On Traveling Salesperson Problems for a double integrator

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***Abstract*— In this paper we propose some novel path planning strategies for a double integrator with bounded velocity and bounded control inputs. First, we study the following version of the Traveling Salesperson Problem (TSP): given a set of points in $\mathbb{R}^d$, find the fastest tour over the point set for a double integrator. We first give asymptotic bounds on the time taken for to complete such a tour in the worst-case. Then, we study a stochastic version of the TSP for double integrator where the points are randomly sampled from a uniform distribution in a compact environment in $\mathbb{R}^2$ and $\mathbb{R}^3$. We propose novel algorithms that perform within a constant factor of the optimal strategy with high probability. Second, we study a dynamic TSP: given a stochastic process that generates targets, is there a policy which guarantees that the number of unvisited targets does not diverge over time? If such stable policies exist, what is the minimum wait for a target? We propose novel stabilizing algorithms whose performances are within a constant factor from the optimum, in $\mathbb{R}^2$ as well as in $\mathbb{R}^3$. We also argue that these algorithms give similar performances for a particular nonholonomic vehicle, the Dubins vehicle.**

I. Introduction

The Traveling Salesperson Problem (TSP) with its variations is one of the most widely known combinatorial optimization problems. While extensively studied in the literature, these problems continue to attract great interest from a wide range of fields, including Operations Research, Mathematics and Computer Science. The Euclidean TSP (ETSP) [1], [2] is formulated as follows: given a finite point set P in $\mathbb{R}^d$ for $d \in \mathbb{N}$, find the minimum-length closed path through all points in P. It is quite natural to formulate this problem in the context of other dynamical vehicles. The focus of this paper is the analysis of the TSP for a vehicle with double integrator dynamics or simply a double integrator; we shall refer to it as DITSP. Specifically, DITSP will involve finding the *fastest* tour for a double integrator through a set of points.

Exact algorithms, heuristics and polynomial-time constant factor approximation algorithms are available for the Euclidean TSP, see [3], [4], [5]. However, unlike most other variations of the TSP, it is believed that the DITSP cannot be formulated as a problem on a finite-dimensional graph, thus preventing the use of well-established tools in combinatorial optimization. On the other hand, it is reasonable to expect that exploiting the geometric structure of feasible paths for a double integrator, one can gain insight into the nature of the solution, and possibly provide polynomial-time approximation algorithms.

The motivation to study the DITSP arises in robotics and uninhabited aerial vehicles (UAVs) applications. In particular, we envision applying our algorithm to the setting of an UAV monitoring a collection of spatially distributed points of interest. Additionally, from a purely scientific viewpoint, it is of general interest to bring together the work on dynamical vehicles and that on TSP. UAV applications also motivate us to study the Dynamic Traveling Repairperson Problem (DTRP), in which the aerial vehicle is required to visit a dynamically generated set of targets. This problem was introduced by Bertsimas and van Ryzin in [6] and then decentralized policies achieving the same performances were proposed in [7]. Variants of these problems have attracted much attention recently [7], [8], [9], [10], [11]. However, as with the TSP, the study of DTRP in conjunction with vehicle dynamics has eluded attention from the research community.

The contributions of this paper are threefold. First, we analyze the minimum time taken to traverse DITSP in $\mathbb{R}^d$ for $d \in \mathbb{N}$. We show that the minimum time taken to traverse DITSP belongs[1] to $O(n^{1-\frac{1}{2d}})$ and in the worst case, it also belongs to $\Omega(n^{1-\frac{1}{d}})$. Second, we study the *stochastic* DITSP, i.e., the problem of finding the fastest tour through a set of target points that are uniformly randomly generated. We show that the minimum time to traverse the tour for the stochastic DITSP belongs to $\Omega(n^{2/3})$ in $\mathbb{R}^2$ and $\Omega(n^4/5)$ in $\mathbb{R}^3$. Drawing inspiration from our earlier work [12], we propose two novel algorithms for the stochastic DITSP: the Recursive Bead Tiling Algorithm for $\mathbb{R}^2$ and the Recursive Cylinder Covering Algorithm for $\mathbb{R}^3$. We prove that these algorithms provide a constant-factor approximation to the optimal DITSP solution with high probability. Third, we propose two algorithms for the DTRP in the heavy load case based on the fixed-resolution versions of the corresponding algorithms for stochastic DITSP. We show that the performance guarantees for the stochastic DITSP translate into stability guarantees for the average performance of the DTRP problem for a double integrator. Specifically, the performances of the algorithms for the DTRP are within a constant factor of the optimal policies. We contend that the successful application to the DTRP problem does indeed demonstrate the significance of the DITSP problem from a control viewpoint. As a final minor contribution, we also show that the results obtained for

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[1]For $f, g : \mathbb{N} \to \mathbb{R}$, we say that $f \in O(g)$ (respectively, $f \in \Omega(g)$) if there exist $N_0 \in \mathbb{N}$ and $k \in \mathbb{R}_+$ such that $|f(N)| \leq k|g(N)|$ for all $N \geq N_0$ (respectively, $|f(N)| \geq k|g(N)|$ for all $N \geq N_0$). If $f \in O(g)$ and $f \in \Omega(g)$, then we use the notation $f \in \Theta(g)$.

stochastic DITSP carry over to the stochastic TSP for the Dubins vehicle, i.e., for a nonholonomic vehicle moving along paths with bounded curvature, without reversing direction. We present all proofs in a technical report available at `http://arxiv.org/abs/cs.RO/0609097`.

This work completes the generalization of the known combinatorial results on the ETSP and DTRP (applicable to systems with single integrator dynamics) to double integrators and Dubins vehicle models. It is interesting to compare our results with the setting where the vehicle is modeled by a single integrator; this setting corresponds to the so-called Euclidean case in combinatorial optimization. The results are summarized as follows:

	Single integrator	Double integrator	Dubins vehicle
Min. time for TSP tour (worst-case)	$\Theta(n^{1-\frac{1}{d}})$ [2]	$\Omega(n^{1-\frac{1}{d}})$, $O(n^{1-\frac{1}{2d}})$	$\Theta(n)$ [13] $(d=2,3)$
Exp. min. time for TSP tour (stochastic)	$\Theta(n^{1-\frac{1}{d}})$ [2]	$\Theta(n^{1-\frac{1}{2d-1}})$ w.h.p. $(d=2,3)$	$\Theta(n^{1-\frac{1}{2d-1}})$ w.h.p. $(d=2,3)$
System time for DTRP	$\Theta(\lambda^{d-1})$ [6] $(d=1)$	$\Theta(\lambda^{2(d-1)})$ $(d=2,3)$	$\Theta(\lambda^{2(d-1)})$ $(d=2,3)$

Remarkably, the differences between these various bounds for the TSP play a crucial role when studying the DTRP problem; e.g., stable policies exist only when the minimum time taken for traversing the TSP tour grows strictly sub-linearly with n. For the DTRP problem we propose novel policies and show their stability for a uniform target-generation process with intensity λ. It is clear from the table that motion constraints make the system much more sensitive to increases in the target generation rate λ.

II. Setup and worst-case DITSP

For $d \in \mathbb{N}$, consider a double integrator dynamics:

$$\ddot{p}(t) = u(t), \quad \|u(t)\| \le r_{\text{ctr}}, \quad \|\dot{p}(t)\| \le r_{\text{vel}}, \qquad (1)$$

where $p \in \mathbb{R}^d$ and $u \in \mathbb{R}^d$ are the position and control input of the vehicle, $r_{\text{vel}} \in \mathbb{R}_+$ and $r_{\text{ctr}} \in \mathbb{R}_+$ are the bounds on the attainable speed and control inputs. Let $\mathcal{Q} \subset \mathbb{R}^d$ be the region of interest. Let P be a set of n points in $\mathcal{Q}$ and $\mathcal{P}_n$ be the collection of all point sets $P \subset \mathcal{Q}$ with cardinality n. Let $\text{ETSP}(P)$ denote the minimum time for the Euclidean TSP over P and let $\text{DITSP}(P)$ denote the minimum time of the TSP for a double integrator over P, i.e., the time taken to traverse the fastest closed path for a double integrator through all points in P. We assume r_{vel} and r_{ctr} to be constant and we study the dependence of DITSP: $\mathcal{P}_n \to \mathbb{R}_+$ on n.

Lemma 2.1: (Worst-case lower bound on the DITSP) For $r_{\text{vel}}, r_{\text{ctr}} \in \mathbb{R}_+$ and $d \in \mathbb{N}$, there exists a point set $P \in \mathcal{P}_n$ in $\mathcal{Q} \subset \mathbb{R}^d$ such that $\text{DITSP}(P)$ belongs to $\Omega(n^{1-\frac{1}{d}})$.

We now propose a simple strategy for the DITSP and analyze its performance. The STOP-GO-STOP strategy can be described as follows: The vehicle visits the points in the same order as in the optimal ETSP tour over the same set of points. Between any pair of points, the vehicle starts at the initial point at rest, follows the shortest-time path to reach the final point with zero velocity.

Theorem 2.2: (Upper bound on the DITSP) For any point set $P \in \mathcal{P}_n$ in $\mathcal{Q} \subset \mathbb{R}^d$ and $r_{\text{ctr}} > 0$, $r_{\text{vel}} > 0$ and $d \in \mathbb{N}$, $\text{DITSP}(P)$ belongs to $O(n^{1-\frac{1}{2d}})$.

III. The stochastic DITSP

The results in the previous section showed that based a simple strategy, the STOP-GO-STOP strategy, we are already guaranteed to have sublinear minimum time for tour traversal for the case when the point sets are considered on an individual basis. However, it is reasonable to argue that there might be better algorithms when one is dealing with *average* performance. In particular, one can expect that when n target points are stochastically generated in $\mathcal{Q}$ according to a uniform probability distribution function, the minimum time for the DITSP should be lower than the one given by the STOP-GO-STOP strategy. We shall refer to the problem of studying the average performance of DITSP over this class of point sets as stochastic DITSP. In this section, we present novel algorithms for stochastic DITSP and then establish bounds on their performances.

We make the following assumptions: in $\mathbb{R}^2$, $\mathcal{Q}$ is a rectangle of width W and height H with $W \ge H$; in $\mathbb{R}^3$, $\mathcal{Q}$ is a rectangular box of width W, height H and depth D with $W \ge H \ge D$. Different choices for the shape of $\mathcal{Q}$ affect our conclusions only by a constant. The axes of the reference frame are parallel to the sides of $\mathcal{Q}$. The points $P = \{p_1, \dots, p_n\}$ are randomly generated according to a uniform distribution in $\mathcal{Q}$.

A. Lower bounds

First we provide lower bounds on the expected length of the stochastic DITSP for the 2 and 3 dimensional cases.

Theorem 3.1: (Lower bounds on stochastic DITSP) For a double integrator (1), the expected minimum time for a stochastic DITSP visiting a set of n uniformly-randomly-generated points satisfies the following inequalities:

$$\lim_{n\to+\infty} \frac{\text{E}[\text{DITSP}(P \subset \mathcal{Q} \subset \mathbb{R}^2)]}{n^{2/3}} \ge \frac{3}{4}\Big(\frac{6WH}{r_{\text{vel}} r_{\text{ctr}}}\Big)^{1/3}, \quad \text{and}$$

$$\lim_{n\to+\infty} \frac{\text{E}[\text{DITSP}(P \subset \mathcal{Q} \subset \mathbb{R}^3)]}{n^{4/5}} \ge \frac{5}{6}\Big(\frac{20WHD}{\pi r_{\text{vel}} r_{\text{ctr}}^2}\Big)^{1/5}.$$

B. Constructive upper bounds

In this section, we first recall our earlier work from [12] and use it to propose novel algorithms for the stochastic DITSP: the Recursive Bead Tiling Algorithm for $\mathbb{R}^2$ and Recursive Cylinder Covering Algorithm for $\mathbb{R}^3$. The algorithms' performances will be shown to be within a constant factor of the optimal with high probability.

In [12], we studied stochastic versions of the TSP for Dubins vehicle. Here, a *feasible curve for the Dubins vehicle* or a *Dubins path* is a curve that is twice differentiable almost everywhere and such that the magnitude of its curvature is bounded above by $1/\rho$, where $\rho > 0$ is the minimum turn radius. Feasible curves for a Dubins vehicle and for a double integrator are related as follows.

Lemma 3.2: (Trajectories of Dubins and double integrators) A feasible curve for Dubins vehicle with minimum turn radius $\rho > 0$ is a feasible curve for a double integrator (modeled in equation (1)) moving with a constant speed $\sqrt{\rho r_{\text{ctr}}}$. Conversely, a feasible curve for a double integrator moving with a constant speed $s \le r_{\text{vel}}$ is a feasible curve for Dubins vehicle with minimum turn radius $\frac{s^2}{r_{\text{ctr}}}$.

In [12], we proposed a novel algorithm, the RECURSIVE BEAD TILING ALGORITHM for the stochastic version of the Dubins TSP (DTSP) in $\mathbb{R}^2$; we showed that this algorithm performed within a constant factor of the optimal with high probability. In this paper, taking inspiration from those ideas, we propose algorithms to compute feasible curves for a double integrator moving with the constant speed r_{vel}. Note that moving at the maximum speed r_{vel} is not necessarily the best strategy since it restricts the maneuvering capability of the vehicle. Nonetheless, this strategy leads to efficient algorithms. Next, we proceed towards devising strategies which perform within a constant factor of the optimal for stochastic DITSP in $\mathbb{R}^2$ as well as $\mathbb{R}^3$, both with high probability.

1) The basic geometric construction: Here we define useful geometric objects and study their properties. Given the constant speed r_{vel} for the double integrator let $\rho = \frac{r_{\text{vel}}^2}{r_{\text{ctr}}}$; from Lemma 3.2 this constant corresponds to the minimum turning radius of the *analogous* Dubins vehicle. Consider two points p_- and p_+ on the plane, with $\ell = \|p_+ - p_-\|_2 \le 4\rho$, and construct the bead $\mathcal{B}_\rho(\ell)$ as detailed in Figure 1.

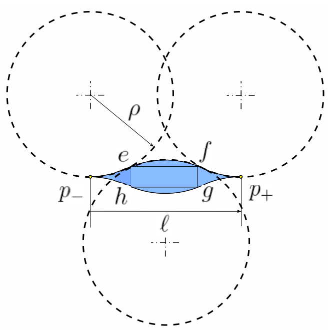

Fig. 1. Construction of the “bead” $\mathcal{B}_\rho(\ell)$. The figure shows how the upper half of the boundary is constructed, the bottom half is symmetric.The figure shows the rectangle $efgh$ which is used to construct the ”cylinder” $\mathcal{C}_\rho(\ell)$.

Associated with the bead is also the rectangle $efgh$. Rotating this rectangle about the line passing through p_- and p_+ gives rise to a cylinder $\mathcal{C}_\rho(\ell)$. The regions $\mathcal{B}_\rho(\ell)$ and $\mathcal{C}_\rho(\ell)$ enjoy the following asymptotic properties as $(l/\rho) \to 0^+$:

(P1) The maximum “thickness” of $\mathcal{B}_\rho(\ell)$ is equal to

$$w(\ell) = 4\rho\left(1 - \sqrt{1 - \frac{\ell^2}{16\rho^2}}\right) = \frac{\ell^2}{8\rho} + \rho \cdot o\left(\frac{\ell^3}{\rho^3}\right).$$

The radius of cross-section of $\mathcal{C}_\rho(\ell)$ is $w(\ell)/4$ and the length of $\mathcal{C}_\rho(\ell)$ is ℓ.

(P2) The area of $\mathcal{B}_\rho(\ell)$ is equal to

$$\text{Area}(\mathcal{B}_\rho(\ell)) = \frac{\ell w(\ell)}{2} = \frac{\ell^3}{16\rho} + \rho^2 \cdot o\left(\frac{\ell^4}{\rho^4}\right).$$

The volume of $\mathcal{C}_\rho(\ell)$ is equal to

$$\text{Volume}[\mathcal{C}_\rho(\ell)] = \pi\left(\frac{w(\ell)}{4}\right)^2 \frac{\ell}{2} = \frac{\pi \ell^5}{2048\rho^2} + \rho^3 \cdot o\left(\frac{\ell^6}{\rho^6}\right).$$

(P3) For any $p \in \mathcal{B}_\rho$, there is at least one feasible curve γ_p through the points $\{p_-, p, p_+\}$, entirely contained within $\mathcal{B}_\rho$. The length of any such path is at most

$$\text{Length}(\gamma_p) \le 4\rho \arcsin\left(\frac{\ell}{4\rho}\right) = \ell + \rho \cdot o\left(\frac{\ell^3}{\rho^3}\right).$$

Analogously, for any $\tilde{p} \in \mathcal{C}_\rho$, there is at least one feasible curve $\gamma_{\tilde{p}}$ through the points $\{p_-, \tilde{p}, p_+\}$, entirely contained within the region obtained by rotating $\mathcal{B}_\rho(\ell)$ about the line passing through p_- and p_+. The length of $\gamma_{\tilde{p}}$ satisfies the same upper bound as the one established for γ_p.

The geometric shapes introduced above can be used to cover $\mathbb{R}^2$ and $\mathbb{R}^3$ in an *organized* way. The plane can be periodically *tiled*[2] by identical copies of $\mathcal{B}_\rho(\ell)$, for any $\ell \in]0, 4\rho]$. The cylinder, however does not enjoy any such special property. For our purpose, we consider a particular covering of $\mathbb{R}^3$ by cylinders described as follows.

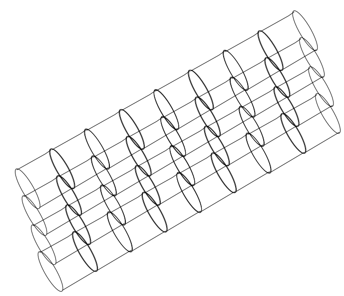

Fig. 2. A typical layer of cylinders formed by stacking rows of cylinders

A *row of cylinders* is formed by joining cylinders end to end along their length. A layer of cylinders is formed by placing rows of cylinders parallel and on top of each other as shown in Figure 2. For covering $\mathbb{R}^3$, these layers are arranged next to each other and with offsets as shown in Figure 3(a), where the cross section of this arrangement is shown. We refer to this construction as the *covering of* $\mathbb{R}^3$.

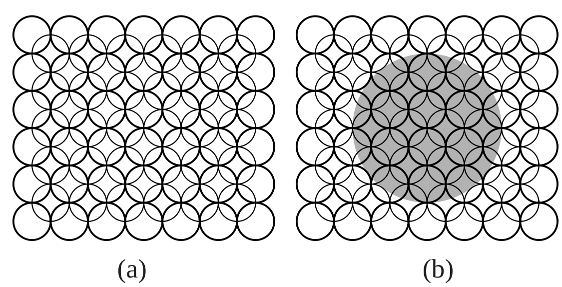

Fig. 3. (a): Cross section of the arrangement of the layers of cylinders used for covering $\mathcal{Q} \subset \mathbb{R}^3$, (b): The relative position of the bigger cylinder relative to smaller ones of the prior phase during the phase transition.

2) The 2D case: The RECURSIVE BEAD TILING ALGORITHM *(*RECBTA*):* Consider a tiling of the plane such that $\text{Area}[\mathcal{B}_\rho(\ell)] = \text{Area}[\mathcal{Q} \subset \mathbb{R}^2]/(2n) = WH/(2n)$; to obtain

[2] A tiling of the plane is a collection of sets whose intersection has measure zero and whose union covers the plane.

this equality we assume ℓ to be a decreasing function of n such that $\ell(n) \leq 4\rho$. Furthermore, we assume the tiling is chosen to be aligned with the sides of $\mathcal{Q} \subset \mathbb{R}^2$, see Figure 4.

The proposed algorithm consists of a sequence of phases; during each of these phases, a feasible curve will be constructed that "sweeps" the set $\mathcal{Q}$. In the first phase, a feasible curve is constructed with the following properties:

(i) it visits all non-empty beads once,
(ii) it visits all rows in sequence top-to-down, alternating between left-to-right and right-to-left passes, and visiting all non-empty beads in a row,
(iii) when visiting a non-empty bead, it services at least one target in it.

In order to visit the outstanding targets, a new phase is initiated. In this phase, instead of considering single beads, we will consider "meta-beads" composed of two beads each, as shown in Figure 4, and proceed in a similar way as the first phase, i.e., a feasible curve is constructed with the following properties:

(i) the curve visits all non-empty meta-beads once,
(ii) it visits all (meta-bead) rows in sequence top-to-down, alternating between left-to-right and right-to-left passes, and visiting all non-empty meta-beads in a row,
(iii) when visiting a non-empty meta-bead, it services at least one target in it.

This process is iterated at most $\log_2 n + 1$ times, and at each phase meta-beads composed of two neighboring meta-beads from the previous phase are considered; in other words, the meta-beads at the i-th phase are composed of 2^{i-1} neighboring beads. After the last phase, the leftover targets will be visited using, for example, a greedy strategy.

The following result is related to a similar result in [14].

Theorem 3.3 (Targets remaining after recursive phases): Let $P \in \mathcal{P}_n$ be uniformly randomly generated in $\mathcal{Q} \in \mathbb{R}^2$. The number of unvisited targets after the last recursive phase of the RecBTA is less than $24 \log_2 n$ with high probability, i.e., with probability approaching one as $n \to +\infty$.

At this point we know that, after a sufficiently large number of phases, almost all targets will be visited with high probability. The key point is that the length of each phase is decreasing at such a rate that the sum of the lengths of all the phases remains bounded. We first state the following result which characterizes the path length for the RecBTA, which we denote as $\mathrm{L}_{\mathrm{RecBTA},\rho}(P)$.

Theorem 3.4 (Path length for the RecBTA): Let $P \in \mathcal{P}_n$ be uniformly randomly generated in the rectangle of width W and height H. For any $\rho > 0$, with high probability

$$\lim_{n\to+\infty} \frac{\mathrm{L}_{\mathrm{RecBTA},\rho}(P)}{n^{2/3}} \leq 24(\rho W H)^{1/3}\left(1 + \frac{7\pi\rho}{3W}\right).$$

In order to obtain an upper bound on the $\mathrm{DITSP}(P)$, we derive the expression for time taken, $\mathcal{T}_{\mathrm{RecBTA}}$, by the RecBTA to execute the path of length $\mathrm{L}_{\mathrm{RecBTA},\rho}(P)$.

Theorem 3.5: (Upper bound on the total time in $\mathbb{R}^2$) Let $P \in \mathcal{P}_n$ be uniformly randomly generated in the rectangle of width W and height H. For any double integrator (1), with high probability

$$\lim_{n\to+\infty} \frac{\mathcal{T}_{\mathrm{RecBTA}}}{n^{2/3}} \leq 24\left(\frac{WH}{r_{\mathrm{vel}} r_{\mathrm{ctr}}}\right)^{1/3}\left(1 + \frac{7\pi r_{\mathrm{vel}}^2}{3W}\right).$$

Remark 3.6: Theorems 3.1 and 3.5 imply that, with high probability, the RecBTA is a $\frac{32}{\sqrt[3]{6}}\left(1 + \frac{7\pi r_{\mathrm{vel}}^2}{3 r_{\mathrm{ctr}} W}\right)$-factor approximation (with respect to n) to the optimal stochastic DITSP in $\mathbb{R}^2$ and that $\mathrm{E}[\mathrm{DITSP}(P \subset \mathcal{Q} \subset \mathbb{R}^2)]$ belongs to $\Theta(n^{2/3})$.

3) The 3D case: The Recursive Cylinder Covering Algorithm *(*RecCCA*):* Consider a covering of $\mathcal{Q} \subset \mathbb{R}^3$ by cylinders such that $\mathrm{Volume}[\mathcal{C}_\rho(\ell)] = \mathrm{Volume}[\mathcal{Q} \subset \mathbb{R}^3]/(4n) = WHD/(4n)$ (Again implying that n is sufficiently large). Furthermore, the covering is chosen in such a way that it is aligned with the sides of $\mathcal{Q} \subset \mathbb{R}^3$.

The proposed algorithm will consist of a sequence of phases; each phase will consist of five sub-phases, all similar in nature. For the first sub-phase of the first phase, a feasible curve is constructed with the following properties:

(i) it visits all non-empty cylinders once,
(ii) it visits all rows of cylinders in a layer in sequence top-to-down in a layer, alternating between left-to-right and right-to-left passes, and visiting all non-empty cylinders in a row,
(iii) it visits all layers in sequence from one end of the region to the other,
(iv) when visiting a non-empty cylinder, it services at least one target in it.

In subsequent sub-phases, instead of considering single cylinders, we will consider "meta-cylinders" composed of 2, 4, 8 and 16 beads each for the remaining four sub-phases, as shown in Figure 5, and proceed in a similar way as the first sub-phase, i.e., a feasible curve is constructed with the following properties:

(i) the curve visits all non-empty meta-cylinders once,
(ii) it visits all (meta-cylinder) rows in sequence top-to-down in a (meta-cylinder) layer, alternating between left-to-right and right-to-left passes, and visiting all non-empty meta-cylinders in a row,
(iii) it visits all (meta-cylinder) layers in sequence from one end of the region to the other,
(iv) when visiting a non-empty meta-cylinder, it services at least one target in it.

A meta-cylinder at the end of the fifth sub-phase, and hence at the end of the first phase will consist of 16 nearby cylinders. After this phase, the transitioning to the next phase will involve enlarging the cylinder to 32 times its current size by increasing the radius of its cross section by a factor of 4 and doubling its length as outlined in Figure 3(b). It is easy to see that this bigger cylinder will contain the union of 32 nearby smaller cylinders. In other words, we are forming the object $\mathcal{C}_\rho(2\ell)$ using a conglomeration of 32 $\mathcal{C}_\rho(\ell)$ objects. This whole process is repeated at most $\log_2 n + 2$ times. After the last phase, the leftover targets will be visited using, for example, a greedy strategy.

The same analysis method as for the RecBTA allows us to study the time $\mathcal{T}_{\mathrm{RecCCA}}$ taken to execute the RecCCA.

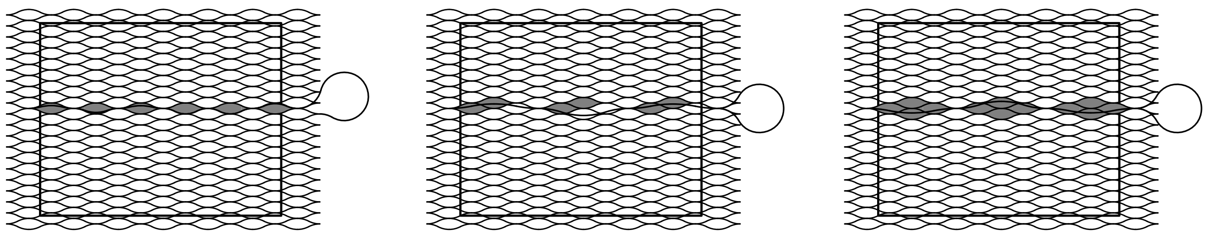

Fig. 4. Sketch of “meta-beads” at successive phases in the recursive bead tiling algorithm.

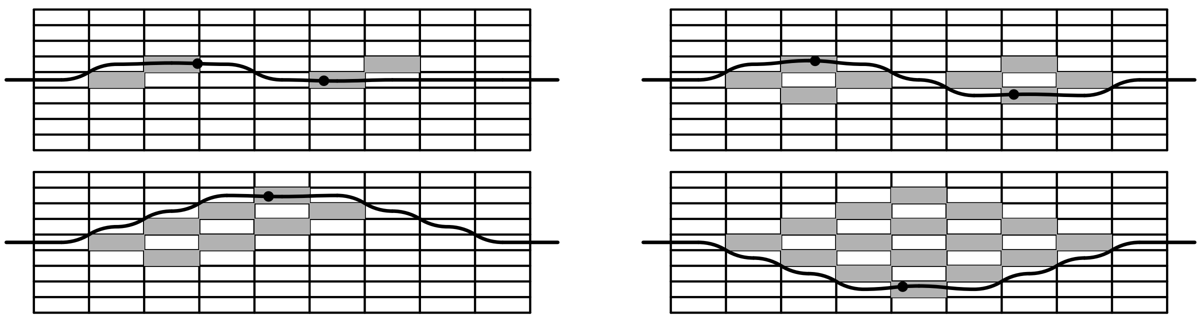

Fig. 5. Starting from top left in the left-to-right, top-to bottom direction, sketch of projection of “meta-cylinders” on the corresponding side of $\mathcal{Q} \subset \mathbb{R}^3$ at second, third, fourth and fifth sub-phases of a phase in the recursive cylinder covering algorithm.

Theorem 3.7: (Upper bound on the total time in $\mathbb{R}^3$) Let $P \in \mathcal{P}_n$ be uniformly randomly generated in the rectangular box of width W, height H and depth D. For any double integrator (1), with high probability

$$\lim_{n \to +\infty} \frac{\mathcal{T}_{\text{RecCCA}}}{n^{4/5}} \leq 61 \left(\frac{WHD}{r_{\text{ctr}}^2 r_{\text{vel}}}\right)^{1/5} \left(1 + \frac{7\pi r_{\text{vel}}^2}{3W r_{\text{ctr}}}\right).$$

Remark 3.8: Theorems 3.1 and 3.7 imply that, with high probability, the RECCCA is a $50\left(1 + \frac{7\pi r_{\text{vel}}^2}{3r_{\text{ctr}}W}\right)$-factor approximation (with respect to n) to the optimal stochastic DITSP in $\mathbb{R}^3$ and that $\mathrm{E}[\mathrm{DITSP}(P \subset \mathcal{Q} \subset \mathbb{R}^3)]$ belongs to $\Theta(n^{4/5})$.

IV. THE DTRP FOR A DOUBLE INTEGRATOR

We now turn our attention to the Dynamic Traveling Repairperson Problem (DTRP) that was introduced in [6] and that we here tackle for a double integrator.

A. Model and problem statement

In the DTRP the double integrator is required to visit a dynamically growing set of targets, generated by some stochastic process. We assume that the double integrator has unlimited range and target-servicing capacity and that it moves at a unit speed with minimum turning radius $\rho > 0$.

Information about the outstanding targets representing the demand at time t is described by a finite set $n(t)$ of positions $\mathcal{D}(t)$. Targets are generated, and inserted into $\mathcal{D}$, according to a time-invariant spatio-temporal Poisson process, with time intensity $\lambda > 0$, and uniform spatial density inside the region $\mathcal{Q}$, which we continue to assume to be a rectangle for two dimensions and a rectangular box for three dimensions. Servicing of a target and its removal from the set $\mathcal{D}$, is achieved when the double integrator moves to the target position. A control policy Φ for the DTRP assigns a control input to the vehicle as a function of its configuration and of the current outstanding targets. The policy Φ is a stable policy for the DTRP if, under its action

$$n_\Phi = \lim_{t \to +\infty} \mathrm{E}[n(t) | \, \dot{p} = \Phi(p, \mathcal{D})] < +\infty,$$

that is, if the double integrator is able to service targets at a rate that is, on average, at least as fast as the rate at which new targets are generated.

Let T_j be the time elapsed from the time the j^{th} target is generated to the time it is serviced and let $T_\Phi := \lim_{j \to +\infty} \mathrm{E}[T_j]$ be the steady-state system time for the DTRP under the policy Φ. (Note that if the system is stable, then it is known [15] that $n_\Phi = \lambda T_\Phi$.) Clearly, our objective is to design a policy Φ with minimal system time T_Φ.

B. Lower and constructive upper bounds

In what follows, we design control policies that provide constant-factor approximation of the optimal achievable performance. Consistently with the theme of the paper, we consider the case of *heavy load*, i.e., the problem as the time intensity $\lambda \to +\infty$. We first provide lower bounds for the system time, and then present novel approximation algorithms providing upper bound on the performance.

Theorem 4.1 (Lower bound on the DTRP system time): For any $\rho > 0$, the system time $T_{\text{DTRP},2}$ and $T_{\text{DTRP},3}$ for the DTRP in two and three dimensions satisfy

$$\lim_{\lambda \to \infty} \frac{T_{\text{DTRP},2}}{\lambda^2} \geq \frac{81}{32} \frac{WH}{r_{\text{vel}} r_{\text{ctr}}}, \quad \lim_{\lambda \to \infty} \frac{T_{\text{DTRP},3}}{\lambda^4} \geq \frac{7813}{972} \frac{WHD}{r_{\text{vel}} r_{\text{ctr}}^2}.$$

We now propose simple strategies, the BEAD TILING ALGORITHM (for $\mathbb{R}^2$) and the CYLINDER COVERING ALGORITHM (for $\mathbb{R}^3$), based on the concepts introduced in the previous section. The BEAD TILING ALGORITHM (BTA) strategy consists of the following steps:

(i) Tile the plane with beads of length $\ell := \min\{C_{\rm BTA}/\lambda, 4\rho\}$, where

$$C_{\rm BTA} = 0.5241 r_{\rm vel}\left(1 + \frac{7\pi\rho}{3W}\right)^{-1}.$$

(ii) Traverse all non-empty beads once, visiting one target per non-empty bead. Repeat this step.

The CYLINDER COVERING ALGORITHM (CCA) strategy is akin to the BTA, where the region is covered with cylinders constructed from beads of length $\ell := \min\{C_{\rm CFA}/\lambda, 4\rho\}$, where

$$C_{\rm CCA} = 0.1615 r_{\rm vel}\left(1 + \frac{7\pi\rho}{3W}\right)^{-1}.$$

The policy is then to traverse all non-empty cylinders once, visiting one target per non-empty cylinder. The following result characterizes the system time for the closed loop system induced by these algorithms and is based on the bounds derived to arrive at Theorems 3.5 and 3.7.

Theorem 4.2 (Upper bound on the DTRP system time): For any $\rho > 0$ and $\lambda > 0$, the BTA and the CCA are stable policies for the DTRP and the resulting system times $T_{\rm BTA}$ and $T_{\rm CFA}$ satisfy:

$$\lim_{\lambda\to\infty} \frac{T_{\rm DTRP,2}}{\lambda^2} \le \lim_{\lambda\to\infty} \frac{T_{\rm BTA}}{\lambda^2} \le 70.5 \frac{WH}{r_{\rm vel} r_{\rm ctr}}\left(1 + \frac{7\pi r_{\rm vel}^2}{3W r_{\rm ctr}}\right)^3,$$

$$\lim_{\lambda\to\infty} \frac{T_{\rm DTRP,3}}{\lambda^4} \le \lim_{\lambda\to\infty} \frac{T_{\rm CFA}}{\lambda^4} \le 2\cdot 10^7 \frac{WHD}{r_{\rm vel} r_{\rm ctr}^2}\left(1 + \frac{7\pi r_{\rm vel}^2}{3W r_{\rm ctr}}\right)^5.$$

Remark 4.3: Note that the achievable performances of the BTA and the CCA provide a constant-factor approximation to the lower bounds established in Theorem 4.1.

V. EXTENSION TO THE TSPS FOR THE DUBINS VEHICLE

In our earlier works [13], [16], [12], we have studied the TSP for the Dubins vehicle in the planar case. In [13], we proved that in the worst case, the time taken to complete a TSP tour by the Dubins vehicle will belong to $\Theta(n)$. One could show that this result holds true even in $\mathbb{R}^3$. In [16], the first known algorithm with strictly sublinear asymptotic minimum time for tour traversal was proposed for the stochastic DTSP in $\mathbb{R}^2$. This algorithm was modified in [12] to give a constant factor approximation to the optimal with high probability. This naturally lead to a stable policy for the DTRP problem for the Dubins vehicle in $\mathbb{R}^2$ which also performed within a constant factor of the optimal with high probability. The RECCCA developed in this paper can naturally be extended to apply to the stochastic DTSP in $\mathbb{R}^3$. It follows directly from Lemma 3.2 that in order to use the RECCCA for a Dubins vehicle with minimum turning radius ρ, one has to simply compute feasible curves for double integrator moving with a constant speed $\sqrt{\rho r_{\rm ctr}}$. Hence the results stated in Theorem 3.7 and Theorem 4.2 also hold true for the Dubins vehicle.

This equivalence between trajectories makes the RECCCA the first known strategy with a strictly sublinear asymptotic minimum time for tour traversal for stochastic DTSP in $\mathbb{R}^3$. The fact that it performs within a constant factor of the optimal with high probability and that it gives rise to a constant factor approximation and stabilizing policy for DTRP for Dubins vehicle in $\mathbb{R}^3$ is also novel.

VI. CONCLUSIONS

In this paper we have proposed novel algorithms for various TSP problems for vehicles with double integrator dynamics. Future directions of research include extensive simulations to support the results obtained in this paper, study of centralized and decentralized versions of the DTRP, and more general task assignment and surveillance problems for vehicles with nonlinear dynamics.

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