Traveling Salesperson Problems for a double integrator

Ketan Savla, Francesco Bullo, Emilio Frazzoli

Abstract

This paper studies the following version of the Traveling Salesperson Problem (TSP) for a double integrator with bounded velocity and bounded control inputs: given a set of points in $\mathbb{R}^d$, find the fastest tour over the point set. We first give asymptotic bounds on the time taken to complete such a tour in the worst-case. Then, we study a stochastic version of the TSP for double integrator in $\mathbb{R}^2$ and $\mathbb{R}^3$, where we propose novel algorithms that perform within a constant factor of the optimal strategy with high probability. Lastly, we study a dynamic TSP in $\mathbb{R}^2$ and $\mathbb{R}^3$, where we propose novel stabilizing algorithms whose performances are within a constant factor from the optimum.

I. Introduction

The Traveling Salesperson Problem (TSP) with its variations is one of the most widely known combinatorial optimization problems. While extensively studied in the literature, these problems continue to attract great interest from a wide range of fields, including Operations Research, Mathematics and Computer Science. The Euclidean TSP (ETSP) [2], [3] is formulated as follows: given a finite point set P in $\mathbb{R}^d$ for $d \in \mathbb{N}$, find the minimum-length closed path through all points in P. It is quite natural to formulate this problem in the context of other dynamical vehicles. The focus of this paper is the analysis of the TSP for a vehicle with double integrator dynamics or simply a double integrator; we shall refer to it as DITSP. Specifically, DITSP will involve finding the *fastest* tour for a double integrator through a set of points.

Submitted to the IEEE Transactions on Automatic Control on November 30, 2006. This version: February 16, 2007. Preliminary versions of this work appeared as a conference paper in [1].

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Exact algorithms, heuristics and polynomial-time constant factor approximation algorithms are available for the Euclidean TSP, see [4], [5]. However, unlike most other variations of the TSP, it is believed that the DITSP cannot be formulated as a problem on a finite-dimensional graph, thus preventing the use of well-established tools in combinatorial optimization.

The motivation to study the DITSP arises in robotics and uninhabited aerial vehicles (UAVs) applications. UAV applications also motivate us to study the Dynamic Traveling Repairperson Problem (DTRP), in which the aerial vehicle is required to visit a dynamically generated set of targets. This problem was introduced by Bertsimas and van Ryzin in [6] and then decentralized policies achieving the same performances were proposed in [7]. Variants of these problems have attracted much attention recently [7], [8]. However, as with the TSP, the study of the DTRP in conjunction with vehicle dynamics has eluded attention from the research community.

The contributions of this paper are threefold. First, we introduce a natural STOP-GO-STOP strategy for the DITSP to show that the minimum time to traverse the tour is upper bounded by a constant times $n^{1-\frac{1}{2d}}$, i.e., it belongs[1] to $O(n^{1-\frac{1}{2d}})$. We also show that, in the *worst* case, this minimum time is lower bounded by a constant times $n^{1-\frac{1}{d}}$, i.e., it belongs to $\Omega(n^{1-\frac{1}{d}})$. Second, we study the *stochastic* DITSP, i.e., the problem of finding the fastest tour through a set of target points that are uniformly randomly generated. We show that the minimum time to traverse the tour for the stochastic DITSP belongs to $\Omega(n^{2/3})$ in $\mathbb{R}^2$ and $\Omega(n^{4/5})$ in $\mathbb{R}^3$. We adapt the RECURSIVE BEAD-TILING ALGORITHM from our earlier work [9] for the stochastic DITSP in $\mathbb{R}^2$ and we propose a novel algorithm, the RECURSIVE CYLINDER-COVERING ALGORITHM, for the stochastic DITSP in $\mathbb{R}^3$. We prove that, with high probability, the tours generated by these algorithms are traversed in time $O(n^{2/3})$ in $\mathbb{R}^2$ and $O(n^{4/5})$ in $\mathbb{R}^3$ with high probability, i.e., these algorithms provide a constant-factor approximation to the optimal DITSP solution with high probability, i.e., with probability approaching one as $n \to +\infty$. Third, for the DTRP problem we propose novel policies based on the fixed-resolution versions of the corresponding algorithms for stochastic DITSP. We show that the performance guarantees for the stochastic DITSP translate into stability guarantees for the average performance of the double integrator DTRP problem. For a uniform target-generation process with intensity λ, the DTRP algorithm

[1]For $f, g : \mathbb{N} \to \mathbb{R}$, we say that $f \in O(g)$ (resp., $f \in \Omega(g)$) if there exist $N_0 \in \mathbb{N}$ and $k \in \mathbb{R}_+$ such that $|f(N)| \leq k|g(N)|$ for all $N \geq N_0$ (resp., $|f(N)| \geq k|g(N)|$ for all $N \geq N_0$). If $f \in O(g)$ and $f \in \Omega(g)$, then we use the notation $f \in \Theta(g)$.

performance is within a constant factor of the optimal policy in the heavy load case, i.e., for $\lambda \to +\infty$. As a final minor contribution, we also show that the results obtained for stochastic DITSP carry over to the stochastic TSP for the Dubins vehicle, i.e., for a nonholonomic vehicle moving along paths with bounded curvature, without reversing direction. In the interest of space, this document contains only sketches of the proofs; all formal proofs are available at `http://arxiv.org/abs/cs.RO/0609097`.

This work completes the generalization of the known combinatorial results on the ETSP and DTRP (applicable to systems with single integrator dynamics) to double integrators and Dubins vehicle models. It is interesting to compare our results with the setting where the vehicle is modeled by a single integrator; this setting corresponds to the so-called Euclidean case in combinatorial optimization. In the following table the single integrator results in the first column are taken from [3], [6]; the double integrator results in the second column are novel; and the Dubins vehicle results in the third column are taken from [9] for $d = 2$ and are novel for $d = 3$:

	Single integrator	Double integrator	Dubins vehicle
Min. time for TSP tour (worst-case)	$\Theta(n^{1-\frac{1}{d}})$	$\Omega(n^{1-\frac{1}{d}})$, $O(n^{1-\frac{1}{2d}})$	$\Theta(n)$ $(d = 2, 3)$
Exp. min. time for TSP tour (stochastic)	$\Theta(n^{1-\frac{1}{d}})$	$\Theta(n^{1-\frac{1}{2d-1}})$ w.h.p. $(d = 2, 3)$	$\Theta(n^{1-\frac{1}{2d-1}})$ w.h.p. $(d = 2, 3)$
System time for DTRP	$\Theta(\lambda^{d-1})$ $(d = 2)$	$\Theta(\lambda^{2(d-1)})$ $(d = 2, 3)$	$\Theta(\lambda^{2(d-1)})$ $(d = 2, 3)$

Remarkably, the differences between the TSP bounds play a crucial role in the DTRP problem; e.g., stable policies exist only when the minimum TSP time grows strictly sub-linearly with n.

II. SETUP AND WORST-CASE DITSP

For $d \in \mathbb{N}$, consider a vehicle with double integrator dynamics:

$$\ddot{p}(t) = u(t), \quad \|u(t)\| \leq r_{\text{ctr}}, \quad \|\dot{p}(t)\| \leq r_{\text{vel}}, \tag{1}$$

where $p, u \in \mathbb{R}^d$ are the position and control input of the vehicle respectively, $r_{\text{vel}}, r_{\text{ctr}} \in \mathbb{R}_+$ are the bounds on the attainable speed and control inputs respectively. Let $\mathcal{Q} \subset \mathbb{R}^d$ be a unit hypercube. Let $P = \{q_1, \dots, q_n\}$ be a set of n points in $\mathcal{Q}$ and $\mathcal{P}_n$ be the collection of all point sets $P \subset \mathcal{Q}$ with cardinality n. Let $\text{ETSP}(P)$ denote the cost of the Euclidean TSP over P and let $\text{DITSP}(P)$ denote the cost of the TSP for double integrator over P, i.e., the time taken to traverse the fastest closed path for a double integrator through all points in P. We assume r_{vel} and r_{ctr} to be constant and we study the dependence of DITSP: $\mathcal{P}_n \to \mathbb{R}_+$ on n.

Lemma 2.1: (Worst-case Lower Bound on the TSP for Double Integrator) For $r_{\text{vel}} > 0$, $r_{\text{ctr}} > 0$ and $d \in \mathbb{N}$, there exists a point set $P \in \mathcal{P}_n$ in $\mathcal{Q} \subset \mathbb{R}^d$ such that $\text{DITSP}(P)$ belongs to $\Omega(n^{1-\frac{1}{d}})$.

Proof Sketch: As shown in [3], there exists a set $\tilde{P}$ of n points whose minimum inter-point distance belongs to $\Omega(n^{-\frac{1}{d}})$. Therefore, $\text{DITSP}(\tilde{P})$ belongs to $n \times \Omega(n^{-\frac{1}{d}})$, i.e., $\Omega(n^{1-\frac{1}{d}})$. ■

We now propose a simple strategy for the DITSP and analyze its performance. The STOP-GO-STOP strategy can be described as follows: The vehicle visits the points in the same order as in the optimal ETSP tour over the same set of points. Between any pair of points, the vehicle starts at the initial point at rest and follows the shortest-time path to reach the final point with zero velocity. Analyzing this STOP-GO-STOP strategy, one can show the following upper bound.

Theorem 2.2: (Upper Bound on the TSP for Double Integrator) For any point set $P \in \mathcal{P}_n$ in $\mathcal{Q} \subset \mathbb{R}^d$, $r_{\text{ctr}} > 0$, $r_{\text{vel}} > 0$ and $d \in \mathbb{N}$, $\text{DITSP}(P)$ belongs to $O(n^{1-\frac{1}{2d}})$.

III. The stochastic DITSP

The results in the previous section showed that based on a simple strategy, the STOP-GO-STOP strategy, we are already guaranteed to have sublinear cost for the DITSP when the point sets are considered on an individual basis. However, it is reasonable to argue that there might be better algorithms when one is concerned with *average* performance. In particular, one can expect that when n target points are stochastically generated in $\mathcal{Q}$ according to a uniform probability distribution function, the cost of DITSP should be lower than the one given by the STOP-GO-STOP strategy. We shall refer to the problem of studying the average performance of DITSP over this class of point sets as *stochastic* DITSP. In this section, we present novel algorithms for stochastic DITSP in $\mathbb{R}^2$ and $\mathbb{R}^3$ and then establish bounds on their performances.

We make the following assumptions: in $\mathbb{R}^2$, $\mathcal{Q}$ is a rectangle of width W and height H with $W \geq H$; in $\mathbb{R}^3$, $\mathcal{Q}$ is a rectangular box of width W, height H and depth D with $W \geq H \geq D$.

Different choices for the shape of $\mathcal{Q}$ affect our conclusions only by a constant. The axes of the reference frame are parallel to the sides of $\mathcal{Q}$. The points $P = (q_1, \dots, q_n)$ are randomly generated according to a uniform distribution in $\mathcal{Q}$.

A. Lower bounds

First, we provide lower bounds on the expected length of the stochastic DITSP for $d = 2, 3$.

Theorem 3.1: (Lower bounds on stochastic DITSP) For all $r_{\text{vel}} > 0$, $r_{\text{ctr}} > 0$, the expected minimum time in a stochastic DITSP to visit a set of n uniformly-randomly-generated points satisfies the following inequalities:

$$\lim_{n\to+\infty} \frac{\mathrm{E}[\mathrm{DITSP}(P \subset \mathcal{Q} \subset \mathbb{R}^2)]}{n^{2/3}} \geq \frac{3}{4}\Big(\frac{6WH}{r_{\text{vel}} r_{\text{ctr}}}\Big)^{1/3} \quad \text{and}$$

$$\lim_{n\to+\infty} \frac{\mathrm{E}[\mathrm{DITSP}(P \subset \mathcal{Q} \subset \mathbb{R}^3)]}{n^{4/5}} \geq \frac{5}{6}\Big(\frac{20WHD}{\pi r_{\text{vel}} r_{\text{ctr}}^2}\Big)^{1/5}.$$

Proof Sketch: In $\mathbb{R}^2$, the area of the set reachable in time t from a random initial state belongs to $O(t^3)$. Therefore, the expected value of the time between two successive points in the tour belongs to $\Omega(n^{-1/3})$. Hence, the minimum time to traverse the total tour belongs to $n \times \Omega(n^{-1/3})$, i.e., $\Omega(n^{2/3})$. The proof for $\mathbb{R}^3$ follows on similar lines. ∎

B. Relation with the Dubins vehicle

In [9], we studied stochastic versions of TSP for a Dubins vehicle. Though conventionally a Dubins vehicle is restricted to be a *planar* vehicle, one can easily generalize the model even for the three (and higher) dimensional case. Correspondingly, a Dubins vehicle can be defined as a vehicle that is constrained to move with a constant speed along paths of bounded curvature, without reversing direction. Accordingly, a *feasible curve for a Dubins vehicle* or a *Dubins path* is defined as a curve that is twice differentiable almost everywhere, and such that the magnitude of its curvature is bounded above by $1/\rho$, where $\rho > 0$ is the minimum turn radius. Based on this, one can immediately come up with the following analogy between feasible curves for a Dubins vehicle and a double integrator.

Lemma 3.2: (Trajectories of Dubins vehicles and double integrators) For all $\rho > 0$ such that $\sqrt{\rho r_{\text{ctr}}} \leq r_{\text{vel}}$, a feasible curve for a Dubins vehicle with minimum turn radius ρ is a feasible curve for a double integrator (modeled in eqn. (1)) moving with a constant speed $\sqrt{\rho r_{\text{ctr}}}$. Conversely, a feasible curve for a double integrator moving with a constant speed $s \leq r_{\text{vel}}$ is a feasible curve for Dubins vehicle with minimum turn radius s^2/r_{ctr}.

In [9], we proposed a novel algorithm, the RECURSIVE BEAD-TILING ALGORITHM (RECBTA) for the stochastic version of the Dubins TSP (DTSP) in $\mathbb{R}^2$; we showed that this algorithm performed within a constant factor of the optimal with high probability. In this paper, taking inspiration from those ideas, we propose an algorithm to compute feasible curves for a double integrator moving with constant speed r_{vel}. Note that moving at the maximum speed r_{vel} is not necessarily the best strategy since it restricts the maneuvering capability of the vehicle. Nonetheless, this strategy leads to efficient algorithms. We adopt the RECBTA for the stochastic DITSP in $\mathbb{R}^2$ and based on the same ideas, we propose the RECURSIVE CYLINDER-COVERING ALGORITHM (RECCCA) for stochastic DITSP in $\mathbb{R}^3$. We prove that these algorithms perform within a constant factor of the optimal with high probability.

C. The basic geometric construction

Here we define useful geometric objects and study their properties. Given the constant speed r_{vel} for the double integrator let $\rho = \frac{r_{\text{vel}}^2}{r_{\text{ctr}}}$; from Lemma 3.2 this constant corresponds to the minimum turning radius of the *analogous* Dubins vehicle. Consider two points p_- and p_+ on the plane, with $\ell = \|p_+ - p_-\|_2 \leq 4\rho$, and construct the bead $\mathcal{B}_\rho(\ell)$ as detailed in Figure 1.

Associated with the bead is also the rectangle *efgh*. Rotating this rectangle about the line passing through p_- and p_+ gives rise to a cylinder $\mathcal{C}_\rho(\ell)$. $\mathcal{C}_\rho(\ell)$ enjoys the following asymptotic properties as $(l/\rho) \to 0^+$ (properties of the bead, $\mathcal{B}_\rho(\ell)$ are listed in [9]):

(P1) The length of $\mathcal{C}_\rho(\ell)$ is ℓ and its radius of cross-section is $w(\ell)/4$, where $w(\ell)$ is the maximum *thickness* of the bead $\mathcal{B}_\rho(\ell)$ and it is equal to

$$w(\ell) = 4\rho\left(1 - \sqrt{1 - \frac{\ell^2}{16\rho^2}}\right) = \frac{\ell^2}{8\rho} + \rho \cdot o\left(\frac{\ell^3}{\rho^3}\right).$$

(P2) The volume of $\mathcal{C}_\rho(\ell)$ is equal to

$$\text{Volume}[\mathcal{C}_\rho(\ell)] = \pi\left(\frac{w(\ell)}{4}\right)^2 \frac{\ell}{2} = \frac{\pi\ell^5}{2048\rho^2} + \rho^3 \cdot o\left(\frac{\ell^6}{\rho^6}\right).$$

(P3) For any $p \in \mathcal{C}_\rho$, there is at least one feasible curve γ_p through the points $\{p_-, p, p_+\}$, entirely contained within the region obtained by rotating $\mathcal{B}_\rho(\ell)$ about the line passing through p_- and p_+. The length of any such path is at most

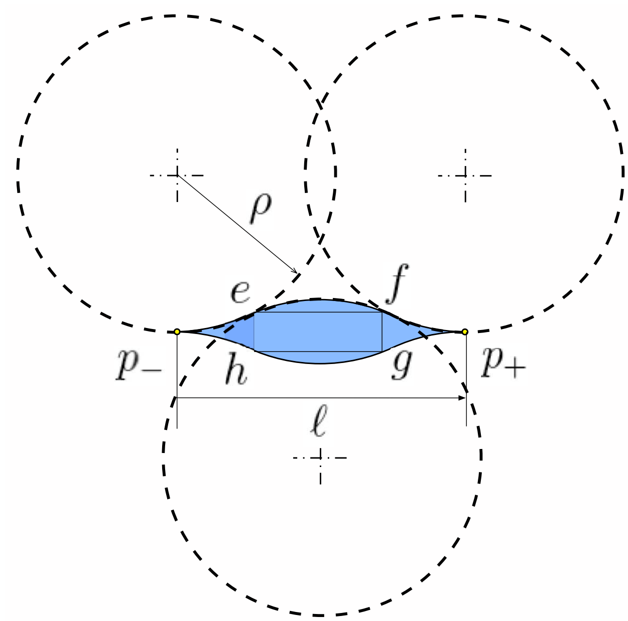

Fig. 1. Construction of the "bead" $\mathcal{B}_\rho(\ell)$. The figure shows how the upper half of the boundary is constructed, the bottom half is symmetric.The figure shows the rectangle *efgh* which is used to construct the "cylinder" $\mathcal{C}_\rho(\ell)$.

$$\mathrm{Length}(\gamma_p) \leq 4\rho \arcsin\left(\frac{\ell}{4\rho}\right) = \ell + \rho \cdot o\left(\frac{\ell^3}{\rho^3}\right).$$

The geometric shapes introduced above can be used to cover $\mathbb{R}^2$ and $\mathbb{R}^3$ in an *organized* way. The plane can be periodically *tiled*[2] by identical copies of $\mathcal{B}_\rho(\ell)$, for any $\ell \in]0, 4\rho]$. The cylinder, however does not enjoy any such special property. For our purpose, we consider a particular covering of $\mathbb{R}^3$ by cylinders described as follows.

A *row of cylinders* is formed by joining cylinders end to end along their length. A layer of cylinders is formed by placing rows of cylinders parallel and on top of each other as shown in Figure 2. For covering $\mathbb{R}^3$, these layers are arranged next to each other and with offsets as shown in Figure 3(a), where the cross section of this arrangement is shown. We refer to this construction as the *covering of* $\mathbb{R}^3$.

[2]A tiling of the plane is a collection of sets whose intersection has measure zero and whose union covers the plane.

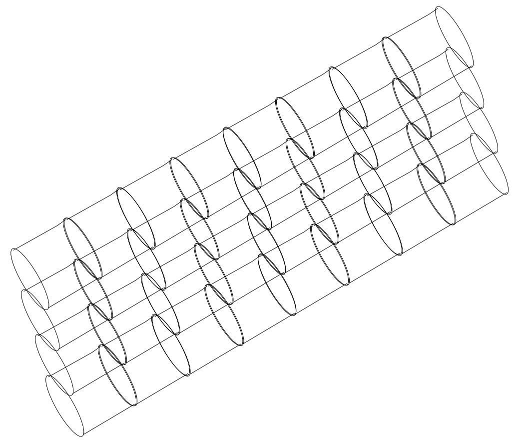

Fig. 2. A typical layer of cylinders formed by stacking rows of cylinders

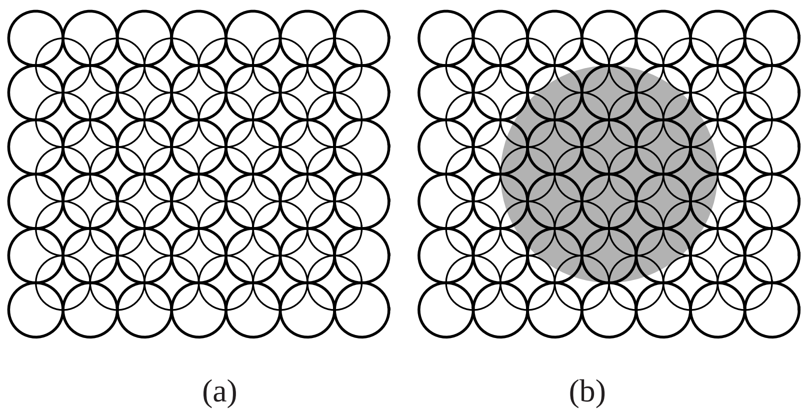

Fig. 3. (a): Cross section of the arrangement of the layers of cylinders used for covering $\mathcal{Q} \subset \mathbb{R}^3$, (b): The relative position of the bigger cylinder relative to smaller ones of the prior phase during the phase transition.

D. The algorithm

We adopt the RECURSIVE BEAD-TILING ALGORITHM (RECBTA) [9] for the stochastic DITSP in $\mathbb{R}^2$. Let $\mathcal{T}_{\mathrm{RecBTA}}$ be the time taken by a double integrator to traverse a stochastic DITSP tour according to the RECBTA. The RECBTA performance is analyzed as follows.

Theorem 3.3: (Upper bound on the total time in $\mathbb{R}^2$) Let $P \in \mathcal{P}_n$ be uniformly randomly generated in the rectangle of width W and height H. For any double integrator (1), with high probability,

$$\lim_{n \to +\infty} \frac{\mathcal{T}_{\mathrm{RecBTA}}}{n^{2/3}} \leq 24 \left(\frac{WH}{r_{\mathrm{vel}} r_{\mathrm{ctr}}} \right)^{1/3} \left(1 + \frac{7\pi r_{\mathrm{vel}}^2}{3W r_{\mathrm{ctr}}} \right).$$

Remark 3.4: Theorems 3.1 and 3.3 imply that, with high probability, the RecBTA is a $\frac{32}{\sqrt[3]{6}}\left(1+\frac{7\pi r_{\mathrm{vel}}^2}{3W r_{\mathrm{ctr}}}\right)$-factor approximation (with respect to n) to the optimal stochastic DITSP in $\mathbb{R}^2$ and that $\mathrm{E}[\mathrm{DITSP}(P \subset \mathcal{Q} \subset \mathbb{R}^2)]$ belongs to $\Theta(n^{2/3})$.

Taking inspiration from the RecBTA, we now propose the Recursive Cylinder-Covering Algorithm (RecCCA) for the stochastic DITSP in $\mathbb{R}^3$. Consider a covering of $\mathcal{Q} \in \mathbb{R}^3$ by cylinders such that $\mathrm{Volume}[\mathcal{C}_\rho(\ell)] = \mathrm{Volume}[\mathcal{Q} \subset \mathbb{R}^3]/(4n) = WHD/(4n)$ (assuming that n is sufficiently large). Furthermore, the covering is chosen in such a way that it is aligned with the sides of $\mathcal{Q} \subset \mathbb{R}^3$.

The proposed algorithm will consist of a sequence of phases; each phase will consist of five sub-phases, all similar in nature. For the first sub-phase of the first phase, a feasible curve is constructed with the following properties:

(i) it visits all non-empty cylinders once,
(ii) it visits all rows of cylinders in a layer in sequence top-to-down in a layer, alternating between left-to-right and right-to-left passes, and visiting all non-empty cylinders in a row,
(iii) it visits all layers in sequence from one end of the region to the other,
(iv) when visiting a non-empty cylinder, it services at least one target in it.

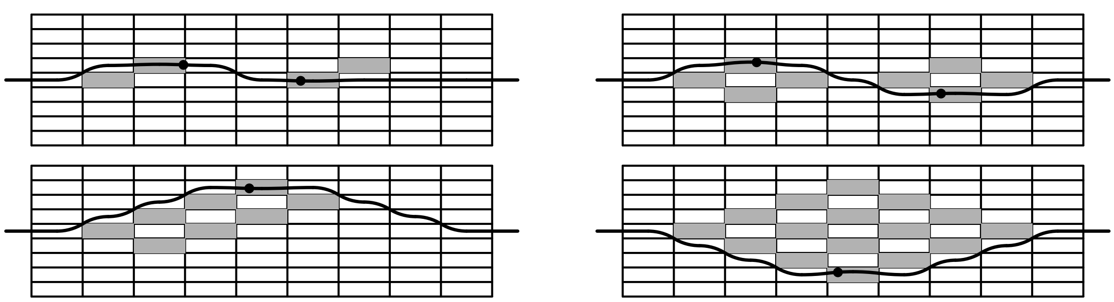

Fig. 4. From top left in the left-to-right, top-to bottom direction, sketch of projection of "meta-cylinders" on the corresponding side of $\mathcal{Q} \subset \mathbb{R}^3$ at second, third, fourth and fifth sub-phases of a phase in the recursive cylinder covering algorithm.

In subsequent sub-phases, instead of considering single cylinders, we will consider "meta-cylinders" composed of 2, 4, 8 and 16 beads each for the remaining four sub-phases, as shown in Figure 4, and proceed in a similar way as the first sub-phase, i.e., a feasible curve is constructed with the following properties:

(i) the curve visits all non-empty meta-cylinders once,

(ii) it visits all (meta-cylinder) rows in sequence top-to-down in a (meta-cylinder) layer, alternating between left-to-right and right-to-left passes, and visiting all non-empty meta-cylinders in a row,

(iii) it visits all (meta-cylinder) layers in sequence from one end of the region to the other,

(iv) when visiting a non-empty meta-cylinder, it services at least one target in it.

A meta-cylinder at the end of the fifth sub-phase, and hence at the end of the first phase will consist of 16 nearby cylinders. After this phase, the transitioning to the next phase will involve enlarging the cylinder to 32 times its current size by increasing the radius of its cross section by a factor of 4 and doubling its length as outlined in Figure 3(b). It is easy to see that this bigger cylinder will contain the union of 32 nearby smaller cylinders. In other words, we are forming the object $\mathcal{C}_\rho(2\ell)$ using a conglomeration of 32 $\mathcal{C}_\rho(\ell)$ objects. This whole process is repeated at most $\log_2 n+2$ times. After the last phase, the leftover targets will be visited using, for example, a greedy strategy. We have the following result for the leftover targets after the last phase which is similar to the result for RECBTA [9].

Theorem 3.5 (Targets remaining after recursive phases): Let $P \in \mathcal{P}_n$ be uniformly randomly generated in $\mathcal{Q} \subset \mathbb{R}^3$. The number of unvisited targets after the last phase of the RECURSIVE CYLINDER-COVERING ALGORITHM over P is less than $24\log_2 n$ with high probability.

We now give a bound on the path length required to execute the first sub-phase.

Lemma 3.6 (Path length for the first sub phase): Consider a covering of the space with cylinders $\mathcal{C}_\rho(\ell)$. For any $\rho > 0$ and for any set of target points, the length L_I of a path executing the first sub-phase of the RECURSIVE CYLINDER-COVERING ALGORITHM in a rectangular box $\mathcal{Q}$ of width W, height H and depth D satisfies

$$L_I \leq \frac{1024\rho^2 WHD}{\ell^4}\left(1+\frac{7\pi\rho}{3W}\right)+\rho\cdot o\left(\frac{\rho^3}{\ell^3}\right).$$

Proof Sketch: We first derive bounds on the length of paths required to *sweep* a row of cylinders from one end to the other and to make a *u-turn* when going from one row to another. The results follows from counting the total number of rows required to cover the domain $\mathcal{Q}$. ■

Similar calculations give the following bounds for the path lengths in subsequent sub-phases.

$$L_{II} \leq \frac{1024\rho^2 WHD}{\ell^4}\left(1+\frac{7\pi\rho}{3W}\right)+\rho\cdot o\left(\frac{\rho^3}{\ell^3}\right), L_{III} \leq \frac{512\rho^2 WHD}{\ell^4}\left(1+\frac{7\pi\rho}{3W}\right)+\rho\cdot o\left(\frac{\rho^3}{\ell^3}\right),$$

$$L_{IV} \leq \frac{512\rho^2 WHD}{\ell^4}\left(1+\frac{7\pi\rho}{3W}\right)+\rho\cdot o\left(\frac{\rho^3}{\ell^3}\right), L_{V} \leq \frac{256\rho^2 WHD}{\ell^4}\left(1+\frac{7\pi\rho}{3W}\right)+\rho\cdot o\left(\frac{\rho^3}{\ell^3}\right).$$

The path length for the first phase is then the sum of the path lengths for the five sub-phases.

Lemma 3.7 (Path length for the first phase): Consider a covering of the space with cylinders $\mathcal{C}_\rho(\ell)$. For any $\rho > 0$ and for any set of target points, the length L_1 of a path visiting once and only once each cylinder with a non-empty intersection with a rectangular box $\mathcal{Q}$ of width W, height H and depth D satisfies

$$L_1 \leq \frac{3328\rho^2 WHD}{\ell^4}\left(1 + \frac{7\pi\rho}{3W}\right) + \rho \cdot o\left(\frac{\rho^3}{\ell^3}\right).$$

Since we increase the length of cylinders by a factor of two while doing the phase transition from one phase to the another, the length of path for the subsequent i^{th} phase is given by:

$$L_i \leq \frac{3328\rho^2 WHD}{16^i \ell^4}\left(1 + \frac{7\pi\rho}{3W}\right) + \rho \cdot o\left(\frac{\rho^3}{\ell^3}\right).$$

We now state the following result which characterizes the total path length for the RecCCA, which we denote as $\mathrm{L}_{\mathrm{RCCA},\rho}(P)$.

Theorem 3.8 (Path length for the Recursive Cylinder-Covering Algorithm*):* Let $P \in \mathcal{P}_n$ be uniformly randomly generated in the rectangle of width W, height H and depth D. For any $\rho > 0$, with high probability

$$\lim_{n\to+\infty} \frac{\mathrm{DITSP}(P \subset \mathcal{Q} \subset \mathbb{R}^3)}{n^{4/5}} \leq \lim_{n\to+\infty} \frac{\mathrm{L}_{\mathrm{RCCA},\rho}(P)}{n^{4/5}} \leq \frac{3328}{15}\left(\frac{\pi}{16}\right)^{4/5}(\rho^2 WHD)^{1/5}\left(1 + \frac{7\pi\rho}{3W}\right).$$

Proof Sketch: There are at most $\log_2 n + 2$ phases. By summing the expression for the path length for the i^{th} phase, L_i, over $\log_2 n + 2$ phases and expressing ℓ in terms of the other parameters, we get the desired result. ■

In order to obtain an upper bound on the $\mathrm{DITSP}(P)$ in $\mathbb{R}^3$, we derive the expression for time taken, $\mathcal{T}_{\mathrm{RecCCA}}$, by the RecCCA to execute the path of length $\mathrm{L}_{\mathrm{RCCA},\rho}(P)$.

Theorem 3.9: (Upper bound on the total time in $\mathbb{R}^3$*)* Let $P \in \mathcal{P}_n$ be uniformly randomly generated in the rectangular box of width W, height H and depth D. For any double integrator (1), with high probability,

$$\lim_{n\to+\infty} \frac{\mathcal{T}_{\mathrm{RecCCA}}}{n^{4/5}} \leq 61\left(\frac{WHD}{r_{\text{ctr}}^2 r_{\text{vel}}}\right)^{1/5}\left(1 + \frac{7\pi r_{\text{vel}}^2}{3W r_{\text{ctr}}}\right).$$

Proof Sketch: We substitute $\rho = \frac{r_{\text{vel}}^2}{r_{\text{ctr}}}$ in the bound for $\mathrm{L}_{\mathrm{RCCA},\rho}(P)$ given by Theorem 3.8 and evaluate the time required to traverse the total path of length $\mathrm{L}_{\mathrm{RCCA},\rho}(P)$ at speed r_{vel}. ■

Remark 3.10: Theorems 3.1 and 3.9 imply that, with high probability, the RecCCA is a $50\left(1 + \frac{7\pi r_{\text{vel}}^2}{3W r_{\text{ctr}}}\right)$-factor approximation (with respect to n) to the optimal stochastic DITSP in $\mathbb{R}^3$ and that $\mathrm{E}[\mathrm{DITSP}(P \subset \mathcal{Q} \subset \mathbb{R}^3)]$ belongs to $\Theta(n^{4/5})$.

IV. THE DTRP FOR DOUBLE INTEGRATOR

We now turn our attention to the Dynamic Traveling Repairperson Problem (DTRP) for the double integrator modeled in eqn. (1). In the DTRP, the double integrator is required to visit a dynamically growing set of targets, generated by some stochastic process. We assume that the double integrator has unlimited range and target-servicing capacity. We let $\mathcal{D}(t)$ denote the set of $n(t)$ outstanding target positions representing the demand at time t. Targets are generated and inserted into $\mathcal{D}$ according to a time-invariant spatio-temporal Poisson process with time intensity $\lambda > 0$ and with uniform spatial density inside the region $\mathcal{Q}$. As before, $\mathcal{Q}$ is a rectangle in two dimensions and a rectangular box in three dimensions. Servicing of a target and its removal from the set $\mathcal{D}$ is achieved when the double integrator moves to the target position. A control policy Φ for the DTRP assigns a control input to the vehicle as a function of its configuration and of the current outstanding targets. The policy Φ is a stable policy for the DTRP if, under its action

$$n_\Phi = \lim_{t \to +\infty} \mathrm{E}[n(t) | \, \dot{p} = \Phi(p, \mathcal{D})] < +\infty,$$

i.e., if the double integrator is able to service targets at a rate that is, on average, at least as large as the target generation rate λ. Let T_j be the time elapsed from the time the j^{th} target is generated to the time it is serviced and let $T_\Phi := \lim_{j \to +\infty} \mathrm{E}[T_j]$ be the steady-state system time for the DTRP under the policy Φ. (If the system is stable, then it is known [10] that $n_\Phi = \lambda T_\Phi$.)

In what follows, we design a control policy Φ whose system time T_Φ is within a constant-factor approximation of the optimal achievable performance. Consistently with the theme of the paper, we consider the case of *heavy load*, i.e., the problem as the time intensity $\lambda \to +\infty$. We first provide lower bounds for the system time, and then present novel approximation algorithms providing upper bound on the performance.

Theorem 4.1 (Lower bound on the DTRP system time): For a double integrator (1), the system time $T_{\mathrm{DTRP},2}$ and $T_{\mathrm{DTRP},3}$ for the DTRP in two and three dimensions satisfy

$$\lim_{\lambda \to \infty} \frac{T_{\mathrm{DTRP},2}}{\lambda^2} \geq \frac{81}{32} \frac{WH}{r_{\mathrm{vel}} r_{\mathrm{ctr}}}, \quad \lim_{\lambda \to \infty} \frac{T_{\mathrm{DTRP},3}}{\lambda^4} \geq \frac{7813}{972} \frac{WHD}{r_{\mathrm{vel}} r_{\mathrm{ctr}}^2}.$$

Proof Sketch: For a stable policy, the average time, $t^*(n^*)$, needed to service a target must be no greater than the average time interval in which a new target is generated, i.e., $\mathrm{E}[t^*(n^*)] \leq 1/\lambda$, where n^* is the average number of outstanding targets. This gives a bound on n^*. Using Little's formula [10], one obtains the result. ∎

In [9], we proposed a simple strategy, the BEAD TILING ALGORITHM (BTA) for the DTRP for Dubins vehicle in $\mathbb{R}^2$. We adapt the BTA for the DTRP problem for a double integrator in $\mathbb{R}^2$ and based on those ideas, we propose the CYLINDER COVERING ALGORITHM (CCA) for $\mathbb{R}^3$. The BTA strategy consists of the following steps:

(i) Tile the plane with beads of length $\ell := \min\{C_{\text{BTA}}/\lambda, 4\rho\}$, where $C_{\text{BTA}} = 0.5241 r_{\text{vel}} \left(1 + \frac{7\pi\rho}{3W}\right)^{-1}$.

(ii) Traverse all non-empty beads once, visiting one target per bead. Repeat this step.

The CCA strategy is akin to the BTA, where the region is covered with cylinders constructed from beads of length $\ell := \min\{C_{\text{CFA}}/\lambda, 4\rho\}$, where $C_{\text{CCA}} = 0.1615 r_{\text{vel}} \left(1 + \frac{7\pi\rho}{3W}\right)^{-1}$. The policy is then to traverse all non-empty cylinders once, visiting one target per cylinder. The following result characterizes the system time for the closed loop system induced by these algorithms and is based on the bounds derived to arrive at Theorems 3.3 and 3.9.

Theorem 4.2 (Upper bound on the DTRP system time): For a double integrator (1) and $\lambda > 0$, the BTA and the CCA are stable policies for the DTRP and the resulting system times T_{BTA} and T_{CFA} satisfy:

$$\lim_{\lambda\to\infty} \frac{T_{\text{DTRP},2}}{\lambda^2} \le \lim_{\lambda\to\infty} \frac{T_{\text{BTA}}}{\lambda^2} \le 70.5 \frac{WH}{r_{\text{vel}} r_{\text{ctr}}} \left(1 + \frac{7\pi r_{\text{vel}}^2}{3W r_{\text{ctr}}}\right)^3,$$

$$\lim_{\lambda\to\infty} \frac{T_{\text{DTRP},3}}{\lambda^4} \le \lim_{\lambda\to\infty} \frac{T_{\text{CFA}}}{\lambda^4} \le 2 \cdot 10^7 \frac{WHD}{r_{\text{vel}} r_{\text{ctr}}^2} \left(1 + \frac{7\pi r_{\text{vel}}^2}{3 r_{\text{ctr}}}\right)^5.$$

Proof Sketch: For the given policies, we derive bounds on the target generation rate and servicing rate for a bead/cylinder. The bead/cylinder is then modeled as a standard $M/D/1$ queue and we use the known result [10] for the system time for such a queue. ■

Remark 4.3: Note that the achievable performances of the BTA and the CCA provide a constant-factor approximation to the lower bounds established in Theorem 4.1.

V. EXTENSION TO THE TSPS FOR THE DUBINS VEHICLE

In our earlier work [9], we have studied the Dubins Traveling Salesperson Problem (DTSP) for the planar case. In that paper, we proposed an algorithm that gave a constant factor approximation to the optimal stochastic DTSP with high probability. This naturally led to a stable policy for the DTRP problem for the Dubins vehicle in $\mathbb{R}^2$ which also performed within a constant factor of the optimal with high probability. The RECCCA developed in this paper can naturally be extended to apply to the stochastic DTSP in $\mathbb{R}^3$. It follows directly from Lemma 3.2 that in order to use the RecCCA for a Dubins vehicle with minimum turning radius ρ, one has to simply compute feasible curves for double integrator moving with a constant speed $\sqrt{\rho r_{\text{ctr}}}$. Hence the results stated in Theorem 3.9 and Theorem 4.2 also hold true for the Dubins vehicle.

This equivalence between trajectories makes the RecCCA the first known strategy with a strictly sublinear asymptotic minimum time for the stochastic DTSP in $\mathbb{R}^3$. Also novel is that the RecCCA performs within a constant factor of the optimal with high probability and gives rise to a constant factor approximation and stabilizing policy for DTRP for Dubins vehicle in $\mathbb{R}^3$.

VI. Conclusions

In this paper we have proposed novel algorithms for various TSP problems for vehicles with double integrator dynamics. We showed that the $\mathrm{DITSP}(P)$ belongs to $O(n^{1-\frac{1}{2d}})$ and in the *worst case* also belongs to $\Omega(n^{1-\frac{1}{d}})$. We further proposed novel approximation algorithm and showed that the *stochastic* $\mathrm{DITSP}(P)$ belongs to $\Theta(n^{2/3})$ in $\mathbb{R}^2$ and to $\Theta(n^{4/5})$ in $\mathbb{R}^3$, both with high probability. The policy proposed in this paper for the DTRP for a double integrator help in proving that the system time belongs to $\Theta(\lambda^2)$ in $\mathbb{R}^2$ and to $\Theta(\lambda^4)$ in $\mathbb{R}^3$. Comparing our results with those for the single integrator [6], we argue that our analysis rigorously establishes the following intuitive fact: higher order dynamics make the system much more sensitive to increases in the target generation rate.

It is interesting to note that the results presented in the paper hold true even in the presence of small damping in the double integrator dynamics: the lower bounds are the same because the damping only slows down the vehicle; the upper bounds also remain the same as long as the damping coefficient is *relatively small* as compared to r_{ctr}. Future directions of research include study of centralized and decentralized versions of the DTRP and more general task assignment and surveillance problems for vehicles with nonlinear dynamics.

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