The coverage problem for loitering Dubins vehicles

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***Abstract*— In this paper we study a facility location problem for groups of Dubins vehicles, i.e., nonholonomic vehicles that are constrained to move along planar paths of bounded curvature, without reversing direction. Given a compact region and a group of Dubins vehicles, the coverage problem is to minimize the worst-case traveling time from any vehicle to any point in the region. Since the vehicles cannot hover, we assume that they fly along static closed curves called *loitering curves*. The paper presents *circular loitering patterns* for a Dubins vehicle and for a group of Dubins vehicles that minimize the worst-case traveling time in sufficiently large regions. We do this by establishing an analogy to the disk covering problem.**

I. Introduction

One of the prototypical missions for Uninhabited Aerial Vehicles, e.g., in environmental monitoring, security, or military setting, is wide-area surveillance. A low-altitude UAV in such a mission must provide coverage of a certain region and investigate events of interest ("targets") as they manifest themselves. In particular, we are interested in cases in which close-range information is required on targets detected by high-altitude aircraft, spacecraft, or ground spotters, and the UAVs must proceed to the location of the detected targets to gather on-site information.

Variations of problems falling in this class have been studied in a number of papers in the recent past, e.g., see [1], [2], [3], [4]. In these papers, the problem is set up in such a way that the location of targets is known a priori and a strategy is computed that attempts to optimize the coverage cost of servicing the known targets. Coordination algorithms for distributed sensing task were proposed and analyzed in [5]. A limitation of the results presented in [5] is the fact that omni-directional or locally controllable vehicles were considered in the problem formulation. Because of this assumption, the results are not applicable to many vehicles of interest, such as aircraft and car-like robots.

In [6] we presented the results of our work for designing closed tours through a set of given points for a nonholonomic vehicle that is constrained to move along planar paths of bounded curvature, without reversing direction. This model is also known as the Dubins vehicle in literature [7]. Path planning for groups of Dubins vehicles has gained considerable interest in recent past [8], [9].

This material is based upon work supported in part by AFOSR MURI Award F49620-02-1-0325.

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In contrast to *simpler* vehicles [5] which can wait at a single location while they are idle, Dubins vehicles have to *loiter* while they are waiting for targets to appear in the region. As a consequence, we need to characterize the configuration of the vehicles at the appearance of new targets in terms of Dubins paths, that we will call *loitering patterns*.

The main contributions of this paper are as follows. First, we study the reachable set of Dubins vehicle and characterize some of its properties that are particularly useful for the problem at hand. Most importantly, we introduce a certain "covering problem" where a circle or a sector with given parameters is to be contained in the Dubins reachable set of minimal time. Second, we characterize optimal circular loitering for a single Dubins vehicle by exploiting the rotational symmetry of the problem and the simple-connectedness of the Dubins reachable set. Third, we design efficient circular loitering patterns for a single team of multiple Dubins vehicle and provide a bound on the achievable performance for sufficiently large environments. Finally, we consider the case of multiple teams composed of the same number of vehicles. We propose a computational approach to computing loitering patterns based on (1) partitioning the environment into Voronoi partitions generated by virtual centers, (2) moving the virtual centers in such a way as to solve a minimum-radius disk-covering problem, and (3) designing efficient loitering patterns for each team in its corresponding Voronoi cell.

The paper is organized as follows. In Section II, we setup the problem and introduce notations that will be used in the rest of the paper. Section III carries a discussion on the reachable set of the Dubins vehicle. The single vehicle and the single team case are considered in Sections IV and V, respectively. In Section VI, we consider the multiple uniform team case. Finally, we conclude with a few remarks about future work in Section VII.

II. Problem Setup and notations

In this section we setup the main problem of the paper and review some basic required notation. A *Dubins vehicle* is a planar vehicle that is constrained to move along paths of bounded curvature, without reversing direction and maintaining a constant speed. We will design loitering patterns for n Dubins vehicle with unlimited sensing range in a compact region $\mathcal{Q} \subset \mathbb{R}^2$. Given a duration $T > 0$, let $\gamma : [0, T] \to \mathbb{R}^2$ be a *closed feasible curve for the Dubins vehicle* or a *closed Dubins path*, i.e., γ is a curve that is twice differentiable almost anywhere, $\|\gamma'(t)\| = 1$ for all $t \in [0, T]$, and the magnitude of the curvature of γ is bounded above by $1/\rho$, where $\rho > 0$ is the minimum turning radius and $\gamma(0) = \gamma(1)$. The configuration of the Dubins vehice traversing the curve γ will be denoted by $g_\gamma(t)$, where $g_\gamma(t) = (\gamma(t), \mathrm{ArcTan}(\gamma'(t))) \in \mathrm{SE}(2)$, where $\mathrm{SE}(2)$ is the special Euclidean group of dimension 2. Let $\Gamma_\rho = \{\gamma \mid \gamma \text{ is a closed Dubins path}\}$. The loitering curves that are designed in this paper belong to Γ_ρ.

Given n vehicles, a *team composition* can be represented as $\{m_1, \ldots, m_n\}$, where $m_i \in \mathbb{N} \cup \{0\}$ and $\sum_{i=1}^n m_i = n$, where $\mathbb{N}$ is the set of all natural numbers. let $\mathcal{M}(n)$ denote the set of all such possible team compositions. In particular, if there are $\ell \leq n$ teams, then the team composition will be given by $\{m_1, \ldots, m_\ell, 0, \ldots, 0\}$. The idea is to partition $\mathcal{Q}$ into ℓ sub-regions such that each team is responsible for one sub-region. Given ℓ teams, let $\Lambda = (\gamma_1, \ldots, \gamma_\ell) \in \Gamma_\rho^\ell$ be a set of closed Dubins path for the teams. These curves will represent the *loitering curves* for the Dubins vehicle. In this paper we will be concerned with minimizing the worst case traveling time by the *closest* Dubins vehicle to any arbitrary (unknown) target point in $\mathcal{Q}$. Since we constrain the vehicles to move at constant (unit) speed along the curves, one can prove by symmetry that the vehicles that are part of the same team are equally spaced along their common loitering curve and move in the same direction (i.e., clockwise/counter-clockwise). Therefore, given a region $\mathcal{Q}$ and a team composition $M = (m_1, \ldots, m_\ell, 0, \ldots, 0)$, Λ completely specifies the loitering *pattern*.

We now define the coverage cost associated with a given loitering pattern. Let $L_\rho : \mathrm{SE}(2) \times \mathbb{R}^2 \to \mathbb{R}_+$ be the length of the shortest Dubins path from initial position and orientation described by an element of $\mathrm{SE}(2)$ to a point $q \in \mathbb{R}^2$, where $\mathbb{R}$ is the set of real numbers and $\mathbb{R}_+$ is the set of positive real numbers. Recall that L_ρ is continuous almost everywhere [10].

Definition 2.1 (Coverage cost): Given a region $\mathcal{Q}$, a team composition M, and a loitering pattern $\Lambda = (\gamma_1, \ldots, \gamma_\ell)$ with durations $(T_1, \ldots, T_\ell)$, define the *coverage cost* associated with the loitering pattern by

$$\mathcal{T}_{\mathcal{Q},M}(\Lambda) := \sup_{q \in \mathcal{Q}} \min_{i \in \{1,\ldots,\ell\}} \sup_{s \in [0, \frac{T_i}{m_i})} \min_{j \in \{1,\ldots,m_i\}} L_\rho\Big(g_{\gamma_i}\Big(s + \frac{(j-1)T_i}{m_i}\Big), q\Big).$$

The coverage cost gives the worst-case traveling time from any vehicle to any point in the region. In the rest of the paper, we will use *coverage cost* and *cost* interchangeably. The minimum cost associated with the given region $\mathcal{Q}$ and team composition M is defined by

$$\mathcal{T}^*_{\mathcal{Q},M} := \inf_{\Lambda \in \Gamma_\rho^\ell} \mathcal{T}_{\mathcal{Q},M}(\Lambda).$$

Finally, the minimum cost associated with the given region $\mathcal{Q}$ is defined as

$$\mathcal{T}^{**}_{\mathcal{Q}} := \min_{M \in \mathcal{M}(n)} \mathcal{T}^*_{\mathcal{Q},M}.$$

In general, the optimal loitering patterns will have to be computed based on the shape of the region $\mathcal{Q}$. However, we will concentrate on *circular loitering patterns*; the rationale for doing so is that it (simplifies the problem and) allows us to provide algorithms and bounds that are independent of the particular shape of the environment. Furthermore, it seems unlikely that UAVs in the field will be able to compute optimal loitering patterns as their assigned regions change in real time; on the other hand, determining the location of the center, and the radius of a circular loitering patterns are much easier tasks.

For a given center $c \in \mathbb{R}^2$, radius $r \in \mathbb{R}_+$, let $\mathcal{O}^+(c, r) : [0, T] \to \mathbb{R}^2$ represent a circular curve of radius r with center c with counter-clockwise orientation. Similarly let $\mathcal{O}^-(c, r)$ represent one with clockwise orientation. Since we will be concentrating only on circular curves, with a slight abuse of notation, we shall use Γ_ρ to denote the set the circular curves with radii greater than or equal to ρ, i.e.,

$$\Gamma_\rho = \{\mathcal{O}^+(c, r) \mid r \geq \rho\} \cup \{\mathcal{O}^-(c, r) \mid r \geq \rho\}.$$

Accordingly, define a *sub-minimum* cost associated with the given region $\mathcal{Q}$ and team composition M as:

$$\widetilde{\mathcal{T}}^*_{\mathcal{Q},M} := \inf_{\Lambda \in \Gamma_\rho^\ell} \mathcal{T}_{\mathcal{Q},M}(\Lambda), \tag{1}$$

where the set of loitering curves is now a set of circular curves with centers at $c_1, \ldots, c_\ell$ and radii $r_1, \ldots, r_\ell$.

We are now ready to formulate the problem statement: Given n Dubins vehicles with known team composition, design circular loitering patterns that minimize the cost function given by eq. (1).

We need to define a few more notations and concepts. Consider a point $c \in \mathbb{R}^2$ and $r \in \mathbb{R}_+$, let $C(c, r)$ be the circle with center at c and of radius r. For a region $U \subset \mathbb{R}^2$, let $U^{2\pi}(c)$ be the annulus traced by U as it rotated through a 2π angle about the point c, i.e.,

$$U^{2\pi}(c) = \cup_{q \in U} C(c, \|q - c\|).$$

Let $\mathcal{B}_c(r)$ be the closed ball of radius r and centered at c. Given a set of angles $\alpha \in [0, 2\pi)$, $\triangle\alpha \in [0, 2\pi]$, let $\mathcal{S}_c(r, \alpha, \triangle\alpha)$ be the sector traced by a segment of length r and fixed at c as it rotates from the angle α to the angle $\alpha + \triangle\alpha$ in the counter-clockwise direction. With this notation, $\mathcal{B}_c(r) = \mathcal{S}_c(r, 0, 2\pi)$. Let $EB_c(U)$ be the minimum ball enclosing U centered at c, i.e., $EB_c(U) = \mathcal{B}_c(\sup_{q \in U} \|q - c\|)$, where $\|.\|$ represents the Euclidean norm. Let $REB_c(U) = \sup_{q \in U} \|q - c\|$ be the radius of the enclosing ball $EB_c(U)$. Since circumball is the smallest of all the enclosing balls, we will give it a special notation. Accordingly, for the region U, let $\mathrm{D}(U)$ be the circumball, $\mathrm{R}(U)$ be the circumradius and $\mathrm{C}(U)$ be the circumcenter of U. Finally, the symbol $\mathbf{I} \in \mathrm{SE}(2)$ will represent the identity element of the $\mathrm{SE}(2)$ group. Specifically, $\mathbf{I}$ will correspond to that state of the Dubins vehicle where it is positioned at the origin and its heading is aligned with the positive X axis.

III. Reachable set for the Dubins vehicle

In this section we state some properties of the Dubins *reachable set* which shall be useful in the due course of

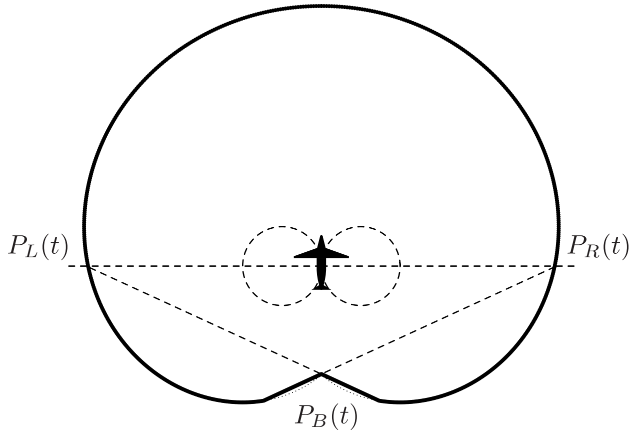

Fig. 2. Truncation of $\mathcal{R}_h(t)$ to form $\widetilde{\mathcal{R}}_h(t)$.

the paper. Given $t \geq 0$ and the current configuration of the Dubins vehicle $h \in \mathrm{SE}(2)$, let $\mathcal{R}_h(t)$ denote the reachable set of the Dubins vehicle in time t starting with state h, i.e.,

$$\mathcal{R}_h(t) = \{q \in \mathbb{R}^2 \mid L_\rho(h, q) \leq t\}.$$

Reachable sets for the Dubins vehicle are shown in Fig. 1. The boundary of the reachable sets consist of arcs of circle involutes and arcs of epicycloids (for further details on these families of curves see, e.g., [11]). We shall also use a *slightly* truncated version of $\mathcal{R}_h(t)$ for *sufficiently large* t. We will denote this set by $\widetilde{\mathcal{R}}_h(t)$. For the sake of clarity we explain the construction of $\widetilde{\mathcal{R}}_h(t)$ from $\mathcal{R}_h(t)$ with the help of Fig. 2 as follows: Consider the axis that is perpendicular to the heading of the Dubins vehicle. Let this axis intersect the boundary of $\mathcal{R}_h(t)$ at $P_L(t)$ and $P_R(t)$. Let $P_B(t)$ be the furthest point that lies exactly *behind* the Dubins vehicle. Let $H_L(t)$ be the half-plane generated by the line passing through $P_L(t)$ and $P_B(t)$ that does not contain the origin. Similarly, let $H_R(t)$ be the half-plane generated by the line passing through $P_R(t)$ and $P_B(t)$ that does not contain the origin. Then $\widetilde{\mathcal{R}}_h(t) = \mathcal{R}_h(t) \setminus (H_L(t) \cap H_R(t))$.

Using the definition of the reachable sets and planar geometry, one can prove that the following properties hold true for any $h \in \mathrm{SE}(2)$.

(P1) $\mathcal{R}_h(t)$ is a monotonic function in t, i.e., $\mathcal{R}_h(t') \subseteq \mathcal{R}_h(t)$ for $t' \leq t$.

There exist constants $\kappa_1 \in [5.7, 5.8]$ and $\kappa_2 \in [6.5, 6.6]$ such that

(P2) $\mathcal{R}_h(t)$ is a simply connected set for all $t \in \mathbb{R}_+ \setminus [\kappa_1\rho, \kappa_2\rho]$.

(P3) For all $t \geq \kappa_2\rho$, $\widetilde{\mathcal{R}}_h(t)$ is star-shaped[1] and the kernel[2] of $\widetilde{\mathcal{R}}_h(t)$ is the set of points that lie on the axis which is perpendicular to the heading direction of the vehicle at h.

[1]A region U is called star-shaped if there is a point $a \in U$ such that the line segment $\bar{ab}$ is contained in U for all $b \in U$. Here $\bar{ab} = \{ta + (1-t)b \mid t \in [0,1]\}$. We then say that U is star-shaped with respect to a.

[2]The kernel of a star-shaped region U is the set of points from which the entire set U is visible.

Next, we introduce a "set covering problem" that will play a key role in the design of efficient loitering patterns. For $m > 0$, define a function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\lambda_m(t) := \max\Big\{r \in \mathbb{R}_+ \mid \exists y \geq \rho,\ \alpha \in [0, 2\pi) \text{ s.t. } \mathcal{S}_{(0,y)}\Big(r, \alpha, \frac{2\pi}{m}\Big) \subset \mathcal{R}_{\mathbf{I}}(t)\Big\}.$$

The function $t \mapsto \lambda_m(t)$ is the radius of the largest sector extending an angle $\frac{2\pi}{m}$ that is centered on the Y axis and away from the point $(0, \rho)$ and can be contained inside $\mathcal{R}_{\mathbf{I}}(t)$. In particular, $\lambda_1(t)$ denotes the radius of the largest disk centered on the Y axis away from $(0, \rho)$ and contained inside $\mathcal{R}_{\mathbf{I}}(t)$.

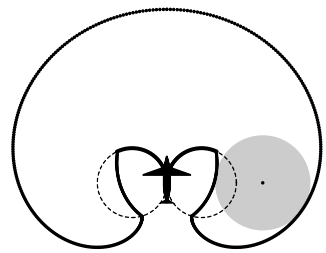

Fig. 3. Finding the value $\lambda_1(5\rho)$.

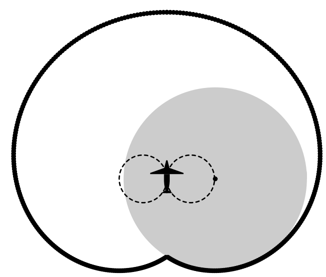

Fig. 4. Finding the value $\lambda_1(7\rho)$.

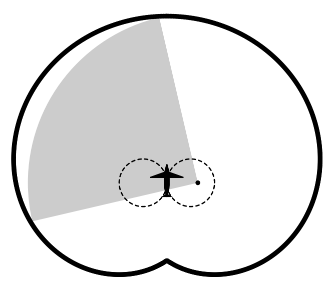

Fig. 5. Finding the value $\lambda_4(7\rho)$.

One can show that, at fixed m, the function $t \mapsto \lambda_m(t)$ is a strictly increasing function in t. This can also be verified from Figs. 6 and 7 where we have plotted $\lambda_1(t)/\rho$ vs. t/ρ and $\lambda_2(t)/\rho$ vs. t/ρ respectively. Hence, the inverse function λ_m^{-1} is also well defined and satisfies

$$\lambda_m^{-1}(r) := \min\Big\{t \mid \exists y \geq \rho,\ \alpha \in [0, 2\pi) \text{ s.t. } \mathcal{S}_{(0,y)}\Big(r, \alpha, \frac{2\pi}{m}\Big) \subset \mathcal{R}_{\mathbf{I}}(t)\Big\}.$$

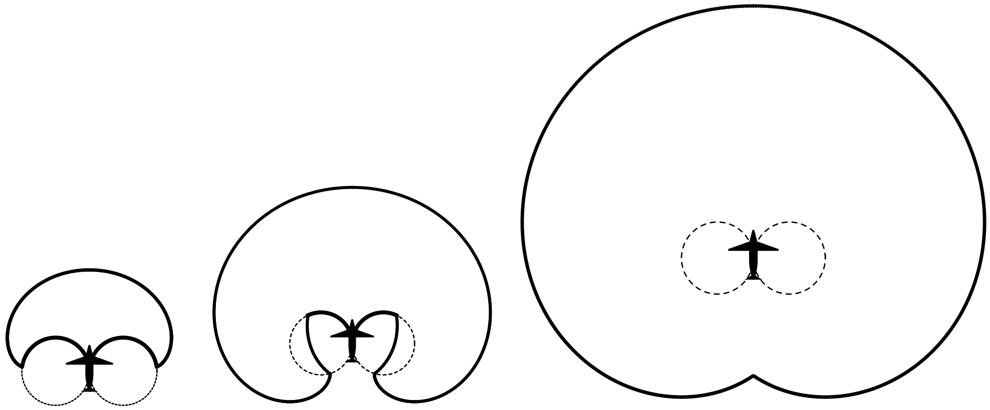

Fig. 1. Reachable sets $\mathcal{R}_{\mathbf{I}}(t)$ for the Dubins vehicle for $t = 3\rho$, 5ρ and 7ρ.

For each $m > 0$, $\lambda_m(t)$ is associated with functions

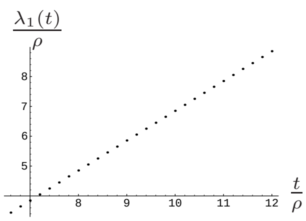

Fig. 6. Plot of $\lambda_1(t)/\rho$ vs. t/ρ.

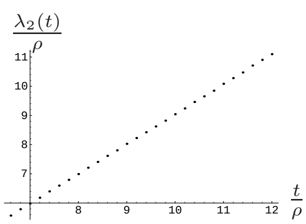

Fig. 7. Plot of $\lambda_2(t)/\rho$ vs. t/ρ.

$\delta_m : \mathbb{R}_+ \to \mathbb{R}_+$ and $\beta_m : \mathbb{R}_+ \to [0, 2\pi)$ which are defined to satisfy the relation $\mathcal{S}_{(0,\delta_m(t))}\Big(r, \beta_m(t), \frac{2\pi}{m}\Big) \subset \mathcal{R}_{\mathbf{I}}(t)$ for all $t \in \mathbb{R}_+$. The functions δ_m and β_m can be computed numerically along with λ_m.

IV. The single vehicle case

In this section we concentrate our attention on the case when $M = (1)$, i.e., only one vehicle is assigned the task to service the region $\mathcal{Q}$. It is easy to see that $\mathcal{T}_{\mathcal{Q},1}(\mathcal{O}^+(c,r)) = \mathcal{T}_{\mathcal{Q},1}(\mathcal{O}^-(c,r))$ for any region $\mathcal{Q}$. Hence, for the rest of the section, we shall omit the explicit mention of the direction of rotation for circular curves.

Lemma 4.1 (Equivalence by rotation): For a region $\mathcal{Q}$, $\rho > 0$, $c \in \mathbb{R}^2$ and radius $r \geq \rho$,

$$\mathcal{T}_{\mathcal{Q},1}(\mathcal{O}(c,r)) = \mathcal{T}_{\mathcal{Q}^{2\pi}(c),1}(\mathcal{O}(c,r)).$$

Proof: For a point $q \in \mathcal{Q}$, define the function $\tau(q) = \sup_{s\in[0,T)} L_\rho(g_{\mathcal{O}(c,r)}(s), q)$. This definition implies that $\tau(q)$ is the minimum t such that q belongs to $\mathcal{R}_{g_{\mathcal{O}(c,r)}(s)}(t)$ for all $s \in [0,T)$. Consider any other point q' (not necessarily in $\mathcal{Q}$) such that $\|q' - c\| = \|q - c\|$. By rotational symmetry about the center c, given any $s \in [0,T)$, one can always find a $s' \in [0,T)$ such that $L_\rho(g_{\mathcal{O}(c,r)}(s'), q') = L_\rho(g_{\mathcal{O}(c,r)}(s), q)$, i.e., q' also belongs to $\mathcal{R}_{g_{\mathcal{O}(c,r)}(s')}(\tau(q))$ for all $s' \in [0,T)$. This implies that the circle of radius $\|c - q\|$ and centered at c belongs to $\mathcal{R}_{g_{\mathcal{O}(c,r)}(s)}(\tau(q))$ for all $s \in [0,T)$. Taking the union over all $q \in \mathcal{Q}$, one arrives at the lemma. ■

Lemma 4.2: For a region $\mathcal{Q}$, $\rho > 0$, $c \in \mathbb{R}^2$ and radius $r \geq \rho$,

$$\mathcal{T}_{\mathcal{Q},1}(\mathcal{O}(c,r)) = \min\{t \in \mathbb{R}_+ \mid \\ \cup_{q\in\mathcal{Q}}\, C((0,r), \|c - q\|) \subset \mathcal{R}_{\mathbf{I}}(t)\}.$$

Proof: Let $\mathcal{T}_{\mathcal{Q},1}(\mathcal{O}(c,r)) = t^*$. Lemma 4.1 implies that $\cup_{q\in\mathcal{Q}} C(c, \|c - q\|)$ belongs to $\mathcal{R}_{g_{\mathcal{O}(c,r)}(s)}(t^*)$ for all $s \in [0,T)$. This property when viewed in the reference frame attached to the Dubins vehicle gives the lemma. ■

Lemma 4.3 (Equivalence): For any region $\mathcal{Q}$, $\rho > 0$, $c \in \mathbb{R}^2$ and radius $r \geq \rho$, if $\mathcal{T}_{\mathcal{Q},1}(\mathcal{O}(c,r)) \in \mathbb{R}_+ \setminus [\kappa_1\rho, \kappa_2\rho]$, then

$$\mathcal{T}_{\mathcal{Q},1}(\mathcal{O}(c,r)) = \mathcal{T}_{EB_c(\mathcal{Q}),1}(\mathcal{O}(c,r)).$$

Proof: Let $\mathcal{T}_{\mathcal{Q},1}(\mathcal{O}(c,r)) = t^*$. Lemma 4.2 combined with the simply connectedness property (P2) of the reachable set implies that a disk of radius $\max_{q\in\mathcal{Q}} \|c - q\|$ centered at $(0,r)$ belongs to $\mathcal{R}_{\mathbf{I}}(t^*)$ if $t^* \in \mathbb{R}_+ \setminus [\kappa_1\rho, \kappa_2\rho]$. An equivalent statement is that for values of time in $\mathbb{R}_+ \setminus [\kappa_1\rho, \kappa_2\rho]$, t^* is the smallest t such that $EB_c(\mathcal{Q}) \subset \mathcal{R}_{g_{\mathcal{O}(c,r)}(s)}(t)$ for all $s \in [0,T)$, that is, $\mathcal{T}_{EB_c(\mathcal{Q}),1}(\mathcal{O}(c,r)) = t^* = \mathcal{T}_{\mathcal{Q},1}(\mathcal{O}(c,r))$. ■

We are now ready to state the main result of this section.

Theorem 4.4 (An optimal circular loitering): Given a region $\mathcal{Q}$ for which $\mathrm{R}(\mathcal{Q}) \in \mathbb{R}_+ \setminus [\lambda_1(\kappa_1\rho), \lambda_1(\kappa_2\rho)]$, the circle of radius $\delta_1\big(\lambda_1^{-1}(\mathrm{R}(\mathcal{Q}))\big)$ with center at $\mathrm{C}(\mathcal{Q})$ is an optimal circular loitering curve over $\mathcal{Q}$. Moreover,

$$\widetilde{\mathcal{T}}^*_{\mathcal{Q},1} = \lambda_1^{-1}\big(\mathrm{R}(\mathcal{Q})\big) \\ = \mathcal{T}_{\mathcal{Q},1}\big(\mathcal{O}(\mathrm{C}(\mathcal{Q}), \delta_1\big(\lambda_1^{-1}(\mathrm{R}(\mathcal{Q}))\big))\big).$$

Proof: We shall consider the case when $\mathrm{R}(\mathcal{Q}) \geq \kappa_2\rho$. The proof for the case when $\mathrm{R}(\mathcal{Q}) \leq \kappa_1\rho$ follows on similar lines. From the definition of λ_1, $\mathrm{R}(\mathcal{Q}) \geq \kappa_2\rho$ implies that

$$\min\{t \in \mathbb{R}_+ \mid \mathcal{B}_{(0,r)}(\mathrm{R}(\mathcal{Q})) \subset \mathcal{R}_{\mathbf{I}}(t)\} \geq \kappa_2\rho.$$

Since $\mathrm{D}(\mathcal{Q})$ is the minimum of all the enclosing balls of $\mathcal{Q}$, we also have that

$$\min\{t \in \mathbb{R}_+ \mid EB_{(0,r)}(REB_c(\mathcal{Q})) \subset \mathcal{R}_{\mathbf{I}}(t)\} \geq \kappa_2\rho.$$

The closedness and the simply connectedness property of $\mathcal{R}_{\mathbf{I}}(t)$ for $t \geq \kappa_2\rho$ implies that

$$\min\{t \in \mathbb{R}_+ \mid C((0,r), REB_c(\mathcal{Q})) \subset \mathcal{R}_{\mathbf{I}}(t)\} \geq \kappa_2\rho.$$

This combined with Lemma 4.2 gives us that

$$\begin{aligned}&\mathcal{T}_{\mathcal{Q},1}(\mathcal{O}(c,r))\\ &= \min\{t \in \mathbb{R}_+ \mid \cup_{q\in\mathcal{Q}} C((0,r), \|c-q\|) \subset \mathcal{R}_{\mathbf{I}}(t)\} \geq \kappa_2\rho.\end{aligned}$$

Since $\mathcal{T}_{\mathcal{Q},1}(\mathcal{O}(c,r)) \geq \kappa_2\rho$, the previous discussion combined with Lemma 4.3 implies that

$$\mathcal{T}_{\mathcal{Q},1}(\mathcal{O}(c,r)) = \mathcal{T}_{EB_c(\mathcal{Q}),1}(\mathcal{O}(c,r)) \geq \mathcal{T}_{\mathrm{D}(\mathcal{Q}),1}(\mathcal{O}(\mathrm{C}(\mathcal{Q}),r)).$$

This proves that the location of the center of rotation for an optimal circular loitering curve is at the circumcenter of $\mathcal{Q}$.

Therefore,

$$\begin{aligned}\widetilde{\mathcal{T}}^*_{\mathcal{Q},1} &= \min_{c\in\mathbb{R}^2,\ r\geq\rho} \mathcal{T}_{\mathcal{Q},1}(\mathcal{O}(c,r))\\ &= \min_{r\geq\rho} \mathcal{T}_{\mathrm{D}(\mathcal{Q}),1}(\mathcal{O}(\mathrm{C}(\mathcal{Q}),r))\\ &= \min\{t \in \mathbb{R}_+ \mid \mathcal{B}_{(0,r)}(\mathrm{R}(\mathcal{Q})) \subset \mathcal{R}_{\mathbf{I}}(t) \text{ for some } r \geq \rho\}\\ &= \lambda_1^{-1}(\mathrm{R}(\mathcal{Q})).\end{aligned}$$

The fact that $\delta_1\big(\lambda_1^{-1}(\mathrm{R}(\mathcal{Q}))\big)$ is the radius of an optimal circular loitering curve follows from the definition of δ. This also proves the second equality in the theorem. ■

Remark 4.5 (Circular loitering patterns are optimal): Although we have been restricting our attention on circular loitering curves, one can prove that, for the single vehicle case, an optimal circular loitering curve is also an optimal loitering curve, i.e.,

$$\begin{aligned}\mathcal{T}^*_{\mathcal{Q},1} &= \lambda_1^{-1}(\mathrm{R}(\mathcal{Q}))\\ &= \mathcal{T}_{\mathcal{Q},1}(\mathcal{O}(\mathrm{C}(\mathcal{Q}), \delta_1\big(\lambda_1^{-1}(\mathrm{R}(\mathcal{Q}))\big))).\end{aligned}$$

We omit this statement's proof in the interest of space. □

V. The single team case

In this section we design a loitering circle for a team of n Dubins vehicles servicing the region $\mathcal{Q}$, i.e., $M = (n, 0, \ldots, 0)$. For brevity in notation, we shall denote this team composition by $M = (n)$. By symmetry, the n vehicles will be placed at an angular distance of $\frac{2\pi}{n}$ from each other. Once again, the direction of the rotation on the curves does not matter. Hence, we shall continue to omit the mention of the direction of rotation for circular curves.

Lemma 5.1 (Equivalence by rotation): For a region $\mathcal{Q}$ and $n > 1$,

$$\mathcal{T}_{\mathcal{Q},n}(\mathcal{O}(c,r)) = \mathcal{T}_{\mathcal{Q}^{2\pi}(c),n}(\mathcal{O}(c,r)).$$

Proof: For the sake of this proof we will interpret $\mathcal{T}_{\mathcal{Q},1}(\mathcal{O}(c,r))$ as the minimum t for which $\mathcal{Q}$ belongs to $\cup_{j\in\{1,\ldots,n\}} \mathcal{R}_{g_{\mathcal{O}(c,r)}(s+(j-1)\frac{T}{n})}(t)$ for all $s \in [0, \frac{T}{n})$. With a slight abuse of notation, for the fixed circular loitering curve $\mathcal{O}(c,r)$, define the function τ by $\tau(q) = \mathcal{T}_{q,n}(\mathcal{O}(c,r))$. This definition implies that $\tau(q)$ is the smallest t which satisfies the property that, given a $s \in [0, \frac{T}{n})$, there exists at least one vehicle $j \in \{1,\ldots,n\}$ such that q belongs to $\mathcal{R}_{g_{\mathcal{O}(c,r)}(s+(j-1)\frac{T}{n})}(t)$. Consider any other point q' (not necessarily in $\mathcal{Q}$) such that $\|q'-c\| = \|q-c\|$. By rotational symmetry about the center c, one can find $j' \in \{1,\ldots,n\}$ such that $L_\rho(g_{\mathcal{O}(c,r)}(s+(j'-1)\frac{T}{n}), q') = L_\rho(g_{\mathcal{O}(c,r)}(s+(j-1)\frac{T}{n}), q)$, i.e., q' also belongs to $\cup_{j\in\{1,\ldots,n\}} \mathcal{R}_{g_{\mathcal{O}(c,r)}(s+(j-1)\frac{T}{n})}(t)$. This implies that the circle of radius $\|c-q\|$ and centered at c belongs to $\cup_{j\in\{1,\ldots,n\}} \mathcal{R}_{g_{\mathcal{O}(c,r)}(s+(j-1)\frac{T}{n})}(t)$ for all $s \in [0, \frac{T}{n})$. Taking the union over all $q \in \mathcal{Q}$, one arrives at the lemma. ■

Given a set $\{g_1,\ldots,g_n\} \subset \mathrm{SE}(2)$ of n distinct positions and orientations, the *Dubins Voronoi partition* generated by $\{g_1,\ldots,g_n\}$ is the collection of sets $\{V_1,\ldots,V_n\}$ defined by

$$V_i = \{q \in \mathbb{R}^2 \mid L_\rho(g_i,q) \leq L_\rho(g_j,q) \text{ for all } j \in \{1,\ldots,n\}\}.$$

We refer to V_i as the *Dubins Voronoi cell* of g_i. For our case, we shall denote the Voronoi partitions in a different way. Given n Dubins vehicles equally spaced along a circle of center c and radius r and all moving counterclockwise, let $\mathcal{V}(g_i,c,r,n) \subset \mathbb{R}^2$ be the Dubins Voronoi cell of the ith Dubins vehicle at state $g_i \in \mathrm{SE}(2)$.

Lemma 5.2: For a region $\mathcal{Q}$ and $n > 1$,

$$\begin{aligned}&\mathcal{T}_{\mathcal{Q},n}(\mathcal{O}(c,r)) = \min\{t \in \mathbb{R}_+ \mid\\ &\qquad \mathcal{V}(g_{\mathcal{O}(c,r)}(s),c,r,n) \cap \mathcal{Q}^{2\pi}(c) \subset \mathcal{R}_{g_{\mathcal{O}(c,r)}(s)}(t)\},\end{aligned}$$

for any $s \in [0,T)$.

Proof: Lemma 5.1 implies that $\mathcal{T}_{\mathcal{Q},n}(\mathcal{O}(c,r)) = \mathcal{T}_{\mathcal{Q}^{2\pi}(c),n}(\mathcal{O}(c,r))$. $\mathcal{T}_{\mathcal{Q}^{2\pi}(c),n}(\mathcal{O}(c,r))$ can be interpreted as the minimum t for which $\mathcal{Q}^{2\pi}(c)$ belongs to $\cup_{j\in\{1,\ldots,n\}} \mathcal{R}_{g_{\mathcal{O}(c,r)}(s+(j-1)\frac{T}{n})}(t)$ for all $s \in [0, \frac{T}{n})$. The lemma then follows from the rotational symmetry about the center c and the definition of Dubins Voronoi partition. ■

Lemma 5.2 suggests how to compute the optimal circular trajectory for a team of Dubins vehicles by converting it into an optimization problem for a single vehicle. However, solving this optimization problem requires the knowledge of the shape of Dubins Voronoi partitions. Even though there is an element of rotational symmetry in our case, the shapes of the Dubins Voronoi partition are not easy enough to lend themselves to analysis. Hence, we shall approximate the Voronoi partitions by sectors. This approximation helps in deriving upper bounds on the cost function.

We are now ready to state the main result of the section.

Theorem 5.3: (An upper bound on the coverage cost for a single team in large environments) For $n > 1$ and for a region $\mathcal{Q}$ with $\mathrm{R}(\mathcal{Q}) \geq \lambda_n(\kappa_2\rho)$,

$$\begin{aligned}\widetilde{\mathcal{T}}^*_{\mathcal{Q},n} &\leq \lambda_n^{-1}(\mathrm{R}(\mathcal{Q})) \\ &\leq \mathcal{T}_{\mathcal{Q},n}(\mathcal{O}(\mathrm{C}(\mathcal{Q}), \delta_n(\lambda_n^{-1}(\mathrm{R}(\mathcal{Q}))))).\end{aligned}$$

Proof: In the interest of space, we will only sketch the proof here. Let $\widetilde{\mathcal{T}}^*_{\mathcal{Q},n} = t^*$. On the lines similar to the proof of Theorem 4.4, one can show that $\mathrm{R}(\mathcal{Q}) \geq \lambda_n(\kappa_2\rho)$ implies that $t^* > \kappa_2\rho$. Lemma 5.2 implies that

$$t^* = \min\{t \mid \mathcal{V}(g_{\mathcal{O}(c,r)}(s), c, r, n) \cap \mathcal{Q}^{2\pi}(c) \subset \mathcal{R}_{g_{\mathcal{O}(c,r)}(s)}(t)\},$$

for any $s \in [0, T)$. Changing the reference frame to the one attached to the Dubins vehicle and approximating the Dubins Voronoi partitions by sectors one can state that t^* is the minimum t such that $\mathcal{S}_{(0,r)}\Big(REB_c(\mathcal{Q}), \alpha, \frac{2\pi}{n}\Big) \setminus \mathcal{S}_{(0,r)}\Big(\min_{q\in\mathcal{Q}} \|q - c\|, \alpha, \frac{2\pi}{n}\Big) \subset \mathcal{R}_{\mathbf{I}}(t)$ for any $\alpha \in [0, 2\pi)$. Since $t^* > \kappa_2\rho$, $\widetilde{\mathcal{R}}_{\mathbf{I}}(t)$ is star-shaped and simply connected for all $t > t^*$. Also $\widetilde{\mathcal{R}}_{\mathbf{I}}(t)$ is an inner approximation of $\mathcal{R}_{\mathbf{I}}(t)$. Combining these observations with the definition of λ_n^{-1}, one arrives at the lemma. ■

Remark 5.4: The bound obtained in Theorem 5.3 is tightest among the bounds possible by approximations of Dubins Voronoi partitions for vehicles moving along circular curves by sectors of circles.

VI. The multiple uniform team case

In this section we consider the multiple team case. We first concentrate on the case when the teams have *uniform* composition. A group of n vehicles comprising of ℓ teams is said to have uniform team composition if n is a multiple of ℓ and the team composition is of the form $(\frac{n}{l}, \dots, \frac{n}{l}, 0, \dots, 0)$. We shall show that, for a *sufficiently large and convex* region $\mathcal{Q}$, an upper bound on the cost of coverage by the l team of loitering Dubins vehicles can be obtained by solving a related disk covering problem.

We first briefly describe the disk-covering problem or, more precisely, the version of the disk covering problem that is relevant for our purposes here. In our context, the disk covering problem can be stated as follows: given a convex region $\mathcal{Q}$ and an integer ℓ, find the smallest real number $RDC_{\mathcal{Q}}(\ell)$ and a set of locations $\{c_1, \dots, c_\ell\}$ such that the ℓ disks, each of radius $RDC_{\mathcal{Q}}(\ell)$ and centered at $\{c_1, \dots, c_\ell\}$ cover $\mathcal{Q}$, that is, $\mathcal{Q} \subset \cup_{i\in\{1,\dots,\ell\}} \mathcal{B}_{c_i}(RDC_{\mathcal{Q}}(\ell))$. We shall refer to $\big(RDC_{\mathcal{Q}}(\ell), \{c_1, \dots, c_\ell\}\big)$ as the solution to the disk covering problem for $\mathcal{Q}$.

Disk covering problems have a long and beautiful history [12]. Many variants of the problem (e.g., geometric minimum disk cover problem) find their applications in numerous engineering applications (e.g., localization in sensor networks).

In [13] distributed algorithms were designed to solve the disk covering problem via a dynamical systems approach. Specifically, the paper proposes the *move toward the furthest* and *move toward the circumcenter* algorithms for a group of ℓ mobile robots. In the *move toward the furthest* algorithm, each "disk center" moves towards the furthest vertex of its Voronoi cell (inside the Voronoi partition generated by all "disk centers"). In the '*move toward the circumcenter*' algorithm, each disk-center moves toward the circumcenter of its Voronoi cell. In both algorithms the Voronoi partition is continuously updated as the disk centers move. Asymptotically, an execution of one of these two algorithms computes a locally optimal solution to the disk covering problem in the sense that the location of these robots correspond to the centers $c_1, \dots, c_\ell$ and the largest of the circumradii of the Voronoi partitions corresponds to $RDC_{\mathcal{Q}}(\ell)$. Moreover, these distributed control laws can be implemented as local interactions between the disk centers. In our setting, this would imply that this would require interaction only between *neighboring* teams of vehicles, i.e., teams whose center of rotations are Voronoi neighbors. An execution of the *move toward the circumcenter* algorithm is illustrated in Figure 8.

We now state the following result which gives an upper bound on the coverage cost for multiple uniform teams of loitering Dubins vehicles.

Theorem 6.1: Consider a group of n Dubins vehicles divided into ℓ teams of uniform composition loitering in a convex region $\mathcal{Q}$. Let $\big(RDC_{\mathcal{Q}}(\ell), \{c_1, \dots, c_\ell\}\big)$ be the solution to the disk covering problem for $\mathcal{Q}$. If $\mathrm{Area}(\mathcal{Q}) \geq \ell\pi\lambda^2_{\frac{n}{\ell}}(\kappa_2\rho)$, then

$$\widetilde{\mathcal{T}}^*_{\mathcal{Q},(\frac{n}{\ell},\dots,\frac{n}{\ell},0,\dots,0)} \leq \lambda^{-1}_{\frac{n}{\ell}}\big(RDC_{\mathcal{Q}}(\ell)\big).$$

Moreover, the loitering pattern which achieves this upper bound is the set of circular curves, each of radius $\delta_{\frac{n}{\ell}}\big(\lambda^{-1}_{\frac{n}{\ell}}\big(RDC_{\mathcal{Q}}(\ell)\big)\big)$, and with centers at $\{c_1, \dots, c_\ell\}$.

Using the control algorithms from [13], one can design a computational approach to computing loitering patterns as follows:

(i) Partition the environment into Voronoi partitions generated by virtual centers.
(ii) Move the virtual centers in such a way as to solve a minimum-radius disk-covering problem
(iii) Designing efficient loitering patterns for each team in its corresponding Voronoi cell.

VII. Conclusion

In this paper, we considered the coverage problem for loitering Dubins vehicles. We have characterized the configuration of the vehicles at the appearance of new targets in terms of Dubins paths, that we call *loitering patterns*. We defined the coverage cost to be the worst-case traveling time from any vehicle to any point in the region. Optimal circular loitering for a single vehicle and efficient circular loitering for a single team of vehicles were characterized. Finally, by establishing an analogy to the disk-covering problem, we proposed a computational approach to characterize efficient loitering patterns for multiple uniform teams.

This paper leaves numerous important extensions open for further research. One needs to study the functions λ_n^{-1} to derive closed form expression for the bounds derived in this paper. It would be interesting to consider the coverage

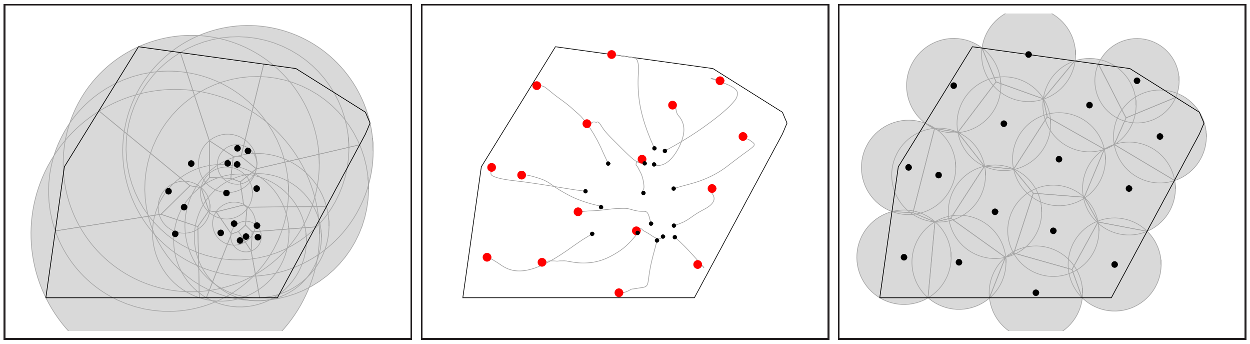

Fig. 8. "Move-toward-the-circumcenter" algorithm for 16 disks in a convex polygonal domain. The left (respectively, right) figure illustrates the initial (respectively, final) locations and Voronoi partition. The central figure illustrates the network evolution. After 20 sec., the disk radius is approximately 0.43273 m. Simulations taken from [13].

problem for other meaningful cost functions. The problem of multi *non-uniform* team of vehicles is also important. Determining the ideal team composition for a given region provides an exciting challenge too.

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