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The fractal dimension of the singular set for solutions of the Navier–Stokes system

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Abstract
We consider suitable weak solutions of the Navier–Stokes system in a bounded space-time domain $D$. We prove that the parabolic fractal dimension of the singular set is less than or equal to $135/82$. We also introduce the concept of the parabolic fractal measure $F_\alpha^p$ and prove that the fractal measure $F_{135/82}^p$ of the singular set is zero. For the Leray–Hopf weak solutions, we prove $F_{1/2}(\Sigma_T) = 0$, where $\Sigma_T$ denotes the set of singular times on $[0, T]$ and $F_{1/2}$ stands for the 1/2-dimensional fractal measure.

Mathematics Subject Classification: 35Q30, 76D05, 35K55

1. Introduction

In this paper, we address the partial regularity of solutions of the 3D Navier–Stokes equations

$$\begin{align*}
\partial_t u - \Delta u + \partial_j (u_j u) + \nabla p &= f \\
\nabla \cdot u &= 0.
\end{align*}$$

Classical results due to Leray and Hopf provide existence of weak solutions of the 3D Navier–Stokes equation provided the initial datum $u(\cdot, 0) = u_0$ belongs to $L^2$ and under suitable assumptions on the force (say $f \in L^2_t H^{-1}_x$) (cf [CF, Ho, Le, T]). It is not known, however, whether solutions may develop singularities even if the initial datum is smooth. In [S1, S2, S3], Scheffer initiated a study of partial regularity by estimating the size of the singular sets. In a classical paper [CKN], Caffarelli et al showed that the parabolic Hausdorff dimension of the possible singular set is at most 1. In [Li, V] (see also [KP, LS]), two alternative proofs were given, while in [K1, K2] (see also [K3]), a simpler proof was given under weaker assumptions on the force.

In order to obtain more information on the singular set, we address here the size of the singular sets as measured in terms of the fractal (also known as the box-counting or Kuratowski)
dimension (cf [EFNT, F]). The concept of fractal dimension is more restrictive than the concept of Hausdorff dimension since it is based on coverings of sets by balls of equal rather than variable radius. Namely, let $A$ be a relatively compact set in a metric space $(X, d)$. Then the fractal dimension of $A$ is defined by

$$\dim_f(A) = \limsup_{r \to 0^+} \frac{\log N(r)}{-\log r},$$

where $N(r)$ is the minimal number of balls of radius $r > 0$ needed to cover $A$.

In [RS1], Robinson and Sadowski proved that the set of singular times has the fractal dimension at most $1/2$, while in [RS2], they proved that the fractal dimension of the space/time singular set is at most $5/3$. (They used this fact to prove almost everywhere uniqueness of Lagrangian trajectories for arbitrary suitable weak solutions.) For $f = 0$, this is based on the following fact proven in [CKN, Li]: there exists a constant $\epsilon^* > 0$ such that if a suitable weak solution satisfies

$$\liminf_{r \to 0^+} \frac{1}{r^{5/3}} \int_{Q^r(x_0, t_0)} (|u|^{10/3} + |p|^{5/3}) \, dx \, dt \leq \epsilon^*,$$

where $Q^r(x_0, t_0) = B_r(x_0) \times (t_0 - r^2, t_0 + r^2)$ is the centred parabolic cube, then $(x_0, t_0)$ is regular (i.e. the solution $u$ is bounded in a neighbourhood). Since it is not known whether weak solutions satisfy $u \in L^{10/7}(D)$ for any $p > 10/3$, it is not clear how to lower the estimate on the fractal dimension below $5/3$. The main result of this paper (theorem 2.1 below) states that the fractal dimension of the singular set less than or equal to $135/82 < 5/3$. It remains an interesting question whether the fractal dimension of the singular set is at most one.

In the spirit of the parabolic Hausdorff measure, we have the concept of the parabolic fractal measure $\mathcal{F}_p^d$ (cf [EFNT, p 142]). In theorem 2.3, we prove that $\mathcal{F}_{135/82}^d(A) = 0$, where $A$ is any compact subset of the singular set. Using the concept of fractal measure, we show in theorem 2.10 that $\mathcal{F}_p^1(\Sigma) = 0$, where $\Sigma$ is any compact subset of the set of singular times. This provides a connection between the results of Scheffer [S2] (cf also [FT]) and Robinson–Sadowski [RS1].

2. Notation and the main theorem

Let $D \subseteq \mathbb{R}^3 \times \mathbb{R}$ be open, bounded and connected. A pair $(u, p)$ is a suitable weak solution of the Navier–Stokes equations if it satisfies

(i) $u \in L^\infty_t L^2_x(D) \cap L^3_t H^1_x(D)$ and $p \in L^{5/3}_t(D)$,

(ii) $f \in L^{10/7}_t(D)$ is divergence free,

(iii) the Navier–Stokes equations (1.1) are satisfied in $D$ and

(iv) the local energy inequality holds in $D$, i.e.

$$\int |u|^2 \phi_T + 2 \int_{\mathbb{R}^3 \times (-\infty,T]} |\nabla u|^2 \phi \leq \int_{\mathbb{R}^3 \times (-\infty,T]} (|u|^2 (\phi_t + \Delta \phi) + (|u|^2 + 2p) u \cdot \nabla \phi + 2(u \cdot f) \phi)$$

for all $\phi \in C_0^\infty(D)$ such that $\phi \geq 0$ in $D$ and all $T \in \mathbb{R}$.

The norms in $L^p_t L^q_x(D)$ are defined as follows: for a measurable function $f$ on $\mathbb{R}^3 \times \mathbb{R}$ and $1 \leq p, q \leq \infty$, we denote

$$\|f\|_{L^p_t L^q_x} = \|f(x, t)\|_{L^p_x(\mathbb{R}^3)} \|f(x, t)\|_{L^q_t(\mathbb{R}^3)}$$
Theorem 2.1. \( \dim \) dimension from \([K1, K2]\) that \( \dim \) dimension equal to zero \([EFNT]\).

Let \( A \) be a bounded subset of \( \mathbb{R}^3 \times \mathbb{R} \). Here we state the definition of the parabolic fractal dimension \( \dim_{pf}(A) \). For \( r > 0 \), denote by \( N(r) \) the minimal number of centred parabolic cubes \( Q^*_r(x, t) = B_r(x) \times (t - r^2, t + r^2) \) necessary to cover \( A \), i.e.

\[
N(r) = \min \left\{ N \in \mathbb{N}_0 : \exists (x_1, t_1), \ldots, (x_N, t_N) \in \mathbb{R}^3 \times \mathbb{R} \text{ such that } A \subseteq \bigcup_{j=1}^N Q^*_r(x_j, t_j) \right\}.
\]

Then the parabolic fractal dimension is defined by

\[
\dim_{pf}(A) = \limsup_{r \to 0} \frac{\log N(r)}{-\log r}.
\]

For example, we have \( \dim_{pf}(Q^*_r(0, 0)) = 5 \), \( \dim_{pf}(B_r(0) \times \{0\}) = 3 \) and \( \dim_{pf}([0] \times [0, 1]) = 2 \). Also, \( \dim_{pf}(\emptyset) = -\infty \). The classical fractal (or the box-counting) dimension \( \dim(f)(A) \) is defined by the same formula except that we use balls instead of centred parabolic cubes. (Alternatively, the parabolic fractal dimension is a special case of the fractal dimension when using the parabolic distance \( \text{dist}((x_1, t_1), (x_2, t_2)) = |x_1 - x_2| + \sqrt{t_1 - t_2} \). Clearly,

\[
\dim_{pf}(A) - \dim_{pf}(\emptyset) = \dim_{pf}(A).
\]

Note that the difference between the parabolic Hausdorff dimension \( \dim_{pH} \) (cf \([S1, S2, S3, CKN]\)) and the parabolic fractal dimension is that in the Hausdorff case, the size of the parabolic cubes is allowed to vary, while in the fractal case it stays fixed. In particular, \( \dim_{pH}(A) \leq \dim_{pf}(A) \) for all \( A \subseteq \mathbb{R}^3 \times \mathbb{R} \). The Hausdorff dimension may be a lot smaller than the fractal dimension. For instance, it is possible to construct a compact countable set of full fractal dimension, while all nonempty countable sets have the (parabolic) Hausdorff dimension equal to zero \([EFNT]\).

Let \( S \) denote the set of singular points for a suitable weak solution \( (u, p) \). (Recall from \([K1, K2]\) that \( (x_0, t_0) \in S \) if \( u \) does not belong to \( L^5_{\text{loc}}(V) \) for any neighbourhood \( V \) of \( (x_0, t_0) \).) A modification of an argument in \([CKN]\) implies that

\[
\dim_{pf}(S \cap K) \leq \frac{5}{3}
\]

for every compact subset \( K \) of \( D \). Our main result is stated next.

**Theorem 2.1.** Assume that \( f \in L^{5/3}(D) \), and let \( (u, p) \) be a suitable weak solution of the Navier–Stokes equations in \( D \subseteq \mathbb{R}^3 \times \mathbb{R} \). Then

\[
\dim_{pf}(S \cap K) \leq \frac{135}{82}
\]

for every compact subset \( K \) of \( D \).

For this purpose, it is sufficient to prove the following theorem.

**Theorem 2.2.** Assume that \( f \in L^{5/3}(D) \). There exists a sufficiently small universal constant \( \epsilon_* > 0 \) with the following property: if \( p \in (0, 1] \) and

\[
\int \int_{Q_r} |u|^{10/3} + \int \int_{Q_r} |p|^{5/3} + \int \int_{Q_r} |\nabla u|^2 + \int \int_{Q_r} |f|^{5/3} \leq \epsilon_* \rho^{135/82}
\]

then \( (0, 0) \) is a regular point.

Note that the regularity condition is not scale invariant (cf \([CKN]\)), and this is what allows an improvement of the estimate on the fractal dimension.
Proof of theorem 2.1 assuming theorem 2.2. Fix $K \subseteq D$ as in the statement, and let $r \in (0, \text{dist}(K, D'))$ be such that $r \leq 1$. We start with a Vitali-type argument adapted to the balls of equal radii. First, find disjoint cubes $Q^*_{r/3}(x_j, t_j)$, where $(x_1, t_1), \ldots, (x_n, t_n) \in S \cap K$, and assume that $n$ is the maximal integer with this property. By the maximality of $n$, we get

$$S \cap K \subseteq \bigcup_{j=1}^n Q^*_{r}(x_j, t_j)$$

(here we used the following: if $Q^*_{r/3}(x, t) \cap Q^*_{r/3}(x', t') \neq \emptyset$, then $Q^*_{r/3}(x', t') \subseteq Q^*_{r}(x, t)$). Hence,

$$n \geq N(r). \quad (2.5)$$

Now, by theorem 2.2, we have

$$\int \int Q^*_{r/3}(x_j, t_j) \left( |u|^{10/3} + |p|^{5/3} + |\nabla u|^2 + |f|^{5/3} \right) \geq \epsilon_* \left( \frac{r}{3} \right)^{135/82}$$

for every $j = 1, 2, \ldots, n$. Since the parabolic cubes $Q^*_{r/3}(x_j, t_j)$ are disjoint, we get

$$I = \int \int_D \left( |u|^{10/3} + |p|^{5/3} + |\nabla u|^2 + |f|^{5/3} \right)
\geq \sum_{j=1}^n \int \int Q^*_{r/3}(x_j, t_j) \left( |u|^{10/3} + |p|^{5/3} + |\nabla u|^2 + |f|^{5/3} \right)
\geq n \epsilon_* \left( \frac{r}{3} \right)^{135/82} \quad (2.6)$$

whence, by (2.5),

$$N(r) \leq n \leq \frac{I}{\epsilon_*(r/3)^{135/82}}$$

which implies \( \dim_{\text{pf}}(S \cap K) \leq 135/82 \). \qed

Recall that the partial regularity theorem in [CKN] not only gives that the parabolic Hausdorff measure is at most 1, but it also asserts that the parabolic Hausdorff length is zero. Here we have a similar result. Namely, for $d \in [0, n+2]$, a relatively compact set $A \subseteq \mathbb{R}^n \times \mathbb{R}$, and $\delta > 0$, define

$$\mathcal{F}^d_{\text{pf}}(A) = \sup_{r \in (0, A)} \inf_{n \in \mathbb{N}_0} \left\{ n^d : \exists n \in \mathbb{N}_0 \exists (x_1, t_1), \ldots, (x_n, t_n) \text{ such that } A \subseteq \bigcup_{j=1}^n Q^*_{r}(x_j, t_j) \right\} \quad (2.7)$$

and

$$\mathcal{F}^d_{\text{pf}}(A) = \lim_{\delta \to 0} \mathcal{F}^d_{\text{pf}}(A) = \inf_{\delta \to 0} \mathcal{F}^d_{\text{pf}}(A). \quad (2.8)$$

Note that for a bounded set $A$, we have $\dim_{\text{pf}}(A) = \inf\{d \leq n + 2 : \mathcal{F}^d_{\text{pf}}(A) = 0\}$. (A similar definition can be given for a relatively compact subset of any metric space.) Since the expression $I$ defined in (2.6) above can be made as small as we wish provided $D$ is a sufficiently small neighbourhood of $S \cap K$ (note that the Lebesgue measure of $S \cap K$ is zero), we obtain in addition to (2.3) the next statement.

**Theorem 2.3.** Under the assumptions of theorem 2.1, we have

$$\mathcal{F}^{135/82}_{\text{pf}}(S \cap K) = 0$$

for every compact subset $K$ of $D$. 
In order to prove theorem 2.2, we first describe the test function \( \phi \) used in the local energy inequality (2.2). First, let
\[
0 < r \leq \bar{\rho} \leq \bar{\rho} \quad (2.9)
\]
such that \( 4r \leq \bar{\rho} \) and \( 4\bar{\rho} \leq \rho \). Then choose
\[
\psi(x, t) = r^2 G(x, r^2 - t),
\]
where \( G \) is the heat kernel. Denote \( Q_r(x, t) = B_r(x) \times (t - r^2, t) \) and \( Q_r = Q_r(0, 0) \). Throughout the paper, fix \( \tilde{\eta} \in C_c^\infty(\mathbb{R}^3 \times [0, 1]) \) such that \( \tilde{\eta} = 1 \) on \( Q_{3/5} \) and \( \text{supp} \tilde{\eta} \subseteq Q_{4/5}^\ast \), where \( Q_r^\ast = Q_r(0, 0) \), and let
\[
\eta(x, t) = \tilde{\eta} \left( \frac{x}{\bar{\rho}}, \frac{t}{\bar{\rho}^2} \right).
\]
The test function used in the following is
\[
\phi(x, t) = \eta(x, t) \psi(x, t) \bar{\eta}(t),
\]
where \( \bar{\eta} \in C_c^\infty(\mathbb{R}, [0, 1]) \) is such that \( \bar{\eta} = 1 \) on \( (-\infty, r^2/4] \) and \( \bar{\eta} = 0 \) on \( [r^2/2, \infty) \).

First, we collect useful upper and lower bounds for the function \( \phi \). Let \( 0 < r \leq \bar{\rho}/2 \).

From [CKN], recall that
\[
\phi(x, t) \geq \frac{1}{Cr} \chi_q(x, t) \quad (x, t) \in Q_r
\]
while by [K1, K2] we have
\[
|\phi_t(x, t) + \Delta \phi(x, t)| \leq \frac{Cr^2}{\bar{\rho}^5} \quad (x, t) \in Q_{\bar{\rho}}.
\]

In addition, we need the following estimates on Green’s function.

**Lemma 2.4.** Let \( 0 < r \leq R \leq 2R \), where \( R \leq \bar{\rho} \). Then
\[
\psi(x, t) \leq \frac{Cr^2}{R^3}
\]
and
\[
|\nabla \psi(x, t)| \leq \frac{Cr^2}{R^4}
\]
for all \( (x, t) \in Q_{2R} \setminus Q_R \).

**Proof of lemma 2.4.** For \( (x, t) \in (B_{2R} \setminus B_{R}) \times (-R^2, 0) \), we have
\[
\psi(x, t) = \frac{r^2}{(4\pi (r^2 - t))^{3/2}} \exp \left( -\frac{|x|^2}{4(r^2 - t)} \right) \leq \frac{1}{r} \exp \left( -\frac{R^2}{4r^2} \right) \leq \frac{Cr^2}{R^3}
\]
while for \( (x, t) \in B_{2R} \setminus (-4R^2, -R^2) \), we have
\[
\psi(x, t) \leq \frac{r^2}{(r^2 - t)^{3/2}} \leq \frac{Cr^2}{R^3}
\]
and (2.13) is proven. The inequality (2.14) is obtained analogously. \( \square \)

For the proof of theorem 2.2, we need several auxiliary results.

**Lemma 2.5.** Let \( 10/3 \leq p \leq 6 \) and \( 2 \leq q \leq 10/3 \) be such that \( 2/q + 3/p = 3/2 \). Then
\[
\|u\|_{L_q^{1/2} L_p^{1/2}(Q_r)} \leq C \|u\|_{L_q^{10/3} L_p^{10/3}(Q_r)}^{15/2p - 5/4} \|\nabla u\|_{L_q^{10/3} L_p^{10/3}(Q_r)}^{15/2p + 3/2} + C \|u\|_{L_q^{10/3} L_p^{10/3}(Q_r)} \quad (2.15)
\]
for all \( u \in L_q^{10/3} L_p^{10/3}(Q_r) \) such that \( \nabla u \in L_q^{10/3} L_p^{10/3}(Q_r) \). In addition, if \( \int_{Q_r} u(\cdot, t) = 0 \) for almost all \( t \in [-r^2, 0] \), then the second term on the right side of (2.15) may be omitted.
For any $v \in L^1(B_1)$, denote
$$A_r(v) = \frac{1}{|B_r|} \int_{B_r} v.$$ If $p$ and $q$ are as in the statement of lemma 2.5, then we have by lemma 2.5
$$\|u - A_r(u)\|_{L^p_t L^q_x(B_t)} \leq C \|u\|_{H^1(B_1)} \|\nabla u\|_{L^p_t L^q_x(B_t)}$$
for all $u \in L^\infty_t L^2_x(Q_r)$ such that $\nabla u \in L^p_t L^q_x(Q_r)$.

**Proof of lemma 2.5.** For every $t \in [-r^2, 0]$, we have by the Gagliardo–Nirenberg inequality
$$\|u(\cdot, t)\|_{L^p(B_r)} \leq C \|u(\cdot, t)\|_{L^{10/3}(B_r)}^{15/2p-5/4} \|\nabla u(\cdot, t)\|_{L^2(B_r)}^{9/4-15/2p} + \frac{C}{p^{7/3}} \|u(\cdot, t)\|_{L^{10/3}(B_r)}$$
and the second term may be omitted if $\int_{B_r} u(\cdot, t) = 0$. The rest follows by applying Hölder’s inequality in $t$.

**Lemma 2.6.** Let $5/3 \leq p \leq 15/7$ and $5/4 \leq q \leq 5/3$ be such that $2/q + 3/p = 3$. Then
$$\|u^2 - A_r(u^2)\|_{L^p_t L^q_x(Q_r)} \leq C \|u\|_{H^1(B_1)} \|\nabla u\|_{L^p_t L^q_x(B_t)}$$
for all $u$ such that the right side is finite.

**Proof of lemma 2.6.** Let $v$ be sufficiently regular function on $Q_r$ such that $\int_{B_r} v(\cdot, t) = 0$ for all $t \in [-r^2, 0]$. Then for all $t \in [-r^2, 0]$, we have
$$\|v(\cdot, t)\|_{L^p(B_r)} \leq C \|v(\cdot, t)\|_{L^{10/3}(B_r)}^{15/2p-7/2} \|v(\cdot, t)\|_{L^2(B_r)}^{9/2-15/2p}$$
and thus
$$\|u\|_{L^p_t L^q_x(Q_r)} \leq C \|u\|_{L^{10/3}(B_r)}^{15/2p-7/2} \|\nabla u\|_{L^2_t L^{3/2}(Q_r)}^{9/2-15/2p}$$
from where, using $\|u^2 - A_r(u^2)\|_{L^p_t L^q_x(B_t)} \leq C \|u\|_{L^{10/3}(B_r)}^{15/2p-7/2} \|\nabla u^2\|_{L^2_t L^{3/2}(Q_r)}^{9/2-15/2p}$
$$\|u^2 - A_r(u^2)\|_{L^p_t L^q_x(Q_r)} \leq C \|u^2 - A_r(u^2)\|_{L^{10/3}(B_r)}^{15/2p-7/2} \|\nabla u^2\|_{L^2_t L^{3/2}(Q_r)}^{9/2-15/2p}$$
and the inequality (2.16) follows.

Let $(x_0, t_0) \in D$ and $r > 0$ be such that $Q_r(x_0, t_0) \subseteq D$. Denote
$$a_{(x_0, t_0)}(r) = \frac{1}{r^{1/2}} \|u\|_{L^p_t L^q_x(Q_r(x_0, t_0))},$$
$$b_{(x_0, t_0)}(r) = \frac{1}{r^{1/2}} \|\nabla u\|_{L^p_t L^q_x(Q_r(x_0, t_0))},$$
$$\Gamma_{(x_0, t_0)}(r) = \frac{1}{r^{1/2}} \|u\|_{L^{10/3}_t L^2_x(Q_r(x_0, t_0))},$$
$$\Delta_{(x_0, t_0)}(r) = \frac{1}{r^{1/2}} \|p\|_{L^{1/2}_t L^2_x(Q_r(x_0, t_0))}.$$ If the label $(x_0, t_0)$ is omitted, it is understood to be $(0, 0)$. Next, we state an important pressure estimate.
Lemma 2.7. Let $0 < R \leq \rho/2$. Then

$$
\|p\|_{L_{t}^{1,0} L_{x}^{1,5/3}(Q_{R})} \leq C \rho \beta(\rho)^{1/2} \Gamma(\rho)^{3/2} + \frac{C R^{9/5}}{\rho^{3/5}} \Delta(\rho)^{2}
$$

(2.17)

provided $Q_{\rho} \subseteq D$.

It is not immediate that the left side of (2.17) is finite—this follows from the proof below.

Proof of lemma 2.7. For $i, j \in \{1, 2, 3\}$, denote

$$
U_{ij} = -(u_{i} - A_{\rho}(u)) (u_{j} - A_{\rho}(u_{j})).
$$

Let $\eta \in C_{0}^{\infty}(\mathbb{R}^{3} \times \mathbb{R}, [0, 1])$ be such that $\eta \equiv 1$ on $Q_{3\rho/4}$ and $\eta \equiv 0$ on $Q_{\rho} \cap \{(x, t) : t < 0\}$ with

$$
|\partial_{\rho}^{a} \partial_{\rho}^{b} \eta(x, t)| \leq \frac{C(a, b)}{\rho^{2a+|\beta|}} (x, t) \in \mathbb{R}^{3} \times \mathbb{R} \quad a \in \mathbb{N}_{0} \quad \beta \in \mathbb{N}^{3}.
$$

Using $\Delta p = -\partial_{j} (u_{i} u_{j}) = \partial_{j} U_{ij}$ and $U_{ij} = U_{ji}$, we get

$$
\Delta(\eta p) = \partial_{j} (\eta U_{ij}) + U_{ij} \partial_{j} \eta - 2 \partial_{j} (U_{ij} \partial_{i} \eta) - p \Delta \eta + 2 \partial_{j} (p \partial_{j} \eta)
$$

and thus

$$
\eta p = -R_{i} R_{j} (\eta U_{ij}) + N * (U_{ij} \partial_{i} \eta) - 2 \partial_{j} N * (U_{ij} \partial_{i} \eta) - N * (p \Delta \eta) + 2 \partial_{j} N * (p \partial_{j} \eta)
$$

$$
= p_{1} + p_{2} + p_{3} + p_{4} + p_{5},
$$

where $N$ is the Newtonian potential and $R_{i}$ is the $i$th Riesz transform. Regarding $p_{1}$, we have by the Calderón–Zygmund theorem

$$
\|p_{1}\|_{L_{t}^{2} L_{x}^{2}(Q_{R})} \leq \|p_{1}\|_{L_{t}^{15/8} L_{x}^{15/8}(\mathbb{R}^{3} \times (-r^{2}, 0))} \leq C \sum_{i=1}^{3} \|\eta U_{ij}\|_{L_{t}^{15/8} L_{x}^{15/8}(Q_{R})}
$$

$$
\leq C \sum_{i=1}^{3} \|U_{ij}\|_{L_{t}^{15/8} L_{x}^{15/8}(Q_{R})} \leq C \sum_{i=1}^{3} \|u_{i} - A_{\rho}(u_{i})\|_{L_{t}^{20/7} L_{x}^{2}(Q_{R})}^{2}
$$

$$
\leq C \|u\|_{L_{t}^{20/7} L_{x}^{2}(Q_{R})}^{3/2} \|\nabla u\|_{L_{t}^{2} L_{x}^{2}(Q_{R})}^{1/2}
$$

where we used lemma 2.5 in the last step. Therefore,

$$
\|p_{1}\|_{L_{t}^{20/7} L_{x}^{2}(Q_{R})} \leq C r^{1/2} \rho \beta(\rho)^{1/2} \Gamma(\rho)^{3/2}.
$$

For $p_{2}$, we use the estimates as in [K1] (exploiting that the convolution defining $p_{2}$ is not singular, cf [CKN, K1, L]). Namely,

$$
\|p_{2}\|_{L_{t}^{10/7} L_{x}^{3/2}(Q_{R})} \leq C r^{8/5} \|p_{2}\|_{L_{t}^{10/7} L_{x}^{3/2}(Q_{R})} \leq \frac{C r^{8/5}}{\rho^{3/5}} \sum_{i,j=1}^{3} \|U_{ij}\|_{L_{t}^{10/7} L_{x}^{3/2}(Q_{R})}
$$

$$
\leq \frac{C r^{8/5}}{\rho^{3/5}} \|u - A_{\rho}(u)\|_{L_{t}^{20/7} L_{x}^{2}(Q_{R})}^{2} \leq \frac{C r^{8/5}}{\rho^{3/5}} \|u - A_{\rho}(u)\|_{L_{t}^{20/7} L_{x}^{2}(Q_{R})}^{2}.
$$

Using lemma 2.5, we then obtain

$$
\|p_{2}\|_{L_{t}^{10/7} L_{x}^{3/2}(Q_{R})} \leq \frac{C r^{8/5}}{\rho^{3/5}} \|u\|_{L_{t}^{20/7} L_{x}^{2}(Q_{R})}^{3/2} \|\nabla u\|_{L_{t}^{2} L_{x}^{2}(Q_{R})}^{1/2} \leq \frac{C r^{8/5}}{\rho^{3/5}} \beta(\rho)^{1/2} \Gamma(\rho)^{3/2}.
$$
which is a better bound than that for $p_1$. The estimate for $p_3$ is the same as the one for $p_2$ leading to the same upper bound. In order to bound $p_4$, we use similar arguments as for $p_2$ and get

$$
\|p_4\|_{L^{9/7}_tL^{15/8}(Q_{\rho})} \leq C \rho^{9/5} \|p_4\|_{L^{7/5}_tL^{7/4}(Q_{\rho})} \leq C \rho^{9/5} \|p\|_{L^{7/5}_tL^{7/4}(Q_{\rho})} \leq C \rho^{9/5} \|p\|_{L^{9/7}_tL^{15/8}(Q_{\rho})}.
$$

The bound for $p_5$ is the same as the one for $p_4$, and the lemma follows by collecting the inequalities for $p_1$, $p_2$, $p_3$, $p_4$ and $p_5$. \hfill \Box

**Lemma 2.8.** For every $t_0 \in (-r^2, 0)$, we have

$$
\left| \int \int_{[t \leq t_0]} |u|^2 u_j \partial_j \phi \right| \leq C \rho^{3/2} \frac{\Gamma(\rho)^{5/2}}{r} \beta(\rho)^{1/2},
$$

where $\phi$ is as in (2.10).

Note an improvement of the exponent of $(\rho/r)^{3/2}$ compared with [CKN, K1]. This is done by slicing the region of integration into dyadic cubes and exploring the decay of the heat kernel.

**Proof of lemma 2.8.** Let $\tilde{\eta}$ and $\eta$ be as in the paragraph before lemma 2.4. Denote

$$
\eta_0(x, t) = \tilde{\eta} \left( \frac{x}{2r}, \frac{t}{(2r)^2} \right)
$$

and

$$
\eta_m(x, t) = \tilde{\eta} \left( \frac{x}{2^{m+1}r}, \frac{t}{(2^{m+1}r)^2} \right) - \tilde{\eta} \left( \frac{x}{2^mr}, \frac{t}{(2^mr)^2} \right)
$$

for $m = 1, 2, \ldots$ By the telescopic property of the functions $\eta_m$, we have

$$
\phi = \sum_{m=0}^{m_0} \phi \eta_m = \sum_{m=0}^{m_0} \phi \eta_m,
$$

where $m_0$ is the largest integer $m$ such that

$$
\frac{3 \cdot 2^{m_0}r}{5} < \frac{4 \tilde{\rho}}{5}.
$$

In particular, we have $2^m \leq 2 \tilde{\rho}/r$ and $\eta_m \equiv 0$ for $m \geq m_0 + 1$. Note that $\eta_0 \equiv 1$ on $Q_{3r/5}$ and that supp $\eta_0 \subseteq Q_{5r/5}$. Also,

$$
\eta_m \equiv 0 \quad (x, t) \in Q_{3r/5} \setminus \left( Q_{4r} \cap \{ t \leq 0 \} \right).
$$

We write

$$
I = \int \int \|u|^2 u_j \partial_j \phi = \sum_{m_0}^{m_0} \int \int \|u|^2 u_j \partial_j (\psi \eta_m) = \sum_{m_0}^{m_0} I_m.
$$

First, consider $I_0 = \int \int \|u|^2 u_j \partial_j (\psi \eta_0) = \int \int \|u|^2 - A_{2\Gamma}(\|u\|^2)) u_j \partial_j (\psi \eta_0)$. Then, by lemmas 2.4 and 2.6,

$$
|I_0| \leq \frac{C}{r^{3/2}} \|u|^2 - A_{2\Gamma}(\|u\|^2) \|L^{9/7}_tL^{15/8}(Q_{\rho}) \|L^{10/9}_tL^{15/8}(Q_{\rho})
$$

$$
\leq \frac{C}{r^{3/2}} \|u|^{3/2} L^{9/7}_tL^{15/8}(Q_{\rho}) \|\nabla u|^{1/2} L^{10/9}_tL^{15/8}(Q_{\rho}) \|u|^{5/2} L^{15/8}_tL^{15/8}(Q_{\rho}) \|\nabla u|^{1/2} L^{10/9}_tL^{15/8}(Q_{\rho}).
$$

For $m \in \{1, \ldots, m_0\}$, we have

$$
I_m = \int \int \|u|^2 u_j \partial_j (\psi \eta_m) = \int \int (\|u|^2 - A_{2\Gamma}(\|u\|^2)) u_j \partial_j (\psi \eta_m)
$$
and thus, using lemmas 2.4 and 2.6,

\[ |I_m| \leq \frac{C}{(2^m r)^2} \| u \|^2 - A_{2^m r}(\| u \|^2) \| L_{L_t}^{m/3} L_{x}^{15/4} (Q_{2^m r}) \| u \| L_{L_t}^{m/3} L_{x}^{15/4} (Q_{2^m r}) \]

\[ \leq \frac{C}{(2^m r)^{3/2}} \| u \|^{3/2} L_{L_t}^{m/3} (Q_{2^m r}) \| \nabla u \|^{1/2} L_{L_t}^{1/2} (Q_{2^m r}) \| u \|^{10/7} L_{x}^{1/2} (Q_{2^m r}) \]

\[ \leq \frac{C}{(2^m r)^{3/2}} \| u \|^{5/2} L_{L_t}^{m/3} (Q_{r}) \| \nabla u \|^{1/2} L_{L_t}^{1/2} (Q_{r}) \| u \|^{10/7} L_{x}^{1/2} (Q_{r}). \]

The lemma then follows by summing up the series. \( \square \)

**Lemma 2.9.** For every \( t_0 \in (-r^2, 0) \), we have

\[ I = \int \int_{\{t \leq t_0\}} p u_j \partial_j \phi \leq C \rho^{3/2} \beta(\rho) \Gamma(\rho)^{3/2} + C \beta^{3/10} \rho^{3/10} \Delta(\rho)^2 \Gamma(\rho), \]

where \( \phi \) is as in (2.10).

**Proof of lemma 2.9.** As in the previous lemma, we write

\[ \int \int_{\{t \leq t_0\}} p u_j \partial_j \phi = \sum_{m=0}^{m_0} \int \int_{\{t \leq t_0\}} p u_j \partial_j (\psi \eta_m) = \sum_{m=0}^{m_0} I_m. \]

First, by lemmas 2.4 and 2.7,

\[ |I_0| \leq \frac{C}{r} \| p \| L_{L_t}^{m/3} (Q_{r}) \| u \| L_{L_t}^{m/3} (Q_{r}) \]

\[ \leq \frac{C}{r^{3/2}} \| p \| L_{L_t}^{m/3} L_{x}^{15/4} (Q_{r}) \| u \| L_{L_t}^{m/3} (Q_{r}) \]

\[ \leq \frac{C}{r^{3/2}} \left( \rho \beta(\rho) \Gamma(\rho)^{3/2} + \frac{r^{9/5}}{\rho^{3/5}} \Delta(\rho)^2 \right) \| u \| L_{L_t}^{m/3} (Q_{r}), \]

Similarly, as in the previous proof, we have for \( 1 \leq m \leq m_0, \)

\[ |I_m| \leq \frac{C}{(2^m r)^2} \| p \| L_{L_t}^{m/3} (Q_{2^m r}) \| u \| L_{L_t}^{m/3} (Q_{2^m r}) \]

\[ \leq \frac{C}{(2^m r)^{3/2}} \| p \| L_{L_t}^{m/3} L_{x}^{15/4} (Q_{2^m r}) \| u \| L_{L_t}^{m/3} (Q_{2^m r}) \]

\[ \leq \frac{C}{(2^m r)^{3/2}} \left( \rho \beta(\rho) \Gamma(\rho)^{3/2} + \frac{(2^m r)^{9/5}}{\rho^{3/5}} \Delta(\rho)^2 \right) \| u \| L_{L_t}^{m/3} (Q_{2^m r}), \]

where we used lemma 2.7. Therefore,

\[ |I_m| \leq \frac{C}{(2^m r)^{3/2}} \rho \beta(\rho) \Gamma(\rho)^{3/2} \| u \| L_{L_t}^{m/3} (Q_{2^m r})^2 + \frac{C(2^m r)^{3/10}}{\rho^{3/5}} \Delta(\rho)^2 \| u \| L_{L_t}^{m/3} (Q_{2^m r}), \]

\[ \leq \frac{C}{(2^m r)^{3/2}} \rho \beta(\rho) \Gamma(\rho)^{3/2} \| u \| L_{L_t}^{m/3} (Q_{r})^2 + \frac{C(2^m r)^{3/10}}{\rho^{3/5}} \Delta(\rho)^2 \| u \| L_{L_t}^{m/3} (Q_{r}), \]

Summing up the bounds on \( |I_m| \) for \( m = 0, 1, \ldots, m_0 \), we get

\[ |I| \leq \frac{C \rho}{r^{3/2}} \rho \beta(\rho) \Gamma(\rho)^{3/2} \| u \| L_{L_t}^{m/3} (Q_{r}) + \frac{C \beta^{3/10}}{\rho^{3/5}} \Delta(\rho)^2 \| u \| L_{L_t}^{m/3} (Q_{r}), \]

and the lemma follows. \( \square \)
Proof of theorem 2.2. Let $0 < r < \tilde{\rho} < \rho$ be such that $4r \leq \tilde{\rho}$ and $4\tilde{\rho} \leq \rho$. Using $\phi$ from (2.10) in the local energy inequality, we obtain for all $t_0 \in [-r^2, 0]$

\[
\int |u|^2 \phi |_{t_0} + 2 \int \int_{\mathbb{R}^3 \times (-\infty, -t_0]} |\nabla u|^2 \phi \leq \int \int_{\mathbb{R}^3 \times (-\infty, -t_0]} \left( |u|^2 (\phi_t + \Delta \phi) + (|u|^2 + 2p)u \cdot \nabla \phi + 2(u \cdot f) \phi \right)
\]

\[
\leq C r^2 \frac{\rho^2}{\rho^5} \|u\|^2_{L_{t,x}^{1,0}(Q_\rho)} + C \frac{r^{3/2}}{\rho^{3/2}} \|u\|^{5/2}_{L_{t,x}^{10,3}(Q_\rho)} \|\nabla u\|_{L_{t,x}^{3,0}(Q_\rho)}^{1/2}
\]

\[
+ C \frac{\rho^{3/10}}{r^{3/10}} \|u\|_{L_{t,x}^{10,3}(Q_\rho)} \|p\|_{L_{t,x}^{3,0}(Q_\rho)} + C \frac{\rho^{1/2}}{r} \|u\|_{L_{t,x}^{10,3}(Q_\rho)} \|f\|_{L_{t,x}^{3,0}(Q_\rho)}.
\]

by lemmas 2.8 and 2.9. Using Hölder’s inequality on the first term on the far right and noting that

\[
\int |u|^2 \phi |_{t_0} \geq \frac{1}{Cr} \|u(\cdot, t_0)\|^2_{L_t^2(B_1)}
\]

and

\[
\int \int_{\mathbb{R}^3 \times (-\infty, 0]} |\nabla u|^2 \phi \geq \frac{1}{Cr} \|\nabla u\|^2_{L_{t,x}^{1,0}(Q_\rho)},
\]

which holds by lemma 2.4, we obtain

\[
\alpha (r)^2 + \beta (r)^2 \leq \frac{C r^2}{\rho^5} \|u\|^2_{L_{t,x}^{1,0}(Q_\rho)} + C \frac{r^{3/2}}{\rho^{3/2}} \|u\|^{5/2}_{L_{t,x}^{10,3}(Q_\rho)} \|\nabla u\|_{L_{t,x}^{3,0}(Q_\rho)}^{1/2}
\]

\[
+ C \frac{\rho^{3/10}}{r^{3/10}} \|u\|_{L_{t,x}^{10,3}(Q_\rho)} \|p\|_{L_{t,x}^{3,0}(Q_\rho)} + C \frac{\rho^{1/2}}{r} \|u\|_{L_{t,x}^{10,3}(Q_\rho)} \|f\|_{L_{t,x}^{3,0}(Q_\rho)}.
\]

Now, fix $\rho \in (0, 1]$ and then choose

\[
r = \frac{1}{16} \rho^{45/41}
\]

and

\[
\tilde{\rho} = \frac{1}{4} \rho^{87/82}.
\]

Then

\[
\alpha \left(\frac{1}{16} \rho^{45/41}\right)^2 + \beta \left(\frac{1}{16} \rho^{45/41}\right)^2 \leq \frac{C}{\rho^{87/82}} \|u\|^2_{L_{t,x}^{1,0}(Q_\rho)} + \frac{C}{\rho^{135/82}} \|u\|^{5/2}_{L_{t,x}^{10,3}(Q_\rho)} \|\nabla u\|_{L_{t,x}^{3,0}(Q_\rho)}^{1/2}
\]

\[
+ \frac{C}{\rho^{243/164}} \|u\|_{L_{t,x}^{10,3}(Q_\rho)} \|p\|_{L_{t,x}^{3,0}(Q_\rho)} + \frac{C}{\rho^{49/82}} \|u\|_{L_{t,x}^{10,3}(Q_\rho)} \|f\|_{L_{t,x}^{3,0}(Q_\rho)}.
\]

(2.18)

Now, assume (2.4), i.e.

\[
\|u\|^2_{L_{t,x}^{10,3}(Q_\rho)} \|p\|_{L_{t,x}^{3,0}(Q_\rho)} \|\nabla u\|^2_{L_{t,x}^{3,0}(Q_\rho)} + \|f\|_{L_{t,x}^{3,0}(Q_\rho)} \leq \epsilon \rho^{135/82}.
\]

Then

\[
\|u\|_{L_{t,x}^{10,3}(Q_\rho)} \leq \rho^{81/164} \epsilon^{3/10},
\]

\[
\|p\|_{L_{t,x}^{3,0}(Q_\rho)} \leq \rho^{81/82} \epsilon^{3/5},
\]

\[
\|\nabla u\|_{L_{t,x}^{3,0}(Q_\rho)} \leq \rho^{135/164} \epsilon^{1/2},
\]

\[
\|f\|_{L_{t,x}^{3,0}(Q_\rho)} \leq \rho^{81/82} \epsilon^{3/5}.
\]
and we obtain
\[ \alpha \left( \frac{1}{16} \rho^{45/41} \right)^2 + \beta \left( \frac{1}{16} \rho^{45/41} \right)^2 \leq C (\epsilon_*^{3/5} + \epsilon_* + \epsilon_*^{9/10} + \epsilon_*^{145/164}) \leq C (\epsilon_*^{3/5} + \epsilon_*). \]

(2.19)

If \( \epsilon_* \in (0, 1) \) is sufficiently small, the condition (2.19) together with \( \| f \|_{L^2_{x,t}}(Q_{r^*}, u, \mu) \leq \epsilon_* \) implies that \( (0, 0) \) is a regular point by [K1, K2].

Using the fractal measure, it is possible to connect the results in [RS1] and [FT, S2] so that the fractal result implies the estimate on the time singularities by Scheffer [S2]. It was proven in [RS1] that \( \dim_{f}(\Sigma_T) \leq 1/2 \).

**Theorem 2.10.** Let \( u \) be a weakly continuous Leray–Hopf weak solution of the Navier–Stokes system on \([0, T]\), where \( T \in (0, \infty) \), with the periodic or Dirichlet boundary conditions. Denote by \( \Sigma_T = \{ t \in [0, T] : u(\cdot, t) \in H^1 V \} \) the set of singular times. Then \( \mathcal{F}^{1/2}(\Sigma_T) = 0 \).

The fractal measure \( \mathcal{F}^d \) is defined by \( \mathcal{F}^d(A) = \lim_{\delta \to 0} \mathcal{F}^{d, \delta}(A) \), where
\[
\mathcal{F}^{d, \delta}(A) = \sup_{r \in (0, \delta)} \inf_{\mathcal{O}} \left\{ \lambda^d : \exists n \in \mathbb{N}_0 \exists t_1, t_2, \ldots, t_n \text{ such that } A \subseteq \bigcup_{j=1}^N Q_r(x_j, t_j) \right\}.
\]

**Proof of theorem 2.10.** First, observe that if
\[
\int_{r^* - r}^{r^* + r} \| u(\cdot, t) \|^2_V dt \leq \epsilon_* r^{1/2}
\]
for some \( r > 0 \) where \( \epsilon_* \) is a sufficiently small constant, then \( t_0 \notin \Sigma_T \), i.e. \( t_0 \) is not a singular time. This holds since (2.20) implies the existence of \( t_1 \in [t_0 - r, t_0 - r/2] \) such that \( \| u(\cdot, t) \|^2_V \leq \epsilon_* r^{1/2} \), and this implies the regularity of \( u \) on \( (t_0, t_0 + r) \) if \( \epsilon_* > 0 \) is a sufficiently small constant.

Now, let \( \mathcal{O} \) be an open set containing \( \Sigma_T \) such that \( \int_{\mathcal{O}} \| u(\cdot, t) \|^2_V dt \leq \epsilon \), and choose \( r_0 > 0 \) such that \( (t - r_0, t + r_0) \subseteq \mathcal{O} \).

Next, let \( \{ (t_j - r/3, t_j + r/3) \}_{j=1}^n \), where \( 0 < r \leq r_0 \), be disjoint intervals such that \( t_j \in \Sigma_T \).

Assume that \( n \) is the largest integer with this property. Then
\[
\int_{t_j - r/3}^{t_j + r/3} \| u(\cdot, t) \|^2_V dt > \epsilon_* \left( \frac{r}{3} \right)^{1/2},
\]
from where
\[
n \epsilon_* \left( \frac{r}{3} \right)^{1/2} \leq \int_{\mathcal{O}} \| u(\cdot, t) \|^2_V dt \leq \epsilon
\]
which implies \( n \leq C \epsilon / \epsilon_* r^{1/2} \). On the other hand, by the maximality of \( n \), the collection \( \{ (t_j - r, t_j + r) \}_{j=1}^n \) covers \( \Sigma_T \) and \( n \epsilon^{1/2} \leq C \epsilon / \epsilon_* \). Therefore \( \mathcal{F}^{1/2}(\Sigma_T) \leq C \epsilon / \epsilon_* \) with \( \epsilon > 0 \) arbitrary.

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References


