2 \times \{0\}\). While on the bottom, we assume \(v = 0\) and \(u_3 = 0\) for \((x_1, x_2, x_3) \in \Gamma_b = \Omega_2 \times \{-h\}\). On the side, we have \(v = 0\) for \((x_1, x_2, x_3) \in \Gamma_s = \partial \Omega_2 \times [-h, 0]\). The full primitive system also contains the equations for the temperature and the salinity, and those can be added without any additional difficulties. Certain modifications are required in the case of the varying bottom, as well as in the case of physical boundary conditions on the top \(\partial v/\partial x_3 + \alpha v = 0\) and \(u_3 = 0\) for \((x_1, x_2, x_3) \in \Gamma_t = \Omega_2 \times \{0\}\). These modifications are given in [KZ]. Let

\[
H = \{ v \in (L^2(\Omega))^2 : \text{div}_2 \int_{-h}^0 v \, dx_3 = 0 \text{ on } \Omega_2, \left( \int_{-h}^0 v \, dx_3 \right) \cdot n = 0 \text{ on } \Gamma_s \}
\]

and \(V = \{ v \in H \cap H^1 : v = 0 \text{ on } \Gamma_3 \cup \Gamma_s \}\). The norms on \(H\) and \(V\) are denoted by \(\| \cdot \|_H = \| \cdot \|_{L^2}\) and \(\| \cdot \|_V\) respectively. We denote by \(A\) the Stokes-type operator associated with the primitive equations; that is \(Av = -P \Delta v\), where \(P\) is the \(L^2\)-orthogonal projection onto \(H\). Let \(f \in L^2_{\text{loc}}([0, \infty), L^2(\Omega)^2)\).

Then for all \(v_0 \in H\), there exists a weak solution \(v \in L^\infty(0, \infty, H) \cap L^2_{\text{loc}}(0, \infty, V)\) [TZ]. Also, the solution satisfies the energy inequality

\[
\frac{1}{2} \int_\Omega |v|^2 \, dx + \nu \sum_{j=1}^{3} \sum_{k=1}^{2} \int_0^{t_1} \int_\Omega \partial_j v_k \partial_j v_k \leq \frac{1}{2} \int_\Omega |v|^2 \, dx_0 + \int_0^{t_1} (v, f)_{L^2}\]

for almost every \(t_0 \geq 0\) (\(t_0 = 0\) included) and every \(t_1 \geq t_0\). By [TZ,GMR], for every \(v_0 \in V\), there exists a maximal \(T_{\text{max}} > 0\) such that there exists a strong solution \(v \in L^\infty_0([0, T_{\text{max}}), V) \cap L^2_{\text{loc}}([0, T_{\text{max}}), D(A))\) of the primitive equations. Also, if \(T_{\text{max}} < \infty\), then \(\lim_{t \to T_{\text{max}}} \| v(\cdot, t) \|_V = \infty\).

**Theorem 0.1** Assume that \(f \in L^2_{\text{loc}}([0, \infty), L^2(\Omega))\) and \(v_0 \in V\). Then there exists a unique strong solution \(v \in L^\infty_0([0, \infty), V) \cap L^2_{\text{loc}}([0, \infty), D(A))\) of the primitive equations with the initial datum \(u_0\).

**Proof (sketch).** Without loss of generality, \(\nu = 1\). Assume contrary to the assertion that \(T_{\text{max}} \in (0, \infty)\), and let \(T \in (0, T_{\text{max}}]\). Denote

\[
E(t) = \left( \sum_{k=1}^{2} \| \nabla u_k(\cdot, t) \|_{L^2}^2 \right)^{1/2}.
\]

Choose \(\delta > 0\) (which depends on \(T_{\text{max}}\)) such that \(\| E \|_{L^2(t, t+2 \delta)}^2 \leq 1/\gamma\), where \(\gamma\) is a sufficiently large constant. Next, we find \(t_j \in (j \delta, (j+1) \delta)\), where \(j = 1, \ldots, l\), such that \(\| \nabla v(\cdot, t_j) \|_{L^2}^2 \leq \delta^{-1} \int_{j \delta}^{(j+1) \delta} \| \nabla v(\cdot, \tau) \|_{L^2}^2 \, d\tau \leq 1/\delta \gamma\)

where \(l\) is the largest integer such that \((l+1) \delta \leq T\). Also, set \(t_0 = 0\) and \(t_{l+1} = T\). The proof consists on estimating
\[ J(t) = \left( \sum_{k=1}^{2} \| u_k(t, \tau) \|_{L^6}^6 \right)^{1/6}, \quad K(t) = \left( \sum_{k=1}^{2} \| \partial_3 u_k(t, \tau) \|_{L^2}^2 \right)^{1/2} \]

and

\[ \overline{J}(t) = \left( \sum_{k=1}^{2} \int_{\Omega} |\nabla (u_k(t, \tau)^3)|^2 \right)^{1/6}, \quad \overline{K}(t) = \left( \sum_{k=1}^{2} \| \nabla \partial_3 u_k(t, \tau) \|_{L^2}^2 \right)^{1/2} \]

on \((t_j, t_{j+1})\), where \(j \in \{0, \ldots, l\}\) is arbitrary. In order to get an estimate for \(J\), we multiply (PE\(_k\)) with \(u_k^2\), where \(k = 1, 2\), integrate over \(\Omega\), and add. The pressure term \(- \sum_{k=1}^{2} \int u_k^2 \partial_k p = -h \sum_{k=1}^{2} \int M(u_k^2) \partial_k p\) is bounded from above by \(C J^2 \overline{J} \| \nabla 2 p \|_{L^{3/2}}\). (Note that we used independence of \(p\) on \(x_3\).) Here, \(M\) is the vertical averaging operator \(Mw(x_1, x_2) = h^{-1} \int_{-h}^{0} w(x_1, x_2, x_3) \, dx_3\). We get \((d/dt) J^0 \leq C \| \nabla 2 p \|_{L^{3/2}}^2 J^4 + CF^2 J^4\) and thus

\[
\frac{d}{dt} J^4 \leq C \| \nabla 2 p \|_{L^{3/2}}^2 J^4 + CF^2 J^2
\]

where \(F(t) = \left( \sum_{k=1}^{2} \| f_k(t, \tau) \|_{L^2}^2 \right)^{1/2}\). In order to obtain an estimate for \(K\) and \(\overline{K}\), we multiply (PE\(_k\)) with \(-\partial_3 u_k\), where \(k = 1, 2\), integrate over \(\Omega\), and add. We get

\[
\frac{1}{2} \frac{d}{dt} K^2 + \overline{K}^2 = \sum_{j,k=1}^{2} \int_{\Omega} \left( \partial_3 u_j u_j \partial_3 u_k + \int_{\Omega} \partial_3 u_j u_k \partial_3 u_k - 2 \int_{\Omega} u_j \partial_3 u_k \partial_3 u_k \right)
+ \sum_{k=1}^{2} \left( \int_{\Omega} \partial_3 p \partial_3 u_k - \int_{\Omega} f_k \partial_3 u_k \right).
\]

The first three integrals can be bounded from above by \(C J K^{1/2} \overline{K}^{3/2}\). In order to estimate the pressure term, we write

\[
\sum_{k=1}^{2} \int_{\Omega} \partial_3 p(x_1, x_2) \int_{-h}^{0} \partial_3 u_k(x_1, x_2, x_3) \, dx_3 \, dx_1 \, dx_2 \leq \sum_{k=1}^{2} \| \partial_3 p \|_{L^{3/2}(\Omega_2)} \| \partial_3 u_k(\cdot, \cdot, -h) \|_{L^3(\Omega_2)}
\]

and use the trace theorem to estimate \(\| \partial_3 u_k(\cdot, \cdot, -h) \|_{L^3} \leq C(K + \overline{K}) \leq C \overline{K}\). Now, we need to bound \(\| \nabla 2 p \|_{L^{3/2}_t L^2_{x}}\). For this purpose, we average the primitive equations in the third direction and obtain

\[
\partial_t (Mu_1) - \Delta_2 Mu_1 + \partial_k p = M \partial_3 u_k - \sum_{j=1}^{2} M \partial_j (u_j u_k) + M f_k
\]

for \(k = 1, 2\), with \(\partial_1 Mu_1 + \partial_2 Mu_2 = 0\). The theorem of Sohr and von Wahl [SW] applied to the equation for \(Mu\) then leads to

\[
\| \nabla 2 p \|_{L^{3/2}_t L^2_{x}} \leq C \sum_{j=1}^{2} \| \partial_3 u_j(\cdot, \cdot, -h, \cdot) \|_{L^2_t L^{3/2}_x} + C \sum_{j,k=1}^{2} \| u_j \partial_3 u_k \|_{L^2_t L^{3/2}_x}
+ C \sum_{k=1}^{2} \| f_k \|_{L^2_t L^2_x} + C \sum_{k=1}^{2} \| \nabla u_k(\cdot, t_j) \|_{L^2}
\]

which is less than or equal to \(C \| K \|_{L^2_t}^{1/2} \| \overline{K} \|_{L^2_t}^{1/2} + C \| J E \|_{L^2_t} + C \| F \|_{L^2_t} + C \sum_{k=1}^{2} \| \nabla u_k(\cdot, t_j) \|_{L^2}\). Using the pressure estimate, we get

\[
J(t)^4 \leq J(t_j)^4 + C \| K \|_{L^2_t} \| \overline{K} \|_{L^2_t} \sup_{t_j \leq t \leq t_{j+1}} J(t)^2 + C \| E \|_{L^2_t}^2 \sup_{t_j \leq t \leq t_{j+1}} J(t)^4
+ C \left( \| F \|_{L^2_t}^2 + \sum_{k=1}^{2} \| \nabla u_k(\cdot, t_j) \|_{L^2}^2 \right) \sup_{t_j \leq t \leq t_{j+1}} J(t)^2
\]

(2)
for $t \in [t_j, t_{j+1})$, while (1) leads to
\[
K(t)^2 + \|\mathbf{K}\|^2_{L^2_t(t_j, t_{j+1})} \leq C\|K\|^2_{L^2_t} \sup_{t_j \leq t \leq t_{j+1}} J(t)^4 + C\|K\|^2_{L^2_t(t_j, t_{j+1})} \\
+ C\|F\|^2_{L^2_t} \sup_{t_j \leq t \leq t_{j+1}} J(t)^2 + C\left(\|F\|^2_{L^2_t} + \sum_{k=1}^2 \|\nabla u_k(\cdot, t_j)\|^2_{L^2}\right) \tag{3}
\]
for $t \in [t_j, t_{j+1})$. After a short computation, we conclude that (2) and (3) imply
\[
\sup_{t_j \leq t \leq t_{j+1}} J(t)^4 + \sup_{t_j \leq t \leq t_{j+1}} K(t)^2 + \|\mathbf{K}\|^2_{L^2_t(t_j, t_{j+1})} \\
\leq J(t_j)^4 + \frac{C}{\gamma} \|\mathbf{K}\|^2_{L^2_t} + \frac{C}{\gamma} \sup_{t_j \leq t \leq t_{j+1}} J(t)^4 + C\|F\|^2_{L^2_t} + C\sum_{k=1}^2 \|\nabla u_k(\cdot, t_j)\|^2_{L^2} \\
+ C\gamma \sup_{t_j \leq t \leq t_{j+1}} J(t)^2 + C\|F\|^2_{L^2_t} + C\sum_{k=1}^2 \|\nabla u_k(\cdot, t_j)\|^2_{L^2}
\]
where we used $K(t) \leq \mathbf{K}(t)$. If $\gamma$ is a large enough positive constant, the sum of the second and the third term on the right hand side can be absorbed in the half of the left hand side. Hence, \[\sup_{t_j \leq t \leq t_{j+1}} J(t)^4 + \sup_{t_j \leq t \leq t_{j+1}} K(t)^2 + \|\mathbf{K}\|^2_{L^2_t(t_j, t_{j+1})} \leq C\delta^{-2} + C + C\|F\|^4_{L^2_t}.\]
Then, by induction, we obtain uniform boundedness of $J(t)$, $K(t)$, and $\int_0^1 \mathbf{K}^2$ up to $T$. From here, it is then not difficult to show that $\|u(\cdot, t)\|_V$ remains bounded on $(0, T_{\text{max}})$, which contradicts $T_{\text{max}} < \infty$. For more details, cf. [KZ].

\begin{thebibliography}{99}

[CT] C. Cao and E.S. Titi, Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics, Annals of Math. (to appear).


\end{thebibliography}