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Citation: J. Math. Phys. 53, 103101 (2012); doi: 10.1063/1.4753991
View online: http://dx.doi.org/10.1063/1.4753991
View Table of Contents: http://jmp.aip.org/resource/1/JMAPAQ/v53/i10
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Local existence of solutions to the free boundary value problem for the primitive equations of the ocean

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(Received 23 January 2012; accepted 23 August 2012; published online 25 September 2012)

Lions, Temam, and Wang in [“Problème à frontière libre pour les modèles couplés de l’océan et de l’atmosphère,” Acad. Sci., Paris, C. R. 318(12), 1165–1171 (1994)] introduced a free surface model for the primitive equations of the ocean. In this paper, we establish the local well-posedness of the model with analytic initial data.

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I. INTRODUCTION

Geophysical dynamics presents an array of interesting and challenging mathematical questions, which have recently drawn the attention of the applied analysis community; see for instance Refs. 2, 3, 5, 8–12, 20, 21, 23, 24, and 26 and the references in Ref. 27.

The main model in geophysical fluid dynamics, meteorology, and climate prediction is governed by the primitive equations; cf. Pedlosky,25 Salmon,32 Richardson.29 These equations are derived from the full compressible Navier-Stokes/Euler equations under the Boussinesq approximation and the hydrostatic balance assumption. The global existence of weak solutions for the viscous primitive equations was established by Lions, Temam, and Wang20, 21, 23 for the case of the atmosphere, and by Temam and Ziane33 for the case of the ocean. The global existence of strong solutions for the viscous primitive equations was established by Cao and Titi in Ref. 4 (see also Ref. 13) in the case of Neumann boundary conditions. The case of the physical Dirichlet boundary conditions on the sides and the bottom boundaries does not follow from the methods of Refs. 4 and 13 due to difficulties arising from the pressure (see Ref. 17 for more details). The global existence for this situation was established in the work of two of the authors.17, 18 These results hold for the full primitive equations of the ocean including the Robin boundary condition on the top, the Dirichlet boundary condition on the sides and on the bottom, and the varying bottom topography. The case of the inviscid primitive equations was studied recently by Kukavica, Temam, Vicol, and Ziane14, 15 in a space of analytic data. They established the local existence of solutions in such spaces.

The general physical equations are posed as a free surface problem where the fluid inside the variable (unknown) domain is governed by the inviscid primitive equations. The model was first derived in the work of Lions, Temam, and Wang.22 The mathematical analysis for this model is in its infancy and the present paper is the first work presenting a rigorous well-posedness result for the free boundary problem (2.1)–(2.4), associated to the primitive equations in a moving domain. We note that there are fundamental differences between the inviscid primitive equations and the Euler equations. For instance, the Euler equations are known to be locally well-posed in Sobolev spaces, while the inviscid primitive equations (even in the case of a fixed domain) present a derivative loss and well-posedness is known only in the analytic case14, 15 and for a certain linearized problem.30, 31 For results on analytic and Gevrey regularity, see for instance Refs. 1, 6, 7, 16, and 19.

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A recent positive result on well-posedness was established by Renardy\textsuperscript{28} for the hydrostatic Euler equations in a thin strip bounded by two free surfaces provided the horizontal velocity is uniform in every cross section of the strip. Due to the thin domain scenario considered in Ref. 28, the pressure term is not present in the evolution equation for the horizontal velocity. Also, compared to Renardy, we include that the expression for the vertical velocity \( w \) depends nonlinearly on the other unknowns (cf. (2.5)).

There are two major obstacles to establishing the well-posedness in the space of real-analytic functions for the initial value boundary problem (2.1)–(2.4). We need to introduce new real-analytic norms, in order to distinguish between the derivatives taken in the horizontal and the vertical (\( x \)- and \( z \)-) directions of the moving domain \( D_t \). The second difficulty is the appearance of the \( \zeta \) variable in the domain of integration. The \textit{a priori} estimate is then closed using a combinatorial result, which we state and prove in Sec. III (cf. the Faà di Bruno type formula and Lemma 3.1).

The paper is organized as follows. In Sec. II, we introduce the basic notion and state our main result on local existence of a real-analytic solution. Sec. III contains the \textit{a priori} estimates for the position of the ocean surface and the horizontal velocity. In Sec. IV, we provide the formal construction of the solution.

\section*{II. NOTATION AND THE MAIN RESULT}

In this paper, we address the free boundary value problem for the ocean model considered in Ref. 22, which consists of the conservation equation for the ocean

\[ \partial_t v + v \partial_x v + w \partial_z v + \partial_z p = 0, \quad (2.1) \]

the diagnostic equations

\[ \partial_x v + \partial_z w = 0, \quad (2.2) \]
\[ \partial_z p + g \rho z = 0, \quad (2.3) \]

and the evolution equation for the ocean surface

\[ \partial_t \zeta + v \partial_x \zeta = w \big|_{z=\zeta}. \quad (2.4) \]

The unknown functions are the horizontal and the vertical velocity \( v(x, z, t) \) and \( w(x, z, t) \), the scalar pressure \( p(x, z, t) \), and the position of the ocean surface \( z = \zeta(x, t) \) with respect to the level \( z = 0 \), where \( \zeta \) is small in magnitude by comparison with the depth of the ocean \( h \). The constant density \( \rho \), the ocean surface pressure \( p(x, \zeta, t) \), and the gravitational constant \( g \) are given. We consider the system (2.1)–(2.4) in a two-dimensional domain \( D_t \) with real-analytic boundary defined by

\[ D_t = \{(x, z) \in \mathbb{R}^2 : x \in \mathbb{R}, -h < z < \zeta(x, t)\}, \]

where \( t \in (0, T) \) for some \( T > 0 \). Note that the two-dimensional domain is considered for the simplicity of presentation, and the three-dimensional case requires only minor modifications. We impose periodic boundary conditions in the \( x \)-direction with a period 1. Also, we assume \( w = 0 \) on the bottom \( \Gamma_b = \{(x, z) \in \mathbb{R}^2 : x \in \mathbb{R}, z = -h\} \) of the domain \( D_t \). Then, integrating (2.2) with respect to \( z \) between \(-h\) and \( \zeta \), we have that \( w \) is given by the formula

\[ w(x, \zeta, t) = -\partial_x \int_{-h}^{\zeta} v(x, z, t) dz + v \partial_x \zeta \quad (2.5) \]

on the ocean surface. Next, we integrate the Eq. (2.3) with respect to the vertical variable between \( \zeta \) and \( z \), which leads to

\[ p(x, z, t) = p(x, \zeta, t) + g \rho \zeta - g \rho z. \quad (2.6) \]
Using (2.5) and (2.6), we may rewrite the system (2.1)–(2.4) as
\[
\partial_t v + v \partial_x v + w \partial_z v + g \rho \partial_x \zeta = \eta, \tag{2.7}
\]
\[
\partial_t v + \partial_z w = 0, \tag{2.8}
\]
\[
\partial_t \zeta + \partial_z \int_{-h}^{z(x,t)} v(x, z, t) \, dz = 0, \tag{2.9}
\]
where \( \eta(x, t) = -\partial_x p(x, \zeta, t) \) is a known function. We consider a real-analytic initial datum
\[
v(x, z, 0) = v_0(x, z), \tag{2.10}
\]
\[
\zeta(x, 0) = \zeta_0(x). \tag{2.11}
\]
in the domain \( D_0 \) and the base interval \( I = (0, 1) \), respectively.

Let \( t > 0 \) be fixed. For \( r \geq 0, M > 0, \) and \( \tau(t) > 0 \), we define the spaces of real-analytic functions with a period 1 in the \( x \)-direction
\[
X_{\tau(t)} = \left\{ v \in C_\text{per}^\infty(D_t) : \|v\|_{X_{\tau(t)}} < \infty \right\}, \tag{2.12}
\]
where the norm \( \|v\|_{X_{\tau(t)}} \) is given by
\[
\|v\|_{X_{\tau(t)}} = \sum_{m=0}^{\infty} \sum_{k=0}^{m} M^k \|\partial_x^{m-k} \partial_z^k v\|_{L^2(D_t)} \frac{\tau(t)^m (m+1)^r}{(m-k)! k!}. \tag{2.13}
\]
and
\[
Y_{\tau(t)} = \left\{ v \in X_{\tau(t)} : \|v\|_{Y_{\tau(t)}} < \infty \right\}, \tag{2.14}
\]
with the semi-norm
\[
\|v\|_{Y_{\tau(t)}} = \sum_{m=1}^{\infty} \sum_{k=0}^{m} M^k \|\partial_x^{m-k} \partial_z^k v\|_{L^2(D_t)} \frac{\tau(t)^m (m+1)^r}{(m-k)! k!}. \tag{2.15}
\]
Similarly, we define the corresponding spaces \( X_{\tau(t)} \) and \( Y_{\tau(t)} \) of 1-periodic functions \( \zeta \in C_\text{per}^\infty(I) \) with the norms
\[
\|\zeta\|_{X_{\tau(t)}} = \sum_{m=0}^{\infty} \|\zeta^{(m)}\|_{L^2(I)} \frac{\tau(t)^m (m+1)^r}{m!}, \tag{2.16}
\]
and
\[
\|\zeta\|_{Y_{\tau(t)}} = \sum_{m=1}^{\infty} \|\zeta^{(m)}\|_{L^2(I)} \frac{\tau(t)^m (m+1)^r}{(m-1)!}. \tag{2.17}
\]
respectively. Note that, throughout the paper, we use the same notation for the real-analytic norms for two different cases, which will be clear from the context.

In the next theorem, we state our main result.

**Theorem 2.1:** Assume that \( (v_0, \zeta_0) \in Y_{\tau_0} \) and let \( r \geq 5 \). Then there exists a local in time real-analytic solution \( (v(t), \zeta(t)) \) to the initial-value free boundary problem (2.7)–(2.11), which satisfies
the estimate
\[
\| (v(t), \xi(t)) \|_{X_{\eta(t)}} + g \rho \int_0^T \| (v(t), \xi(t)) \|_{Y_{\eta(t)}} dt
\]
\[
+ 4C_0 \| (v_0, \xi_0) \|_{X_0} \int_0^T (1 + \tau(t)^{-2}) \| (v(t), \xi(t)) \|_{Y_{\eta(t)}} dt
\]
\[
+ 2C_0 \| (v_0, \xi_0) \|^2 \int_0^T (1 + \tau(t)^{-3/2}) \| (v(t), \xi(t)) \|_{Y_{\eta(t)}} dt \leq 2 \| (v_0, \xi_0) \|_{X_0},
\]
for all times \( t \in [0, T] \), where \( C_0 \) is a constant and \( T \) depends only on the initial data and \( \eta \). Moreover, the radius of analyticity \( \tau : [0, T] \to [\tau_0/2, \tau_0] \) can be determined explicitly by (3.53) below.

Above and in the sequel, the symbols \( C_0 \) and \( C \) denote sufficiently large generic positive constants, which may depend on the depth of the ocean \( h \); any other additional dependence is explicitly noted.

In Sec. III, we provide a priori estimates needed in the proof of the theorem, while Sec. IV contains the explicit construction of solutions.

III. A PRIORI ESTIMATES

The purpose of this section is to establish the a priori estimates for the position of the ocean surface \( \xi \) and the horizontal velocity \( v \), which are leading to (2.18).

By the definition of the analytic norms (2.16) and (2.17), we have
\[
\frac{d}{dt} \| \xi(t) \|_{X_{\eta(t)}} = \dot{\xi}(t) \| \xi(t) \|_{Y_{\eta(t)}} + \sum_{m=0}^{\infty} \frac{d}{dt} \| \xi^{(m)}(t) \|_{L^2(I)} \frac{\tau(t)^{m}(m+1)!}{m!}.
\]
(3.1)

In order to obtain an estimate for \( (d/dt) \| \xi^{(m)}(t) \|_{L^2(I)} \), we apply the operator \( \partial_x^m \) on each side of (2.9). We then multiply the resulting equation by \( \partial_x^m \xi \) and integrate over the interval \( I \). An application of the Cauchy-Schwarz inequality gives
\[
\frac{d}{dt} \| \xi^{(m)}(t) \|_{L^2(I)} \leq \| \partial_x^{m+1} \int_{-h}^z v(x, z, t) dz \|_{L^2(I)}.
\]
(3.2)

In order to treat the right-hand side of (3.2), we need the following version of the Faà di Bruno formula. Let
\[
f(x, z) = \int_{-h}^z v(x, z') dz',
\]
and \( z = \xi(x) \). Then
\[
\frac{d^{m+1}}{dx^{m+1}} (f(x, \xi(x))) = \sum_{j=0}^{m+1} \sum_{k \leq j} A_{j,k} \left( \begin{array}{c} m+1 \\ j \end{array} \right) \frac{j!}{k_1! \cdots k_j!} \partial_x^{m-j+1} \partial_x^j f(x, \xi(x)) \left( \frac{\xi'(x)}{1!} \right)^{k_1} \cdots \left( \frac{\xi^{(j)}(x)}{j!} \right)^{k_j},
\]
where \( A_{j,k} \) is the set of all \( j \)-tuples of nonnegative integers \( (k_1, \ldots, k_j) \) satisfying \( k_1 + \cdots + k_j = k \) and \( 1 \cdot k_1 + \cdots + j \cdot k_j = j \). Hence, from (3.1) and (3.2), we have
\[
\frac{d}{dt} \| \xi(t) \|_{X_{\eta(t)}} \leq \dot{\xi}(t) \| \xi(t) \|_{Y_{\eta(t)}} + \sum_{m=0}^{\infty} \sum_{j=0}^{m+1} \sum_{k \leq j} \sum_{A_{j,k}} \mathcal{U}_{m,k_1,\ldots,k_j}(v, \xi) \frac{\tau(t)^{m}(m+1)!}{m!},
\]
(3.3)
by using the notation
\[
\mathcal{U}_{m,k_1,\ldots,k_j}(v, \xi) = \left( \begin{array}{c} m+1 \\ j \end{array} \right) \frac{j!}{k_1! \cdots k_j!} \left\| \partial_x^{m-j+1} \partial_x^j f(\cdot, \xi(\cdot)) \left( \frac{\xi'(\cdot)}{1!} \right)^{k_1} \cdots \left( \frac{\xi^{(j)}(\cdot)}{j!} \right)^{k_j} \right\|_{L^2(I)}.
\]
(3.4)
where \( k = k_1 + \cdots + k_j \). In Lemma 3.1 below, we establish an estimate for the second term on the right side of (3.3), denoted by

\[
\mathcal{U}(v, \zeta) = \sum_{m=0}^{\infty} \sum_{j=0}^{m+1} \sum_{|\beta|=j} \sum_{\lambda} \mathcal{U}_{m,k_1,\ldots,k_j}(v, \zeta) \frac{\tau(t)^m}{m!},
\]

leading to an \textit{a priori} estimate for \( \zeta \).

Assume that \( \|\zeta_0\|_{X_{\gamma_0}} \leq M_1 \). There exists \( t_0 > 0 \) such that \( \|\zeta(t)\|_{X_{\gamma_0}} \leq 2M_1 \) and \( \tau_0/2 \leq \tau(t) \leq \tau_0 \) for all \( t \in (0, t_0) \). By (2.16), we have

\[
\|\zeta^{(j)}(t)\|_{L^\infty(t)} \leq \frac{2M_1 \tau^{-j} j!}{(j + 1)^{r+1/2}},
\]

and an application of the one-dimensional Agmon’s inequality

\[
\|\zeta\|_{L^\infty} \leq C\|\zeta\|_{L^2}^{1/2} \|\zeta\|_{L^2}^{1/2} + C\|\zeta\|_{L^2}
\]

gives

\[
\|\zeta^{(j)}(t)\|_{L^\infty(t)} \leq \frac{CM_1(1 + \tau^{-1/2}) \tau^{-j} j!}{(j + 1)^{r+1/2}}
\]

for all \( j \in \mathbb{N} \) and \( t \in (0, t_0) \). Note that

\[
\|\zeta^{(j)}(t)\|_{L^\infty(t)} \leq \frac{\|\zeta(t)\|_{X_{\gamma_0}} \tau^{j+1} j!}{(j + 1)^{r+1/2}}
\]

and

\[
\|\zeta^{(j)}(t)\|_{L^\infty(t)} \leq \frac{CM_1(1 + \tau^{-1/2}) \tau^{j+1} j!}{(j + 1)^{r+1/2}}
\]

for all \( j \in \mathbb{N} \) and \( t \in (0, t_0) \).

\textbf{Lemma 3.1:} Let \( t > 0 \) be fixed and let \( v, \zeta \in Y_t \) for some \( \tau = \tau(t), M \geq CM_1 \tau_0^{-1} (1 + \tau_0^{-1/2}) \), and \( r \geq 5 \). Then

\[
\mathcal{U}(v, \zeta) \leq C(1 + \tau^{-1/2}) \left( \|\zeta\|_{X_t} + \|\zeta\|_{Y_t} + \|v\|_{X_t} \right)
\]

for a sufficiently large positive constant \( C \) depending on \( h \).

In the proof, we need the following combinatorial lemma.

\textbf{Lemma 3.2:} Assume

\[
k = k_1 + k_2 + \cdots + k_j,
\]

\[
j = k_1 + 2k_2 + \cdots + jk_j,
\]

and let \( j_0 \in \{1, \ldots, j\} \) be the largest integer such that \( k_{j_0} \neq 0 \). If

\[
j_0 \leq \frac{m}{100}
\]

and

\[
m - j + k \leq \frac{m}{100},
\]

then the lower bound

\[
2^{(n_0-2)k_1} \cdots (j_0 + 1)^{(n_0-2)k_{j_0}} \geq \frac{1}{C} (m + 1)^{n_0+1},
\]

holds provided \( r_0 \geq 5 \).
Thus, Lemma 3.2 is established for \( m \).

Indeed, if the opposite is true, we obtain leading again to a contradiction if \( m \) for the sake of contradiction, suppose that the assertion (3.14) is not true; namely, assume that

\[
2^{(r_0-2)k_j} \cdot (j_0 + 1)^{(r_0-2)k_{j_0}} \leq \frac{1}{C (m + 1)^{n+1}},
\]

with a constant \( C \) to be determined below. If

\[
\sum_{j \in A_q} k_j \geq 4,
\]

then

\[
\prod_{j \in A_q} (1 + j)^{(r_0-2)k_j} \geq \prod_{j \in A_q} (1 + m^{1/2})^{(r_0-2)k_j} \geq (1 + m^{1/2})^{(r_0-2)\sum_{j \in A_q} k_j} \geq (m + 1)^{2(r_0-2)},
\]

which is greater than \((m + 1)^{r_0+1}\) provided \( r_0 \geq 5 \). This contradicts (3.15) if \( C > 1 \). Similarly, if

\[
\sum_{j \in A_q} k_j \geq 2 \cdot 4,
\]

then

\[
\prod_{j \in A_q} (1 + j)^{(r_0-2)k_j} \geq (1 + m^{1/4})^{24(r_0-2)} \geq (m + 1)^{2(r_0-2)},
\]

leading again to a contradiction if \( C > 1 \). Proceeding as above, we conclude

\[
\sum_{j \in A_q} k_j \leq 2^{q-1} \cdot 4,
\]

for all \( q \in \mathbb{N} \). Now, note that \( m^{2-q} < 2 \), if \( q > \log_2 \log_2 m \). Also, we claim

\[
k_1 \leq 2 \log_2 (m + 1).
\]

Indeed, if the opposite is true, we obtain

\[
\prod_{j \leq k_1} (1 + j)^{(r_0-2)k_j} \geq 2^{(r_0-2)k_1},
\]

and the right side of this inequality is greater than or equal to \((m + 1)^{r_0+1}\) if \( k_1 \geq ((r_0 + 1)/(r_0 - 2))\log_2 (m + 1) \). But now,

\[
j = k_1 + 2k_2 \cdots + j_0k_{j_0} = k_1 + \sum_{q=1}^{q_0} \sum_{j \in A_q} jk_j
\]

\[
\leq 2 \log_2 (m + 1) + \frac{m}{100} + m^{1/2} \cdot 4 + m^{1/4} \cdot 4^2 + \cdots + m^{2^{q_0}} \cdot 4^{2^{q_0-1}},
\]

where \( q_0 \) is the smallest integer strictly greater than \( \log_2 \log_2 m \). Therefore,

\[
j \leq 2 \log_2 (m + 1) + \frac{m}{25} + m^{1/2} \cdot 4 + m^{1/2} \cdot 4^2 + \cdots + m^{1/2} \cdot 4^{2^{q_0-1}}
\]

\[
\leq 2 \log_2 (m + 1) + \frac{m}{25} + m^{1/2} \cdot 4 \leq \frac{m}{20},
\]

for \( m \) large enough. Then \( m - j + k \geq m - j \geq m - m/20 \geq 19m/20 \), which contradicts (3.13). Thus, Lemma 3.2 is established for \( m \) larger than a constant. Adjusting \( C \) in (3.14), we can make sure that (3.14) then holds for all \( m \in \mathbb{N} \).

Proof of Lemma 3.1: In order to prove the estimate (3.11), we distinguish between three separate cases for the indices in the formula (3.5): Case 1: \( m - j + k > m/100 \); Case 2: \( j_0 > m/100 \); and
Case 3: \( j_0 \leq m/100, m - j + k \leq m/100 \). As in Lemma 3.2, we denote by \( j_0 \in \{1, \ldots, j\} \) the largest integer such that \( k_{j_0} \neq 0 \).

Case 1. First, let \( m \in \mathbb{N}, m - j + k > m/100 \), and \( k_1, \ldots, k_j \in \mathbb{N} \) be such that \( k_1 + \cdots + k_j = k \) and \( 1 \cdot k_1 + \cdots + j \cdot k_j = j \). By Hölder’s inequality and (3.8), we have

\[
U_{m, k_1, \ldots, k_j}(v, \zeta) \leq \left( \frac{m+1}{j} \right) \prod_{k=1}^j \frac{1}{k_1! \cdots k_j!} \|\partial_x^{m-j+1} \partial_t^j f(\cdot, \zeta(\cdot))\|_{L^1(I)} \left( \frac{\|\zeta^\prime\|_{L^\infty(I)}}{1!} \right)^{k_1} \cdots \left( \frac{\|\zeta^{(j)}\|_{L^\infty(I)}}{j!} \right)^{k_j}
\]

\[
\leq (CM_1(1 + \tau^{-1/2}))^k \|\partial_x^{m-j+1} \partial_t^j f(\cdot, \zeta(\cdot))\|_{L^1(I)}
\]

\[
\times \frac{(m-j+k)!}{(m-j+1)!} \frac{(1 + \tau^{-1/2})^k}{(1 + 1/\sqrt{2}) \cdots (j + 1/\sqrt{2}) \cdots (1 + 1/\sqrt{2}) \cdot (m-j+k)!} \frac{1}{(m-j+1)!}
\]  

(3.20)

If \( j = k = 0 \), we use the estimate \( \|\partial_x^{m+1} f(\cdot, \zeta(\cdot))\|_{L^1(I)} \leq C(h) \|\partial_x^{m+1} v\|_{L^2(D)} \) to obtain

\[
U_{m, k_1, \ldots, k_j}(v, \zeta) \leq C \|\partial_x^{m+1} v\|_{L^2(D)}.
\]

(3.21)

Otherwise, \( 1 \leq k \leq j + 1 \) and we have \( \|\partial_x^{m-j+1} \partial_t^k f(\cdot, \zeta(\cdot))\|_{L^1(I)} \leq \|\partial_x^{m-j+1} \partial_t^j v\|_{L^2(D)} \).

Thus,

\[
U_{m, k_1, \ldots, k_j}(v, \zeta) \leq (CM_1(1 + \tau^{-1/2}))^k \|\partial_x^{m-j+1} \partial_t^j v\|_{L^2(D)}
\]

\[
\times \frac{(m-j+k)!}{(m-j+1)!} \frac{(1 + \tau^{-1/2})^k}{(1 + 1/\sqrt{2}) \cdots (j + 1/\sqrt{2}) \cdots (1 + 1/\sqrt{2}) \cdot (m-j+k)!} \frac{1}{(m-j+1)!}
\]  

(3.22)

By (3.21) and (3.22), we obtain

\[
\sum_{m=0}^{\infty} \sum_{j=0}^{m} \sum_{k \leq j} U_{m, k_1, \ldots, k_j}(v, \zeta) \frac{\tau(t)^m(m+1)^j}{m!} \leq C(h) \|v\|_{Y_v}
\]

(3.23)

\[
+ CM_1(1 + \tau^{-1/2}) \sum_{m=1}^{\infty} \sum_{j=1}^{m} \sum_{k \leq j} M^{k-1} \|\partial_x^{m-j+1} \partial_t^k v\|_{L^2}
\]

\[
\times \frac{(m-j+k)!}{(m-j+1)!} \frac{(m-j+k+1)!}{m!} \frac{(1 + \tau^{-1/2})^k}{(1 + 1/\sqrt{2}) \cdots (j + 1/\sqrt{2}) \cdots (1 + 1/\sqrt{2}) \cdot \cdots \cdot (1 + 1/\sqrt{2})} \frac{1}{(m-j+k+1)!}
\]  

(3.24)

\[
\leq C(h) \|v\|_{Y_v} + CM_1(1 + \tau^{-1/2}) \|v\|_{Y_v},
\]

where the last sum is over the set \( A_{j,k}^1 \) consisting of all the elements of \( A_{j,k} \) for which \( m - j + k > m/100 \); in the last inequality, we also used

\[
CM_1 \tau^{-1/2}(1 + \tau^{-1/2}) \leq M
\]

(3.25)

and

\[
\frac{(m+1)^j}{(m-j+1)!} \frac{(m-j+k+1)!}{(m-j+1)!} \frac{(m-j+k+1)!}{(m-j+k+1)!} \frac{(m-j+k+1)!}{m!} \frac{(1 + \tau^{-1/2})^k}{(1 + 1/\sqrt{2}) \cdots (j + 1/\sqrt{2}) \cdots (1 + 1/\sqrt{2})} \leq C^k
\]

for some large enough constant \( C \) depending only on \( r \), since by the binomial formula

\[
\sum_{k_1+\cdots+k_j=k} \frac{k!}{k_1! \cdots k_j!} \frac{1}{(j+1)^r} \leq C^k
\]

for all \( j \in \mathbb{N} \) and \( r > 3/2 \).

Case 2. Let \( m \in \mathbb{N} \) and \( k_1, \ldots, k_j \in \mathbb{N} \) be such that \( k_1 + \cdots + k_j = k \) and \( 1 \cdot k_1 + \cdots + j \cdot k_j = j \). Also, let \( j_0 > m/100 \). Recall that, as in Lemma 3.2, we denote with \( j_0 \in \{1, \ldots, j\} \) the largest
integer for which \( k_{j_0} \neq 0 \). By Hölder’s inequality, we get

\[
U_{m,k_1,...,k_j}(v, \xi) \leq \left( \frac{m + 1}{j} \right) \frac{1}{k_1! \cdots k_j!} \left\| \partial_x^{m-j+1} \partial_z^k f(\cdot, \xi(\cdot)) \right\|_{L^\infty(v)} \times \left( \left\| \xi' \right\|_{L^\infty(v)} \frac{k_{j_0} - 1}{j_0!} \right) \left\| \xi(j_0) \right\|_{L^\infty(v)} \frac{k_{j_0} - 1}{j_0!}.
\]

(3.26)

An application of the one-dimensional Agmon’s inequality

\[
\left\| f(\cdot, \xi(\cdot)) \right\|_{L^\infty(v)} \leq C \left\| f_0 \right\|_{L^2(v)} \left\| f \right\|_{L^2(v)} + C \left\| \xi' \right\|_{L^\infty(v)} \left\| f \right\|_{L^2(v)} + C \left\| f \right\|_{L^2(v)}
\]

and the estimates (3.8) and (3.9) lead to

\[
U_{m,k_1,...,k_j}(v, \xi) \leq C \left( \left\| \partial_x^{m-j+2} \partial_z^k v \right\|_{L^2(D_x)} \left\| \partial_x^{m-j+1} \partial_z^k v \right\|_{L^2(D_x)} \right.
\]

\[
+ \left\| \xi' \right\|_{L^\infty(v)} \left\| \partial_x^{m-j+1} \partial_z^k v \right\|_{L^2(D_z)} \left( \left\| \partial_x^{m-j+2} \partial_z^k v \right\|_{L^2(D_x)} \left\| \partial_x^{m-j+1} \partial_z^k v \right\|_{L^2(D_x)} \right).
\]

(3.27)

Hence, we may bound

\[
\sum_{m=0}^{\infty} \sum_{j=0}^{m+1} \sum_{k \leq j} \sum_{A_{j,k}} \left( M^{k-1} \left\| \partial_x^{m-j+2} \partial_z^k v \right\|_{L^2} \frac{\tau^m(m + 1)^r}{(m - j + 2)!\left( k - 1 \right)!} \right)^{1/2}
\]

\[
\times \left( M^{k-1} \left\| \partial_x^{m-j+1} \partial_z^k v \right\|_{L^2} \frac{\tau^m(m - j + k)^r}{(m - j + 1)!\left( k - 1 \right)!} \right)^{1/2}
\]

\[
\leq C \tau^{-1/2} \left\| \xi \right\|_{Y} \left\| v \right\|_{X},
\]

where we used (3.25) and

\[
\frac{(m + 1)^r(m - j + 2)!}{(m - j + 1)!(k - 1)!} \frac{(k - 1)!}{m!2^{r-1/2}k_1 \cdots (j_0 + 1)^r(k_0 - 1)!(j_0 + 1)^r\cdots(j_{j_0} - 1)!} \frac{(m - j + 1)!}{m!2^{r-1/2}k_1 \cdots (j_0 + 1)^r(k_0 - 1)!(j_0 + 1)^r\cdots(j_{j_0} - 1)!} \frac{(k - 1)!}{(m - j + 2)^r} \leq C^k
\]

for all \( r > 3/2 \). By the discrete Hölder’s inequality and (3.25), we get

\[
S_2 \leq C \tau^{-1/2} \sum_{m=0}^{\infty} \sum_{j=0}^{m+1} \sum_{k \leq j} \sum_{A_{j,k}} \left( M^{k-1} \left\| \partial_x^{m-j+2} \partial_z^k v \right\|_{L^2} \frac{\tau^m(m - j + k)^r}{(m - j + 1)!\left( k - 1 \right)!} \right)^{1/2}
\]

\[
\times \left( M^{k-1} \left\| \partial_x^{m-j+1} \partial_z^k v \right\|_{L^2} \frac{\tau^m(m - j + k)^r}{(m - j + 1)!\left( k - 1 \right)!} \right)^{1/2}
\]

\[
\leq C \tau^{-1/2} \left\| \xi \right\|_{Y} \left\| v \right\|_{X},
\]
where we used \( \|z\|_{L^\infty} \leq CM_1 \tau^{-1}(1 + \tau^{-1/2}) \leq CM_1 \), which follows from (3.8), and
\[
\frac{(m+1)!}{m!2^{r-1/2k_1} \ldots (j_0 + 1)!^{r-1/2k_0}k_0^{-1}(j_0 + 1)!^r j_0 k_1! \ldots k_{j_0}!(m-j+1)!^{r-1/2k_1}k_1! \ldots (k_{j_0}!)!} \leq \frac{(m+1)!}{m!} \frac{2^{r-1/2k_0} \ldots (j_0 + 1)!^{r-1/2k_0}k_0^{-1}(j_0 + 1)!^r j_0 k_1! \ldots k_{j_0}!(m-j+1)!^r} \leq C^k
\]
for all \( r > 3/2 \). Finally, we have
\[
S_3 \leq C\|\xi\|_Y \sum_{m=0}^{m+1} \sum_{j=0}^{m+1} \sum_{k \leq k_j} M^k \|z_m^{m-j+1}a_k^{k-1}v\|_{L^2} \frac{\tau^{m-j+k}(m-j+k+1)^r}{(m-j+1)!^r(k-1)!} \leq C\|\xi\|_Y \|v\|_{X_t},
\]
by using (3.25) and
\[
\frac{(m+1)!}{m!2^{r-1/2k_1} \ldots (j_0 + 1)!^{r-1/2k_0}k_0^{-1}(j_0 + 1)!^r j_0 k_1! \ldots k_{j_0}!(m-j+1)!^{r-1/2k_1}k_1! \ldots (k_{j_0}!)!} \leq \frac{(m+1)!}{m!} \frac{2^{r-1/2k_0} \ldots (j_0 + 1)!^{r-1/2k_0}k_0^{-1}(j_0 + 1)!^r j_0 k_1! \ldots k_{j_0}!(m-j+1)!^r} \leq C^k
\]
for all \( r > 3/2 \). We conclude that \( S_1 + S_2 + S_3 \leq C(1 + \tau^{-1/2})\|\xi\|_Y \|v\|_{X_t} \).

Case 3. Let \( m \in \mathbb{N} \) and \( k_1, \ldots, k_j \in \mathbb{N} \) be such that \( k_1 + \cdots + k_j = k \) and \( 1 \cdot k_1 + \cdots + j \cdot k_j = j \). Also, let \( j_0 \leq m/100 \) and \( m \leq j + k \leq m/100 \). Then, by Hölder’s inequality, (3.8) and (3.10), we have
\[
U_{m,k_1, \ldots, k_j}(v, \xi) \leq \left( \frac{m+1}{m} \right)^{j!} \sum_{k \leq k_j} M^k \|z_m^{m-j+1}a_k^{k-1}v\|_{L^2} \frac{\tau^{m-j+k}(m-j+k+1)^r}{(m-j+1)!^r(k-1)!} \]
\[
\leq C(1 + \tau^{-1/2})\|\xi\|_Y \left( CM_1(1 + \tau^{-1/2}) \right)^{k-1} \|z_m^{m-j+1}a_k^{k-1}v\|_{L^2(D)} \times \frac{\tau^{m-j+k}(m-j+k+1)^r}{(m-j+1)!^r(k-1)!}.
\]
Therefore,
\[
\sum_{m=0}^{m+1} \sum_{j=0}^{m+1} \sum_{k \leq k_j} U_{m,k_1, \ldots, k_j}(v, \xi) \frac{\tau^{m-j+k}(m-j+k+1)^r}{m!} \leq C(1 + \tau^{-1/2})\|\xi\|_Y \sum_{m=0}^{m+1} \sum_{j=0}^{m+1} \sum_{k \leq k_j} M^k \|z_m^{m-j+1}a_k^{k-1}v\|_{L^2} \frac{\tau^{m-j+k}(m-j+k+1)^r}{(m-j+1)!^r(k-1)!} \leq C(1 + \tau^{-1/2})\|\xi\|_Y \|v\|_{X_t},
\]
where we denote by \( A_{j,k} \) the set of all the elements of \( A_{j,k} \) for which \( j_0 \leq m/100 \) and \( m \leq j + k \leq m/100 \). In (3.31), we used (3.25) and
\[
\frac{(m+1)!}{m!} \frac{(m-j+1)!^r(k-1)!}{(m-j+1)!^r!^{r-1/2}k_0 \ldots (j_0 + 1)!^{r-1/2}k_0! \ldots (k_{j_0}!)!} \leq \frac{(m+1)!}{m!} \frac{2^{r-1/2k_0} \ldots (j_0 + 1)!^{r-1/2k_0}k_0^{-1}(j_0 + 1)!^r j_0 k_1! \ldots k_{j_0}!(m-j+1)!^r} \leq C^k,
\]
which follows by Lemma 3.2 for all \( r \geq 5 \). Combining the estimates (3.23), (3.29), and (3.31) from the three different cases, we conclude the proof of the lemma.

\[\blacksquare\]
the dependence on the derivatives of $\zeta$ is given explicitly. We assume that $\zeta(\cdot, t) \in C^\infty_{\text{per}}(I)$ and $|\zeta(x, 0)| \leq h/4$ for $x \in I$ and $t > 0$.

**Lemma 3.3:** Let $v$ be a smooth function in the domain $D_t$. Then the two dimensional Agmon’s type inequality

$$
\|v\|_{L^2(D_t)} \leq C\|v\|_{L^2(D_t)}^{1/2} \|\partial_x^2 v\|_{L^2(D_t)}^{1/2} + C\|\zeta\|_{L^\infty(I)} \|v\|_{L^2(D_t)}^{1/2} + C\|\zeta\|_{L^\infty(I)} \|\partial_x v\|_{L^2(D_t)}^{1/2} + C\|\zeta\|_{L^\infty(I)} \|\partial_z v\|_{L^2(D_t)}^{1/2} + C\|v\|_{L^2(D_t)} \tag{3.32}
$$

holds for a large enough constant $C$ depending on $h$.

**Proof:** We use the change of coordinates $\phi : D_t \rightarrow I \times I$, defined by $\phi(x, z) = (x, (z + h)/(\zeta + h))$ for $(x, \zeta) \in D_t$. The estimate (3.32) follows by the change of variables formula and the two-dimensional Agmon’s inequality on $I \times I$.

Also, recall the time-differentiation formula

$$
\frac{d}{dt} \int_{D_t} v^2 = 2 \int_{D_t} v \partial_t v + \int_{D_t} \partial_t \zeta \frac{z + h}{\zeta + h} v \partial_z v + \int_{D_t} \partial_t \zeta \frac{z + h}{\zeta + h} v^2, \tag{3.33}
$$

which is obtained by the change of variables as in the proof above.

Now, using the definition of the analytic norms (2.13) and (2.15), we derive an equation for the derivative of $\|v\|_{X_{\zeta(\cdot, t)}}$

$$
\frac{d}{dt} \|v(t)\|_{X_{\zeta(\cdot, t)}} = \tau(t) \|v(t)\|_{Y_{\zeta(\cdot, t)}} + \sum_{m=0}^{\infty} \sum_{k=0}^{m} M^k \frac{d}{dt} \|\partial_x^{m-k} \partial_z^k v\|_{L^2(D_t)} \tau(t)^m (m+1)! \frac{(m-k)!}{(m-k)!} \tag{3.34}
$$

for all $t > 0$. Applying the operator $\partial_x^{m-k} \partial_z^k$ to (2.7) and taking the $L^2(D_t)$-inner product with $\partial_x^{m-k} \partial_z^k v$, we obtain

$$
\frac{1}{2} \frac{d}{dt} \int_{D_t} (\partial_x^{m-k} \partial_z^k v)^2 = -\int_{D_t} \partial_x^{m-k} \partial_z^k (v \partial_x v + w \partial_z v + g \rho \partial_x \zeta - \eta) \partial_x^{m-k} \partial_z^k v

+ \int_{D_t} \partial_x \zeta \frac{z + h}{\zeta + h} \partial_x^{m-k} \partial_z^{k+1} v \partial_x^{m-k} \partial_z^k v + \int_{D_t} \partial_z \zeta \frac{z + h}{\zeta + h} \partial_x^{m-k} \partial_z^{k+1} v \partial_x^{m-k} \partial_z^k v.
$$

Thus,

$$
\frac{d}{dt} \|\partial_x^{m-k} \partial_z^k v\|_{L^2(D_t)} \leq \sum_{m=0}^{\infty} \sum_{k=0}^{m} \left[ \frac{m-k}{i} \right] \left( \frac{m-k}{j} \right) \left( \frac{k}{i} \right) \left( \frac{k}{j} \right) \left( \frac{k}{i+j} \right) (\|\partial_x^{m-k-i} \partial_z^{j} v\|_{L^2(D_t)} + \|\partial_x^{m-k-i} \partial_z^{j} v\|_{L^2(D_t)})

+ g \rho \|\partial_x^{m-k+1} \partial_z^{k+1} \zeta\|_{L^2(D_t)} + \|\partial_x^{m-k+1} \partial_z^{k+1} \zeta\|_{L^2(D_t)}

+ \|\partial_x \zeta \|_{L^\infty(I)} \|\partial_z^{m-k+1} \zeta\|_{L^2(D_t)} + C \|\partial_x \zeta \|_{L^\infty(I)} \|\partial_z^{m-k+1} \zeta\|_{L^2(D_t)},
$$

for $m \in \mathbb{N}$ and $k \in \{0, \ldots, m\}$. Using the estimate (3.35) for the second term on the right side of (3.34), observing the independence of $\zeta$ and $\eta$ on the $z$ variable, we have an *a priori* estimate

$$
\frac{d}{dt} \|v(t)\|_{X_{\zeta(\cdot, t)}} \leq \tau(t) \|v(t)\|_{Y_{\zeta(\cdot, t)}} + \mathcal{U}(v, v) + \mathcal{V}(w, v) + g \rho \|\zeta(t)\|_{Y_{\zeta(\cdot, t)}} + \|\eta(t)\|_{X_{\zeta(\cdot, t)}} \tag{3.36}
$$

$$
+ M^{-1} \|\partial_x \zeta \|_{L^\infty(I)} \|v(t)\|_{Y_{\zeta(\cdot, t)}} + C \|\partial_x \zeta \|_{L^\infty(I)} \|v(t)\|_{X_{\zeta(\cdot, t)}}
$$

where we denoted

$$
\mathcal{U}(v, v) = \sum_{m=0}^{\infty} \sum_{k=0}^{m} M^k \frac{d}{dt} \|\partial_x^{m-k} \partial_z^k v\|_{L^2(D_t)} \frac{\tau(t)^m (m+1)!}{(m-k)!}. \tag{3.37}
$$
employ the estimates and (3.32) from Lemma 3.3 and respectively, obtained by (3.8), (3.10), and the Agmon’s type inequality (3.32) with $M_1(1 + \tau^{-1/2}) r^{-1} \leq M$.

Lemma 3.4: Let $t > 0$ be fixed and let $v, \xi \in Y$, for some $\tau = \tau(t), M \geq M_1(1 + \tau^{-1/2}) r^{-1}$, and $r \geq 2$. Then

\[
\mathcal{U}(v, \xi) \leq C(1 + \tau^{-1})\|v\|_{X_i} \|\xi\|_{Y_i} (3.41)
\]

and

\[
\mathcal{V}(w, v) \leq C(1 + \tau^{-2})\|w\|_{X_i} \|v\|_{Y_i} (3.42)
\]

for a sufficiently large positive constant $C$ depending on $h$.

Proof: In order to estimate $\|\partial_x^{m-k} \partial_\xi^{k-j+1} \partial_z^{k-j} v\|_{L^2(D)}$ and $\|\partial_x^{m-k} \partial_\xi^{k-j+1} \partial_z^{k-j+1} v\|_{L^2(D)}$ for all indices $m \in \mathbb{N}, k \in \{0, \ldots, m\}, i \in \{0, \ldots, m-k\}$, and $j \in \{0, \ldots, k\}$, we distinguish between two separate cases: Case 1: $0 \leq i \leq \lfloor(m - k)/2\rfloor$, $0 \leq j \leq \lfloor k/2 \rfloor$ and Case 2: $\lfloor(m - k)/2\rfloor + 1 \leq i \leq m - k, \lfloor k/2 \rfloor + 1 \leq j \leq k$, and we split $\mathcal{U} = \mathcal{U}_{low} + \mathcal{U}_{high}$ and $\mathcal{V} = \mathcal{V}_{low} + \mathcal{V}_{high}$, respectively. In Case 1, we apply Hölder’s inequality

\[
\|\partial_x^{m-k} \partial_\xi^{k-j+1} \partial_z^{k-j} v\|_{L^2(D)} \leq \|\partial_x^{m-k} \partial_\xi^{k-j+1} \partial_z^{k-j} v\|_{L^2(D)} \leq \|\partial_x^{m-k} \partial_\xi^{k-j+1} \partial_z^{k-j} v\|_{L^2(D)}
\]

and (3.32) from Lemma 3.3

\[
\|\partial_x^{m-k} \partial_\xi^{k-j+1} \partial_z^{k-j} v\|_{L^2(D)} \leq C\|\partial_x^{m-k} \partial_\xi^{k-j+1} \partial_z^{k-j} v\|_{L^2(D)} + M^{1/2}\|\partial_x^{m-k} \partial_\xi^{k-j+1} \partial_z^{k-j} v\|_{L^2(D)}
\]

with the estimate (3.8) on the derivatives of $\xi$ and $M_1(1 + \tau^{-1/2}) r^{-1} \leq M$. After a similar computation as the one below, we obtain $\mathcal{U}_{low} \leq C(1 + \tau^{-1})\|v\|_{X_i} \|\xi\|_{Y_i}$. In Case 2, an application
of Hölder’s and Agmon’s inequalities gives

\[
U_{\text{high}} \leq C \sum_{m=1}^{\infty} \sum_{k=1}^{m} \sum_{j=(m-k)/2+1}^{k} M^k \| \partial_x^i \partial_z^j v \|_{L^2(D_0)} \| \partial_x^{m-k-i-1} \partial_z^{k-j} v \|_{L^{\infty}(D_0)} \frac{\tau^m (m+1)^r}{i! j!} (m-k-i)!(k-j)!
\]

\[
\leq C \sum_{m=1}^{\infty} \sum_{k=1}^{m} \sum_{j=(m-k)/2+1}^{k} \left( M^k \| \partial_x^i \partial_z^j v \|_{L^2(D_0)} \right)^{\frac{m-i-j+1}{2}} \frac{\tau^m (m-j-i+2)^{r/2}}{(m-k-i+1)!(k-j)!} \times \left( \tau^m \left( M^k \| \partial_x^i \partial_z^j v \|_{L^2(D_0)} \right)^{\frac{m-i-j-1}{2}} \frac{\tau^m (m-j-i+4)^{r/2}}{(m-k-i+3)!(k-j)!} \right)
\]

where we also used \((m + 1)^r \leq C(i + j + 1)^r\) and \(r \geq 1\). Hence, by the discrete Hölder’s and Young’s inequalities, we get

\[
U_{\text{high}} \leq C(1 + \tau^{-1}) \| v \|_{X}, \| v \|_{Y}.
\] (3.43)

For the estimate on \(V\), we proceed similarly and consider the same cases as above. Note that, representing \(w\) in terms of \(v\) leads to a loss of one space derivative. Namely, by integrating (2.8) with respect to the vertical variable between \(-h\) and \(z\), we get

\[
w(x, z, t) = - \int_{-h}^{z} \partial_z v(x, z', t) \, dz',
\] (3.44)

which implies the estimates \(\| \partial_x^i \partial_z^j v \|_{L^2(D_0)} \leq C \| \partial_x^i v \|_{L^2(D_0)}\) for \(i \in \mathbb{N}\) and \(j \geq 1\). Thus, for given \(k \in \{0, \ldots, m\}\), it is convenient to consider two separate subcases of Case 1: Case 1.1: \(j = 0\) and Case 1.2: \(j \in \{1, \ldots, [k/2]\}\), and we write \(V_{\text{low}} = V_{\text{low}}^0 + V_{\text{low}}^1\). By the Agmon’s type inequality (3.32), we obtain

\[
\| \partial_x^i v \|_{L^\infty(D_0)} \leq C \| \partial_x^{i+1} v \|_{L^2(D_0)}^{1/2} \left( \| \partial_x^{i+2} v \|_{L^2(D_0)}^{1/2} + M^{1/2} \| \partial_x^{i+1} v \|_{L^2(D_0)}^{1/2} \right)
\] (3.45)

\[
+ M \| \partial_x^{i+1} v \|_{L^2(D_0)} + M^{1/2} \tau^{-1/2} \| \partial_x^{i} v \|_{L^2(D_0)} + C \| \partial_x^{i+1} v \|_{L^2(D_0)}
\]

for all \(i \in \mathbb{N}\) and \(M_1(1 + \tau^{-1/2} \tau^{-1} \leq M\). Then, for Case 1.1, we have

\[
V_{\text{low}}^0 \leq \sum_{m=0}^{\infty} \sum_{k=0}^{(m-k)/2} \sum_{i=0}^{m} M^k \| \partial_x^i \partial_z^{k-i} v \|_{L^\infty(D_0)} \| \partial_x^{m-k-i} \partial_z^k v \|_{L^2(D_0)} \tau^m (m+1)^r / i! j! (m-k-i)! k!
\] (3.46)

\[
\leq C(1 + \tau^{-2}) \| v \|_{X}, \| v \|_{Y},
\]
for \( r \geq 2 \). In Case 1.2, by Hölder’s inequality and (3.32), we deduce

\[
\mathcal{V}_{\text{low}}^1 \leq C \sum_{m=2}^{\infty} \sum_{k=2}^{[m-k/2]} \sum_{i=0}^{[m-k/2]} \sum_{j=1}^{m-k} \frac{M^k \| \partial_t^i \partial_x^j w \|_{L^\infty(D_t)} \| \partial_t^{m-k-i} \partial_x^{k-j+1} v \|_{L^2(D_t)}}{i!j!(m-k-i)(k-j)} \cdot \frac{r^m(m+1)^r}{(m+1)!},
\]

\[
\leq C \sum_{m=2}^{\infty} \sum_{k=2}^{[m-k/2]} \sum_{i=0}^{[m-k/2]} \sum_{j=1}^{m-k} \left( M^{j-1} \| \partial_t^{i+1} \partial_x^{j-1} v \|_{L^2(D_t)} \cdot \frac{r^{i+j}(i+j+1)^r}{(i+1)!} \right)^{1/2} \times \left( \frac{r^{-i} M^{j-1} \| \partial_t^{i+2} \partial_x^{j-1} v \|_{L^2(D_t)} \cdot \frac{r^{i+j+2}(i+j+3)^r}{(i+2)!j!}}{(i+2)!j!} \right)^{1/2}
\]

\[
+ \frac{r^{-i} M^{j} \| \partial_t^{i+1} \partial_x^{j+1} v \|_{L^2(D_t)} \cdot \frac{r^{i+j+1}(i+j+2)^r}{(i+1)!j!}}{(i+1)!j!}\right)^{1/2}
\]

\[
+ M^{j-1} \| \partial_t^{i+1} \partial_x^{j-1} v \|_{L^2(D_t)} \cdot \frac{r^{i+j+1}(i+j+1)^r}{(i+1)!j!} \right) \times \left( M^{k-j+1} \| \partial_t^{m-k-i} \partial_x^{k-j+1} v \|_{L^2(D_t)} \cdot \frac{r^{m-j+2}(m-j-i+1)(m-j-i+2)^r}{(m-k-i)(k-j)!} \right),
\]

where we used \((m+1)^r \leq C(m-i-j+2)^r\) and \( r \geq 2 \). Another application of the discrete Hölder’s and Young’s inequalities leads to

\[
\mathcal{V}_{\text{low}}^1 \leq C(1 + r^{-1}) \| v \|_{X_1} \| v \|_{Y_r}, \quad (3.47)
\]

Finally, in Case 2, we obtain

\[
\mathcal{V}_{\text{high}} \leq C \sum_{m=1}^{\infty} \sum_{k=1}^{[m-k/2]} \sum_{j=1}^{m-k} M^k \| \partial_t^i \partial_x^j w \|_{L^2(D_t)} \| \partial_t^{m-k-i} \partial_x^{k-j+1} v \|_{L^\infty(D_t)} \cdot \frac{r^m(m+1)^r}{(m+1)!j!(m-k-i)(k-j)!},
\]

by Hölder’s inequality, \( \| \partial_t^i \partial_x^j w \|_{L^2(D_t)} \leq \| \partial_t^{i+1} \partial_x^{j-1} v \|_{L^2(D_t)} \), and

\[
\| \partial_t^{m-k-i} \partial_x^{k-j+1} v \|_{L^\infty(D_t)} \leq C \| \partial_t^{m-k-i} \partial_x^{k-j+1} \|_{L^2(D_t)}^{1/2} \| \partial_t^{m-k-i} v \|_{L^2(D_t)}^{1/2} \]

\[
+ M^{1/2} \| \partial_t^{m-k-i} \partial_x^{k-j+1} v \|_{L^2(D_t)}^{1/2} \]

\[
+ M^{1/2} \| \partial_t^{m-k-i} \partial_x^{k-j+1} v \|_{L^2(D_t)}^{1/2} \]

\[
+ C \| \partial_t^{m-k-i} \partial_x^{k-j+1} v \|_{L^\infty(D_t)},
\]

where we also used that \((m+1)^r \leq C(i+j+1)^r\) and \( r \geq 2 \). Thus, the proof of Lemma 3.4 is complete.

Let \( C_0 \) be a large enough fixed constant. By (3.3) and Lemma 3.1, we have the a priori estimate

\[
\frac{d}{dt} \| \xi(t) \|_{X_{r_0}} \leq \tau(t) \| \xi(t) \|_{Y_{r_0}} + C_0(1 + \tau(t)^{-1/2}) \| \xi(t) \|_{X_{r_0}} \| v(t) \|_{Y_{r_0}} + C_0(1 + \tau(t)^{-1/2}) \| \xi(t) \|_{Y_{r_0}} \| v(t) \|_{Y_{r_0}},
\]

\[
(3.50)
\]
Similarly, by (3.36), (3.39), (3.40), and Lemma 3.4, we obtain

\[
\frac{d}{dt} \|v(t)\|_{X_{\tau}} \leq \tau(t)\|v(t)\|_{Y_{\tau}} + C_0(1 + \tau(t)^{-1})\|v(t)\|_{X_{\tau}} + C_0(1 + \tau(t)^{-2})\|\partial_v v(t)\|_{X_{\tau}} + C_0(1 + \tau(t)^{-3/2})\|\xi(t)\|_{Y_{\tau}} + \|\eta(t)\|_{X_{\tau}},
\]

(3.51)

Adding the estimates (3.50) and (3.51) gives

\[
\frac{d}{dt} \|(v(t), \xi(t))\|_{X_{\tau}} \leq \left(\tau(t) + g\rho + 3C_0(1 + \tau(t)^{-2})\|(v(t), \xi(t))\|_{X_{\tau}} + C_0(1 + \tau(t)^{-3/2})\|(v(t), \xi(t))\|_{X_{\tau}}^2\right)\|(v(t), \xi(t))\|_{X_{\tau}} + \|\eta(t)\|_{X_{\tau}},
\]

(3.52)

for all \(t > 0\), provided \(v, \xi \in Y_{\tau}\) with \(M \geq CM_1(1 + \tau_0^{-1})\tau_0^{-1}\) and \(r \geq r_0\).

Let \((v_0, \xi_0) \in Y_{\tau}\) and let \(\tau(t) = \int_0^t \|\eta(s)\|_{X_{\tau}} ds\). We define \(\tau(t)\) to be the solution of the ordinary differential equation

\[
\dot{\tau}(t) + 4g\rho + 18C_0(1 + \tau(t)^{-1})(v_0, \xi_0) + 8C_0(1 + \tau(t)^{-3/2})(v_0, \xi_0) = 0,
\]

(3.53)

satisfying the initial condition \(\tau(0) = \tau_0\). Clearly, \(\tau(t)\) is a decreasing function. Let \(T\) be the time such that \(\tau(t) > \tau_0/2\) for all \(t \in [0, T)\) and \(\tau(T) = \tau_0/2\). Note that \(\tau(t)\) and \(T\) are uniquely determined by the initial data. Then

\[
\tau(t) + g\rho + 3C_0(1 + \tau(t)^{-2})\|(v(t), \xi(t))\|_{X_{\tau}} + C_0(1 + \tau(t)^{-3/2})\|(v(t), \xi(t))\|_{X_{\tau}}^2 < 0
\]

(3.54)

at the time \(t = 0\), and by (3.52), we have

\[
\|(v(t), \xi(t))\|_{X_{\tau}} \leq 2\|(v_0, \xi_0)\|_{X_{\tau}}
\]

(3.55)

for short time. Hence, we conclude that the inequalities (3.54) and (3.55) hold for all \(t < T\).

In Sec. IV, we provide a formal construction of the analytic solution \((v, \xi)\) of the system (2.7)–(2.9) satisfying (3.52), and we prove that \((v, \xi)\) is a priori bounded in \(L^\infty(0, T, X_{\tau}) \cap L^1(0, T, (1 + \tau^{-2})Y_{\tau})\) for all times \(t \in [0, T]\).

IV. CONSTRUCTION OF THE SOLUTIONS

In this section, we construct the solution of the system (2.7)–(2.9) using the Picard iteration method. Starting from \(n = 0\) with given \(v_0 = v(0)\) and \(\xi_0 = \xi(0)\), we define the functions \(v_n(t)\) and \(\xi_n(t)\) iteratively according to

\[
v_{n+1}(t) = v_0 - \int_0^t (v_n(s)\partial_x v_n(s) + w_n(s)\partial_x \xi_n(s)) ds - g\rho \int_0^t \partial_x \xi_n(s) ds + \int_0^t \eta(s) ds,
\]

(4.1)

with \(w_n(t)\) given in terms of \(v_n(t)\) by the formula

\[
w_n(t) = -\int_{-h}^t \partial_x v_n(x, z', t) dz',
\]

(4.2)

and

\[
\xi_{n+1}(t) = \xi_0 - \int_0^t \partial_x \left(\int_{-h}^t v_n(x, z', s) dz'\right) ds
\]

(4.3)

for all \(n \in \mathbb{N}\).
Lemma 4.1: Let \((v_0, \zeta_0) \in Y_0\). Then the sequence of Picard iterates \(\{(v_n, \zeta_n)\}_{n \in \mathbb{N}}\) satisfies

\[
\sup_{t \in [0,T]} \|(v_n(t), \zeta_n(t))\|_{X,0} + g \rho \int_0^T \|(v_n(t), \zeta_n(t))\|_{Y,0} \, dt
\]

\[
+ 4C_0 \|(v_0, \zeta_0)\|_{X,0} \int_0^T (1 + \tau(t)^{-2}) \|(v_n(t), \zeta_n(t))\|_{Y,0} \, dt
\]

\[
+ 2C_0 \|(v_0, \zeta_0)\|^2 \int_0^T (1 + \tau(t)^{-3/2}) \|(v_n(t), \zeta_n(t))\|_{Y,0} \, dt \leq 2 \|(v_0, \zeta_0)\|_{X,0},
\]

for all \(n \in \mathbb{N}\) and \(T = T(v_0, \zeta_0) > 0\) small enough.

Proof: We proceed by induction on \(n \in \mathbb{N}\). First, let \(n = 0\). For the estimate (4.4) to be satisfied, we choose \(T\) such that

\[
g \rho T \|(v_0, \zeta_0)\|_{Y,0} \leq \frac{1}{6} \|(v_0, \zeta_0)\|_{X,0},
\]

\[
4C_0 \|(v_0, \zeta_0)\|_{Y,0} \int_0^T (1 + \tau(t)^{-2}) \, dt \leq \frac{1}{6},
\]

and

\[
2C_0 \|(v_0, \zeta_0)\|_{X,0} \|(v_0, \zeta_0)\|_{Y,0} \int_0^T (1 + \tau(t)^{-3/2}) \, dt \leq \frac{1}{6}.
\]

Clearly, the condition (4.5) holds for \(T \leq T_1\), where \(T_1 = \|(v_0, \zeta_0)\|_{Y,0}/6g \rho \|(v_0, \zeta_0)\|_{Y,0}\). By the definition of \(\tau\), we have \(18C_0(1 + \tau^{-2}) < -\dot{\tau}/\|(v_0, \zeta_0)\|_{X,0}\) and \(8C_0(1 + \tau^{-3/2}) < -\dot{\tau}/\|(v_0, \zeta_0)\|^2_{X,0}\), which implies that (4.6) and (4.7) hold if we choose \(T\) so that

\[
\tau_0 - \tau(T) \leq \frac{C \|(v_0, \zeta_0)\|_{X,0}}{\|(v_0, \zeta_0)\|_{Y,0}},
\]

which is satisfied if \(T \leq T_2\), where \(T_2\) is determined explicitly by (3.53) and (4.8). Taking \(T \leq \min\{T_1, T_2\}\), we conclude the result for \(n = 0\).

Next, we assume the estimate (4.4) is true for some \(n\). By (4.1) and (4.3), we obtain for \(n + 1\) the inequality

\[
\frac{d}{dt} \|(v_{n+1}, \zeta_{n+1})\|_{X,0} \leq \left( \dot{\tau} + C_0(1 + \tau^{-1}) \|(v_n, \zeta_n)\|_{X,0} + C_0(1 + \tau^{-2}) \|(v_n, \zeta_n)\|_{X,0} \right) \|(v_{n+1}, \zeta_{n+1})\|_{Y,0}
\]

\[
+ \left( C_0(1 + \tau^{-3/2}) \|(v_n, \zeta_n)\|_{X,0} \|(v_n, \zeta_n)\|_{Y,0} + C_0(1 + \tau^{-1}) \|(v_n, \zeta_n)\|_{Y,0} \right) \|(v_{n+1}, \zeta_{n+1})\|_{X,0},
\]

\[
+ C_0(1 + \tau^{-1}) \|(v_n, \zeta_n)\|_{X,0} \|(v_n, \zeta_n)\|_{Y,0} + C_0(1 + \tau^{-2}) \|(v_n, \zeta_n)\|_{Y,0} \|(v_n, \zeta_n)\|_{X,0},
\]

\[
+ g \rho \|(v_n, \zeta_n)\|_{Y,0} + \eta \|\eta\|_{X,0},
\]

where we have utilized Lemmas 3.1 and 3.3, and analogous arguments as in the derivation of the a priori estimate (3.52). By (4.9), we have

\[
\frac{d}{dt} \|(v_{n+1}, \zeta_{n+1})\|_{X,0} \leq \left( \dot{\tau} + 2C_0(1 + \tau^{-2}) \|(v_n, \zeta_n)\|_{X,0} \right) \|(v_{n+1}, \zeta_{n+1})\|_{Y,0}
\]

\[
+ \left( C_0(1 + \tau^{-3/2}) \|(v_n, \zeta_n)\|_{X,0} \|(v_n, \zeta_n)\|_{Y,0} + C_0(1 + \tau^{-2}) \|(v_n, \zeta_n)\|_{Y,0} \right) \|(v_{n+1}, \zeta_{n+1})\|_{X,0},
\]

\[
+ 2C_0(1 + \tau^{-2}) \|(v_n, \zeta_n)\|_{X,0} \|(v_n, \zeta_n)\|_{Y,0} + \eta \|\eta\|_{X,0}.
\]
Using the induction hypothesis and the definition of $\tau$, we obtain from (4.9) and Grönwall’s inequality that
\[
\|(v_{n+1}(t), \zeta_{n+1}(t))\|_{X_{ad}} + 4\rho \int_0^t \|(v_{n+1}(s), \zeta_{n+1}(s))\|_{Y_{ad}} \, ds
\]
\[
+ 16C_0\|(v_0, \zeta_0)\|_{X_0} \int_0^t (1 + \tau(s)^{-2})\|(v_{n+1}(s), \zeta_{n+1}(s))\|_{Y_{ad}} \, ds
\]
\[
+ 8C_0\|(v_0, \zeta_0)\|_{X_0}^2 \int_0^t (1 + \tau(s)^{-3/2})\|(v_{n+1}(s), \zeta_{n+1}(s))\|_{Y_{ad}} \, ds
\]
\[
\leq \|(v_0, \zeta_0)\|_{X_0} + \tilde{\eta}(t) + 4\rho \int_0^t \|(v_n(s), \zeta_n(s))\|_{Y_{ad}} \, ds
\]
\[
+ 4C_0\|(v_0, \zeta_0)\|_{X_0} \int_0^t (1 + \tau(s)^{-2})\|(v_n(s), \zeta_n(s))\|_{Y_{ad}} \, ds
\]
\[
+ 2C_0\|(v_0, \zeta_0)\|_{X_0}^2 \int_0^t (1 + \tau(s)^{-3/2})\|(v_n(s), \zeta_n(s))\|_{Y_{ad}} \, ds,
\]
where $\tilde{\eta}(t) = \int_0^t \|\eta(s)\|_{X_{ad}} \, ds$. We establish (4.4) for $n + 1$ by taking the supremum of the inequality (4.10) over $t \in [0, T]$, then using the induction hypothesis and choosing $T$ such that $T \leq \min\{T_1, T_2, T_3\}$, where $T_3$ is the largest time for which $\tilde{\eta}(t) < (1/2)\|(v_0, \zeta_0)\|_{X_0}$ for all $t \in (0, T_3)$. Therefore, the proof is complete.

Finally, using the change of coordinates $\phi : D \to I \times I$ defined as in the proof of Lemma 3.3, we obtain by (4.4) that the approximate solutions $(v_n, \zeta_n)$ are uniformly bounded (together with all their space derivatives) on any compact subset of the domain $I \times I$. Thus, an application of the classical Arzelá-Ascoli theorem implies the existence of a real-analytic solution to (2.7)–(2.9) in the class $L^\infty(0, T, X_I) \cap L^1(0, T, (1 + \tau^{-2})Y_I)$ with radius of analyticity $\tau(t)$ given by (3.53).

**ACKNOWLEDGMENTS**

We thank Vlad Vicol for useful discussions. The authors were supported in part by the NSF grants DMS-115893, DMS-1009769, and DMS-1109562, respectively.

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