On Robustness Analysis of Large-scale Transportation Networks

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***Abstract*— In this paper, we study robustness properties of transportation networks with respect to its pre-disturbance equilibrium operating condition and the agents' response to the disturbance. We perform the analysis within a dynamical system framework over a directed acyclic graph between a single origin-destination pair. The dynamical system is composed of ordinary differential equations (ODEs), one for every edge of the graph. Every ODE is a mass balance equation for the corresponding edge, where the inflow term is a function of the agents' route choice behavior and the arrival rate at the base node of that edge, and the outflow term is function of the congestion properties of the edge. We consider disturbances that reduce the maximum flow carrying capacity of the links and define the *margin of stability* of the network as the minimum capacity that needs to be removed from the network so that the delay on all the edges remain bounded over time. For a given equilibrium operating condition, we derive upper bounds on the margin of stability under local information constraint on the agents' behavior, and characterize the route choice functions that yield this bound. We also setup a simple convex optimization problem to find the most robust operating condition for the network and determine edge-wise tolls that yield such an equilibrium operating condition.**

I. Introduction

Social planning for efficient usage of transportation networks (TNs) is attracting renewed research interest as transportation demand is fast approaching its infrastructure capacity. While there exists an abundant literature on socially optimal traffic assignments, e.g., see [1], robustness analysis of TNs has received very little attention. In this paper, we study the relationship of the robustness properties of a large-scale TN to its pre-disturbance equilibrium operating condition and the agents' response to the disturbance.

We abstract the topology of the transportation network by a directed acyclic graph between a single origin-destination pair. For the analysis, we adopt a dynamical system framework that is composed of ordinary differential equations (ODEs), one for every edge of the graph. Every ODE represents a mass balance equation for the corresponding edge, where the inflow term is a function of the agents' route choice behavior and the arrival rate at the base node of that edge, and the outflow term is function of the congestion properties of the edge. We consider a setup where, before the disturbance, the network is operating at an equilibrium operating condition and information about this equilibrium condition is shared by all the agents. Such an equilibrium condition might be thought of as the outcome of a slower time-scale learning process, e.g., see [2], [3], [4], in presence of incentive mechanisms such as tolls, e.g., see [5], [6]. After the disturbance, we assume that the global knowledge of the agents remains fixed and that the agents act by complementing the fixed global knowledge with real-time local information. Such a setup is meant is give insight into the evolution of the network in the immediate aftermath of a disruption when the availability of accurate global information about the whole network is sparse or it is too time-consuming for the agents to incorporate the real-time information about the whole network because of the huge computations involved.

We consider disturbances that reduce the maximum flow carrying capacities of the edges by affecting their congestion properties. We define the margin of stability of the TN to be the maximum sum of capacity losses, under which the traffic densities on all the edges remain bounded over time. We then prove that, irrespective of the route choice behavior of the agents, the margin of stability is upper-bounded by the minimum of all the *node cuts* of the residual capacities of the TN. We then characterize the route choice behaviors that match this upper bound. Finally, we study the dependence of the margin of stability on the equilibrium, and formulate a simple optimization problem for finding the most robust equilibria. This is, in general, different from the classical socially optimal equilibrium, as well as from the user-optimal equilibrium. We also discuss the utility of tolls in yielding a desired equilibrium operating condition. Our results provide important guidelines for social planners in terms of determining robust equilibrium operating conditions and route choice behaviors for TNs. Alternate notions of robustness for networks have been proposed in [7], [8], [9].

The contributions of the paper are as follows: (i) we formulate a novel dynamical system framework for robustness analysis of transportation networks, (ii) we derive an upper bound on the margin of stability of the network and characterize the features of the agents' route choice behavior under which this bound is tight, and (iii) we postulate the notion of *robustness price of anarchy* to quantify the loss in robustness due to sub-optimal equilibrium operating condition of a network and discuss the use of tolls in removing this gap.

This technical results of this paper rely on tools from several disciplines. The upper bounds on the margin of stability for a given equilibrium operating condition uses graph theory notions from flow networks, e.g., see [10]. The properties of the route choice functions that give maximum margin of stability are reminiscent of cooperative dynamics, e.g., see [11]. The problem of determining tolls for a desired

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equilibrium condition exploits the fact that the associated congestion game is a potential game and that the extremum of the potential function corresponds to the equilibrium.

The rest of the paper is organized as follows. In Section II, we describe basic notations and concepts useful for the paper and formulate the robustness analysis problem. In Sections III and IV, we derive bounds on the margin of stability of the network. Section V discusses the problem of selection of the most robust equilibrium operating point of the network. Finally, we conclude in Section VI with remarks on future research directions.

II. Problem Formulation

We start by defining a few preliminary notations. Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{R}$ be the set of real numbers, $\mathbb{R}_+$ be the set of non-negative real numbers and $\mathbb{R}_{>0}$ be the set of positive real numbers. Let $\mathbf{1}_n$ be the n-dimensional vector, all of whose entries are one. Let $|B|$ be the cardinality of set B and let $\mathrm{int}(B)$ denote the interior of set B. Let $\mathbb{1}_A(x)$ be the indicator function with respect to A, i.e., $\mathbb{1}_A(x) = 1$ if $x \in A$ and zero otherwise. Given a set A, let A^c be its complement. For $p \in [1, \infty]$, $\|.\|_p$ is the p-norm. Specifically, let $\|.\|$ denote the Euclidean norm.

A. Physical properties of the network and Wardrop equilibrium

Let the topology of the transportation network be described by a *directed* graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of nodes and $\mathcal{E}$ is the set of edges. An edge $e \in \mathcal{E}$ from $u \in \mathcal{V}$ to $w \in \mathcal{V}$ is represented by the ordered set (u, w). Given an edge $e = (u, w)$, u will be called the *base* node of edge e. Let $o \in \mathcal{V}$ and $d \in \mathcal{V}$ be the origin and the destination nodes in $\mathcal{G}$. For every node $v \in \mathcal{V} \setminus \{d\}$, we shall denote by $\mathcal{E}_v^+ \subseteq \mathcal{E}$ the set of outgoing edges from v. Similarly, For every node $v \in \mathcal{V} \setminus \{o\}$, we shall denote by $\mathcal{E}_v^- \subseteq \mathcal{E}$ the set of incoming edges to v. A *path* from a vertex $u \in \mathcal{V}$ and $w \in \mathcal{V}$ is an ordered set of vertices $u, v_1, v_2, \ldots, v_l, w$ such that $(u, v_1), (v_1, v_2), \ldots, (v_l, w)$ each belong to $\mathcal{E}$. The *length* of such path will be defined to be $l + 1$, i.e., the number of edges in the path. A path is said to exist between $u \in \mathcal{V}$ and $w \in \mathcal{E}$ if there exists an ordered set of nodes connecting u and w.

Throughout, we shall assume that

(A1) $\mathcal{G}$ is acyclic, and there exists a path from every $v \in \mathcal{V} \setminus \{d\}$ to d.

Let $\mathcal{P}$ be the set of distinct paths from o to d in $\mathcal{G}$, and let Π be the simplex of probability distributions over $\mathcal{P}$.

Traffic arrives at a unit rate at the node o and is destined for the node d. We shall denote the traffic density and flow, on the network at time t by vectors $\rho(t), f(t) \in \mathbb{R}_+^{\mathcal{E}}$ respectively, whose entries, $\rho_e(t)$, and $f_e(t)$, respectively, will denote the traffic density, and flow, on the link $e \in \mathcal{E}$ at time t. Traffic flow and density on a link are related by the functional dependence

$$f_e = \mu_e(\rho_e)\,, \qquad \forall e \in \mathcal{E}\,, \tag{1}$$

on which we shall make the following assumption for all $e \in \mathcal{E}$:

(A2) $\mu_e : \mathbb{R}_+ \to \mathbb{R}_+$ is Lipschitz continuous, strictly increasing, strictly concave, and such that $\mu_e(0) = 0$, $\lim_{\rho_e \downarrow 0} \mathrm{d}\mu_e/\mathrm{d}\rho_e < +\infty$ and $\limsup_{\rho_e \to +\infty} \mu_e(\rho_e) < +\infty$.

Let $\mu(\rho)$ be the vector of the edge-wise flow functions $\mu_e(\rho_e)$. Let Ψ be the set of all functions $\mu : \mathbb{R}_+^{\mathcal{E}} \to \mathbb{R}_+^{\mathcal{E}}$ that satisfy condition (A2).

Example: The following flow function belongs to Ψ.

$$\mu_e(\rho_e) = f_{\mathrm{e}}^{\max}\left(1 - e^{-\alpha_e \rho_e}\right) \quad e \in \mathcal{E}, \tag{2}$$

where $\alpha_e > 0$ for all $e \in \mathcal{E}$.

It follows from assumption **(A2)** that $\mu_e(\,\cdot\,)$ admits a continuous inverse $\mu_e^{-1}(\,\cdot\,)$. We can then consider the delay functions

$$T_e(f_e) := \mu_e^{-1}(f_e)/f_e\,, \tag{3}$$

i.e., $T_e(f_e)$ is the time taken to traverse link e when the flow on it is f_e. Following assumption (A2), let $f_{\mathrm{e}}^{\max} := \limsup_{\rho_e \to +\infty} \mu_e(\rho_e)$ be the maximum flow capacity of edge e. Let $f^{\max} \in \mathbb{R}_+^{\mathcal{E}}$ be the vector of maximum flow capacities of the links in $\mathcal{E}$, ordered lexicographically. Let $\Psi(f^{\max}) := \{\mu \in \Psi \mid \limsup_{\rho_e \to +\infty} \mu_e(\rho_e) = f_{\mathrm{e}}^{\max} \quad \forall e \in \mathcal{E}\}$. For a vector $f \in \mathbb{R}_+^{\mathcal{E}}$, put $\lambda_v^-(f) := \sum_{e \in \mathcal{E}_v^-} f_e$ for all $v \in \mathcal{V} \setminus \{o\}$, $\lambda_v^+(f) := \sum_{e \in \mathcal{E}_v^+} f_e$ for all $v \in \mathcal{V} \setminus \{d\}$. For a given $\mathcal{G}$ and $f^{\max} \in \mathbb{R}_{>0}^{\mathcal{E}}$, define the set of admissible flows through $\mathcal{G}$ as

$$\begin{aligned}\mathbb{F}(\mathcal{G}, f^{\max}) := \{f \in \mathbb{R}_{\geq 0}^{\mathcal{E}} \mid f_e \leq f_{\mathrm{e}}^{\max} \quad \forall\, e \in \mathcal{E};\\ \lambda_u^+(f) = \lambda_u^-(f) + \mathbb{1}_o(u) \quad \forall\, u \in \mathcal{V} \setminus \{d\}\}.\end{aligned}$$

Let $\mathring{\mathbb{F}}(\mathcal{G}, f^{\max})$ be the interior of $\mathbb{F}(\mathcal{G}, f^{\max})$. Throughout this paper, we will assume that $\mathcal{G}$ and $f^{\max}$ are such that $\mathbb{F}(\mathcal{G}, f^{\max}) \neq \emptyset$.

Let $\tau_e \geq 0$ be the toll on edge $e \in \mathcal{E}$, and let $\tau \in \mathbb{R}_{\geq 0}^{\mathcal{E}}$ be the vector of tolls. Assuming a unit dollar value for a unit amount of delay the utility associated with edge e when the flow on it is f_e is $-(T_e(f_e) + \tau_e)$. We now recall the notion of a Wardrop equilibrium [1] that also includes the effect of tolls. Define $\mathcal{P}_f := \{p \in \mathcal{P} \mid f_e > 0\,,\ \forall e \in p\}$. A Wardrop equilibrium is a vector $f \in \mathbb{F}(\mathcal{G}, f^{\max})$, such that, for all $p, q \in \mathcal{P}$,

$$p, q \in \mathcal{P}_f \quad \Longrightarrow \quad \sum_{e \in p} T_e\left(f_e\right) + \tau_e = \sum_{e \in q} T_e\left(f_e\right) + \tau_e\,,$$

$$p \in \mathcal{P}_f,\, q \notin \mathcal{P}_f \quad \Longrightarrow \quad \sum_{e \in p} T_e\left(f_e\right) + \tau_e \leq \sum_{e \in q} T_e\left(f_e\right) + \tau_e\,.$$

The following result guarantees the existence and uniqueness of a Wardrop equilibrium in our setting.

Proposition 2.1: Given a $\mathcal{G}$ satisfying (A1), $\mu(\rho) \in \Psi$ and $\tau \in \mathbb{R}_{\geq 0}^{\mathcal{E}}$, there exists a unique Wardrop equilibrium $f^{\mathrm{eq}}(\tau) \in \mathbb{F}(\mathcal{G}, f^{\max})$.

Proof It follows from assumption **(A2)** that, for all $e \in \mathcal{E}$, the delay function $T_e(\,\cdot\,)$ is continuous, strictly increasing, and such that $T_e(0) > 0$. Then, the claim follows by applying Theorems 2.4 and 2.5 from [1]. ■

In this paper, we assume that $\mathcal{G}$, μ and τ are such that $f_e^{\text{eq}}(\tau) > 0$ for every $e \in \mathcal{E}$. This is for technical reasons and we shall address the case when $f_e^{\text{eq}}(\tau) = 0$ for some $e \in \mathcal{E}$ in a future publication. We shall drop the explicit dependency of f^{eq} on τ, when not required, for brevity in notation.

B. Dynamic Route Choice Behavior

We now describe the behavioral model of agents' routing through the network. We envision a very large population of agents traveling through the network. Agents enter the network from the origin node o at a constant unitary rate, travel through it, and leave the network from the destination node d. Once left the network, agents join a reservoir, from which they will eventually reenter the network. While inside the network, agents occupy some link $e \in \mathcal{E}$. The rate at which they leave link e is $f_e = \mu_e(\rho_e)$, where ρ_e is the current occupancy density of e. When leaving link $e \in \mathcal{E}_v^-$, agents choose which link $j \in \mathcal{E}_v^+$ to join. We shall assume such a choice to depend on two factors: (i) a personal belief on what the best path is, which is updated at a time scale much slower than than the one of the dynamics through the network; (ii) an instantaneous local feedback consisting in the observation of the current traffic flows on the outgoing links. Since the focus of this paper is on robustness properties when the network is disturbed from its equilibrium operating condition, we study the relationship between the robustness properties of the network and the myopic behavior of agents based on instantaneous local feedback for a constant belief by the agents on the best path that also coincides with the equilibrium operating condition.

More precisely, if $f^{\text{eq}} \in \mathbb{F}(\mathcal{G}, f^{\max})$ is the Wardrop equilibrium operating condition of the network then it also constitutes the beliefs shared by all the agents on the best path through the network. Let $\rho^{\text{eq}} = \mu^{-1}(f^{\text{eq}})$ be the corresponding equilibrium agent densities on the links. For a node $u \in \mathcal{V} \setminus \{d\}$, let Γ_u be the simplex of probability distributions over $\mathcal{E}_u^+$. We shall assume that the probability that an agent chooses link $e \in \mathcal{E}_u^+$ when traversing node u and observes the actual density $\rho \in \mathbb{R}_+^{\mathcal{E}}$ is given by $G_e^u(\rho, f^{\text{eq}})$, where $G^u : \mathbb{R}_+^{\mathcal{E}} \times \mathbb{F}(\mathcal{G}, f^{\max}) \to \Gamma_u$ is a Lipschitz continuous function. Let G be the vector of node-wise route choice functions G^u. Let Υ be the set of all such Lipschitz continuous G. In the sequel, we shall consider a certain subclass of Υ by placing restrictions on the amount of local information available to the agents. Accordingly, let $\Upsilon_1 \subset \Upsilon$ be the set of G such that the node-wise G^u have access only to the densities on the links $\mathcal{E}_u^+$, i.e., if the *local* density at node u is denoted by $\rho^u \in \mathbb{R}_+^{\mathcal{E}_u^+}$, then such a $G^u(\rho^u, f^{\text{eq}})$ is the route choice function at node u.

C. Dynamical system formulation

For a given $\mathcal{G}$, flow profiles $\mu \in \Psi$, equilibrium flow $f^{\text{eq}} \in \mathbb{F}(\mathcal{G}, f^{\max})$, route choice function vector $G \in \Upsilon$, let $\mathcal{D}(\mathcal{G}, G, \mu, f^{\text{eq}}; t)$ be the solution of the following system of ordinary differential equations:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\rho_e(t) &= \lambda_u^-(f) G_e^u\left(f^u, f^{\text{eq}}\right) - f_e, \quad \rho_e(0) = \mu_e^{-1}(f_e^{\text{eq}}), \\ f_e &= \mu_e(\rho_e), \qquad \forall e \in \mathcal{E}_u^+ \quad \forall u \in \mathcal{V} \setminus \{d\}, \end{aligned} \tag{4}$$

The stability of the transportation network is formally defined as follows.

Definition 2.2: For a given $\mathcal{G}$, flow profiles μ, equilibrium flow f^{eq}, route choice function vector G, the dynamical system given by Equation (4), or equivalently $\mathcal{D}(\mathcal{G}, G, \mu, f^{\text{eq}}; t)$ is stable if

$$\limsup_{t \to +\infty} \|\mathcal{D}(\mathcal{G}, G, \mu, f^{\text{eq}}; t)\|_\infty < +\infty.$$

Remark 2.3: Equations (1) and (3) imply that, for any $e \in \mathcal{E}$, $\rho_e(t) \to +\infty$ implies that $T_e \to +\infty$. Therefore, definition 2.2 implies that the transportation network is stable if the traffic comes to a standstill at one or more locations in the network.

In this paper, we consider disturbances that affect the flow function, specifically by reducing the maximum flow capacity of the edges. We shall measure the magnitude of the disturbance by the resultant reduction in the flow carrying capacity of the network. Let $\Delta(f^{\max}) = \{\delta \in \mathbb{R}_+^{\mathcal{E}} \mid \delta \leq f^{\max}\}$, where the inequality is element wise, denote the set of feasible disturbances. On application of a disturbance $\delta \in \Delta(f^{\max})$, the flow profile switches to $\mu^\delta \in \Psi(f^{\max} - \delta)$ which is also dominated by the pre-disturbance flow function by no more than δ, i.e., it satisfies $\mu_e^\delta(\rho_e) \in [\mu_e(\rho_e) - \delta_e, \mu_e(\rho_e)]$ for all $\rho_e \geq 0$ and all $e \in \mathcal{E}$.

Example: The following post-disturbance model is admissible with respect to the example given in Equation (2).

$$\mu_e^\delta(\rho_e) = \left(f_{\mathrm{e}}^{\max} - \delta_{\mathrm{e}}\right)\left(1 - e^{-\alpha_e \rho_e}\right) \quad e \in \mathcal{E}. \tag{5}$$

D. Robustness: Margin of stability

We quantify the robustness of transportation network by defining a notion of margin of stability as follows. For a given $\mathcal{G}$, equilibrium flow f^{eq}, route choice function vector G, define the set of destabilizing disturbances as:

$$\begin{aligned} \Delta_U(\mathcal{G}, G, f^{\text{eq}}) := \{\delta \in \Delta(f^{\max}) \mid \exists \mu^\delta \in \Psi(f^{\max} - \delta) \text{ s.t.} \\ \mathcal{D}(\mathcal{G}, G, \mu^\delta, f^{\text{eq}}; t) \text{ is unstable}\}. \end{aligned}$$

Define the the margin of stability of the given transportation network as:

$$\gamma\left(\mathcal{G}, G, f^{\text{eq}}\right) := \inf_{\delta \in \Delta_U(\mathcal{G}, G, f^{\text{eq}})} \|\delta\|_1.$$

E. Problem statement

Given a set of feasible route choice functions Υ, define the maximum attainable margin of stability as:

$$\Gamma\left(\mathcal{G}, \Upsilon, f^{\text{eq}}\right) := \sup_{G \in \Upsilon} \gamma\left(\mathcal{G}, G, f^{\text{eq}}\right). \tag{6}$$

Our primary objective is to find the route choice functions G in the given feasible set Υ that yield a margin of stability as close to Γ as possible. Define maximum margin of stability with respect to the equilibrium flow as follows.

$$\Gamma^* (\mathcal{G}, \Upsilon) := \sup_{f^{\mathrm{eq}} \in \mathbb{F}(\mathcal{G}, f^{\max})} \Gamma (\mathcal{G}, \Upsilon, f^{\mathrm{eq}}). \tag{7}$$

Our secondary objective is to identify the equilibrium operating point for the network, $f^{\mathrm{eq}}(\tau)$ that would yield the margin of stability as close to Γ^* as possible and determine the tolls τ that yield a desired equilibrium flow over the network.

In general it is difficult to analytically solve Equations (6) and (7). We start by deriving bounds on the corresponding quantities.

III. Upper bound on the margin of stability

In this section, we derive a number of upper bounds on the margin of stability.

We start by giving an upper bound that holds true for any $G \in \Upsilon$. We first need to mention a few relevant concepts from flow networks.

An $o-d$ cut (O, D) is a partition of $\mathcal{V}$ such that $o \in O$ and $d \in D$. The cut-set of (O, D) is the set $C(O, D) := \{(u, v) \in \mathcal{E} \mid u \in O, v \in D\}$. The capacity of an $o-d$ cut, $C = (O, D)$, is defined as $\mathcal{C}(O, D) = \sum_{e \in C(O,D)} f_{\mathrm{e}}^{\max}$.

Theorem 3.1: Given a graph $\mathcal{G}$ satisfying (A1), equilibrium flow $f^{\mathrm{eq}} \in \mathbb{F}(\mathcal{G}, f^{\max})$, we have that

$$\gamma (\mathcal{G}, G, f^{\mathrm{eq}}) \leq \min_{\{A \subset \mathcal{V} \mid o \in A,\, d \notin A\}} \mathcal{C}(A, A^c) - 1, \quad \forall\, G \in \Upsilon.$$

Proof Let $e^* := \mathrm{argmax}_{e \in \mathcal{E}_{\mathrm{mc}}} f_e^{\mathrm{eq}}$, where $\mathcal{E}_{\mathrm{mc}}$ is the cut set of $(\tilde{A}, \tilde{A}^c)$ with $\tilde{A} := \mathrm{argmin}_{\{A \subset \mathcal{V} \mid o \in A,\, d \notin A\}} \mathcal{C}(A, A^c)$. For any $\epsilon \in \big(0, \max_{e \in \mathcal{E}_{\mathrm{mc}}} f_e^{\mathrm{eq}}\big)$, consider a feasible disturbance vector $\delta(\epsilon)$, where

$$\delta_e(\epsilon) = \begin{cases} f_{\mathrm{e}}^{\max} - f_e^{\mathrm{eq}} + \epsilon \cdot \mathbb{1}_{e^*}(e) & \text{if } e \in \mathcal{E}_{\mathrm{mc}}, \\ 0 & \text{otherwise,} \end{cases}$$

Note that,

$$\|\delta(\epsilon)\|_1 = \sum_{e \in \mathcal{E}_{\mathrm{mc}}} (f_{\mathrm{e}}^{\max} - f_e^{\mathrm{eq}}) + \epsilon. \tag{8}$$

Let this disturbance be applied to the network at time $t = 0$.

The max-flow min-cut theorem [10] implies that a sufficient condition for $\mathbb{F}(\mathcal{G}, f^{\max} - \delta(\epsilon)) = \emptyset$ is that the minimum of the capacities of all the $o-d$ cuts in $(\mathcal{G}, f^{\max} - \delta(\epsilon))$ is strictly less than 1. Capacity of the $\mathcal{E}_{\mathrm{mc}}$ cut in $(\mathcal{G}, f^{\max} - \delta(\epsilon))$ is $\sum_{e \in \mathcal{E}_{\mathrm{mc}}} f_{\mathrm{e}}^{\max} - \|\delta(\epsilon)\|_1$, which is equal to $1 - \epsilon$. This implies that, the minimum of the capacities of all the $o-d$ cuts in $(\mathcal{G}, f^{\max} - \delta(\epsilon))$ is less than or equal to $1 - \epsilon$, i.e., strictly less than 1 for all $\epsilon > 0$. Therefore, $\mathbb{F}(\mathcal{G}, f^{\max} - \delta(\epsilon)) = \emptyset$ for all $\epsilon > 0$. This implies that, for all $\epsilon > 0$, there is at least one one edge $e \in \mathcal{E}$ such that $\frac{\mathrm{d}}{\mathrm{d}t}\rho_e(t) > 0$ for all $t \geq 0$, which in turn implies that $\lim_{t \to +\infty} \rho_e(t) = \infty$. In other words, $\mathcal{D}(\mathcal{G}, G, \mu^\delta, f^{\mathrm{eq}}; t)$ is unstable for all $\epsilon > 0$ and $G \in \Upsilon$. The result follows by taking infimum over $\epsilon > 0$ in Equation (8). ∎

One can prove a tighter upper bound by restricting attention to route choice functions in Υ_1.

Theorem 3.2: Given a graph $\mathcal{G}$ satisfying (A1), equilibrium flow $f^{\mathrm{eq}} \in \mathring{\mathbb{F}}(\mathcal{G}, f^{\max})$, we have that,

$$\gamma (\mathcal{G}, G, f^{\mathrm{eq}}) \leq \min_{u \in \mathcal{V} \setminus \{d\}} \sum_{e \in \mathcal{E}_{u^+}} (f_{\mathrm{e}}^{\max} - f_e^{\mathrm{eq}}), \quad \forall G \in \Upsilon_1.$$

Proof For an acyclic $\mathcal{G}$ and a $G \in \Upsilon_1$ and hence satisfying (A3), the inflow at any node $u \in \mathcal{V} \setminus \{d\}$ is independent of the agent densities *downstream* from u. Let $\overline{u} \in \mathcal{V} \setminus \{d\}$ be such that $\sum_{e \in \mathcal{E}_{\overline{u}}^+} \big(f_{\mathrm{e}}^{\max} - f_e^{\mathrm{eq}}\big) = \min_{u \in \mathcal{V} \setminus \{d\}} \sum_{e \in \mathcal{E}_u^+} \big(f_{\mathrm{e}}^{\max} - f_e^{\mathrm{eq}}\big)$. Let $e^* := \mathrm{argmax}_{e \in \mathcal{E}_{\overline{u}}^+} f_e^{\mathrm{eq}}$. For any $\epsilon \in \big(0, f_{e^*}^{\mathrm{eq}}\big)$, consider a feasible disturbance vector $\delta(\epsilon)$ defined as

$$\delta_e(\epsilon) = \begin{cases} f_{\mathrm{e}}^{\max} - f_e^{\mathrm{eq}} + \epsilon \cdot \mathbb{1}_{e^*}(e) & \text{if } e \in \mathcal{E}_{\overline{u}}^+, \\ 0 & \text{otherwise,} \end{cases}$$

Note that,

$$\|\delta(\epsilon)\|_1 = \sum_{e \in \mathcal{E}_{\overline{u}}^+} (f_{\mathrm{e}}^{\max} - f_e^{\mathrm{eq}}) + \epsilon. \tag{9}$$

Therefore, the dynamics of the sum of densities on the edges outgoing from the node $\overline{u}$ under the disturbance $\delta(\epsilon)$ can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{e \in \mathcal{E}_{\overline{u}}^+} \rho_e(t)\right) = \sum_{e \in \mathcal{E}_{\overline{u}}^+} f_e^{\mathrm{eq}} - \sum_{e \in \mathcal{E}_{\overline{u}}^+} \mu_e^{\delta(\epsilon)}(\rho_e). \tag{10}$$

For all $e \in \mathcal{E}$, we have that

$$\mu_e^{\delta(\epsilon)}(\rho_e) \leq \limsup_{\rho_e \to +\infty} \mu_e^{\delta(\epsilon)}(\rho_e) = f_{\mathrm{e}}^{\max} - \delta_e(\epsilon), \quad \forall \rho_e \geq 0. \tag{11}$$

Therefore, combining Equations (9), (10) and (11), we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{e \in \mathcal{E}_{\overline{u}}^+} \rho_e(t)\right) \geq \epsilon > 0,$$

i.e., $\sum_{e \in \mathcal{E}_{\overline{u}}^+} \rho_e(t) \to +\infty$ as $t \to +\infty$. Therefore, $\rho_{e'}(t) \to +\infty$ as $t \to +\infty$ for at least one $e' \in \mathcal{E}_{\overline{u}}^+$. The result follows by taking infimum over $\epsilon > 0$ in Equation (9). ∎

Remark 3.3: It is easy to verify that the upper bound in Theorem 3.2 is less than or equal to the upper bound in Theorem 3.1. This, possibly, illustrates the loss in margin of stability due to lack of global information.

For the rest of the paper we shall restrict our attention to the set Υ_1 as the set of feasible route choice functions.

IV. Route choice behavior and lower bound on the margin of stability

In this section, we derive lower bounds on the margin of stability for the specific case when $f_e^{\mathrm{eq}} > 0$ for all $e \in \mathcal{E}$ and the route choice set is Υ_1. Throughout this section, we shall write $G_j^v(\rho^v)$ for $G_j^v(\rho^v; f^{\mathrm{eq}})$.

In particular, we will show that a $G \in \Upsilon_1$ with the following properties will give the maximum possible margin of stability.

(A3) $\mu_e(\rho_e) = f_e^{\mathrm{eq}}, \ \forall e \in \mathcal{E}_v^+ \implies \lambda_v^-(f^{\mathrm{eq}}) G_j^v(\rho^{\mathrm{eq}}) = f_j^{\mathrm{eq}}$;

(A4) G_j^v is differentiable in ρ and $\partial G_j^v(\rho^v)/\partial\rho_k < 0\,, \quad \forall j \neq k \in \mathcal{E}_v^+$.

(A5) $\lim_{\rho_j \to +\infty} G_j(\rho) = 0\,, \qquad \forall j \in \mathcal{E}_v^+$.

Assumption **(A3)** is a consistency assumption, and ensures that, if the current flow coincides locally with the agents' belief, then their route choice coincides with such a belief. Assumption **(A4)** instead captures the fact that, if the flow, and hence the congestion, on a certain link is increased, then each of the other links is chosen with higher probability. Let $\overline{\Upsilon}_1$ be the subset of Υ_1 whose elements satisfy assumptions (A3) and (A4).

An example of function G that satisfies conditions **(A3)-(A5)** when $f^{\text{eq}} \in \mathbb{F}(\mathcal{G}, \mathring{f}^{\max})$ is the i-logit route choice with noise level $\beta > 0$, defined by

$$G_e^v(\rho^v) = \frac{f_e^{\text{eq}} \exp(-\beta(\mu_e(\rho_e) - f_e^{\text{eq}}))}{\sum_{j\in\mathcal{E}_v^+} f_j^{\text{eq}} \exp(-\beta(\mu_j(\rho_j) - f_j^{\text{eq}})}\,,$$

for all $v \in \mathcal{V} \setminus \{d\}$, and $e \in \mathcal{E}_v^+$.

We now present a local result for the behaviour of the system on the set of outgoing edges from a given node. We start with the following result ensuring existence of a local equilibrium.

Lemma 4.1: For every node $v \in \mathcal{V} \setminus \{d\}$, and $0 \leq \lambda_v < \sum_{e\in\mathcal{E}_v^+} f_e^{\max}$, there exists an equilibrium point $\rho^v \in \mathbb{R}^{\mathcal{E}_v^+}$ such that

$$\lambda_v G_e^v(\rho^v) = \mu_e(\rho_e)\,, \qquad \forall e \in \mathcal{E}_v^+\,.$$

Proof We state the proof for the case when $|\mathcal{E}_v^+| = 2$. The proof for the general case follows along similar lines. Let $\mathcal{E}_v^+ = \{e, j\}$. First, if both $\lambda_v \leq f_e^{\max}$, and $\lambda_v \leq f_j^{\max}$, then the map $g(f_e) = \lambda_v G_e^v(f_e, \lambda_v - f_e)$ continuously maps the closed interval $[\lambda_v - f_j^{\max}, f_e^{\max}]$ in itself, and existence of a fixed point is guaranteed by Brouwer's fixed point theorem.

Second, if both $\lambda_v > f_e^{\max}$, and $\lambda_v > f_j^{\max}$, then consider the function $h(f_e) = \lambda_v G_e^v(\mu_e^{-1}(f_e), \mu_j^{-1}(\lambda_v - f_e)) - f_e$. Then, h is continuous over the nonempty interval $(\lambda_v - f_j^{\max}, f_e^{\max})$, and satisfies

$$\lim_{f_e \to \lambda_v - f_j^{\max}} h(f_e) = f_j^{\max}\,, \quad \lim_{f_e \to f_e^{\max}} h(f_e) = -f_e^{\max}\,.$$

Therefore, by the mean-value theorem, there exists some f_e^* such that $h(f_e^*) = 0$. Setting $f_j^* = \lambda_v - f_e^*$ concludes the proof.

Finally, consider the case when $f_e^{\max} < \lambda_v$, while $f_j^{\max} \geq \lambda_v$ (the case when the roles of e and j are reversed can be treated by a symmetric argument). Then, the function $h(f_e) = \lambda_v G_e^v(\mu_e^{-1}(f_e), \mu_j^{-1}(\lambda_v - f_e)) - f_e$ is continuous and decreasing over the interval $(0, f_e^{\max})$, and satisfies

$$\lim_{f_e \to 0} h(f_e) > 0\,, \quad \lim_{f_e \to f_e^{\max}} h(f_e) = -f_e^{\max}\,,$$

and existence of a zero is again guaranteed by the mean-value theorem. ∎

We now prove an important lemma on the node-wise stability and diffusivity properties of $G \in \overline{\Upsilon}_1$.

Lemma 4.2 (Local stability and diffusivity properties): Given a $\delta \in \Delta(f^{\max})$ and a $G \in \overline{\Upsilon}_1$, consider the initial value problem for every node $v \in \mathcal{V} \setminus \{d\}$:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\rho_j = \lambda_v(t) G_j^v(\rho^v) - \mu_j^\delta(\rho_j)\,, \\ \rho_j(0) = \rho_j^{\text{eq}}\,, \qquad\qquad j \in \mathcal{E}_v^+\,. \end{cases}$$

Assume that the input $\lambda_v(t)$ is continuous, and it satisfies the following for every node $v \in \mathcal{V} \setminus \{d\}$:

$$\lambda_v(t) \leq \lambda_v^* \leq \sum\nolimits_{j\in\mathcal{E}_v^+} \left(f_j^{\max} - \delta_j\right)\,, \qquad t \geq 0\,. \tag{12}$$

Then, for all $v \in \mathcal{V} \setminus \{d\}$ and $t \geq 0$, we have that

$$\sum_{j\in J} \mu_j^\delta(\rho_j(t)) - f_j^{\text{eq}} \leq \left[\lambda_v^* - \lambda_v^-(f^{\text{eq}})\right]^+ + \sum_{j\in\mathcal{E}_v^+} \delta_j. \quad \forall J \subseteq \mathcal{E}_v^+. \tag{13}$$

.

Proof Let $\tilde{\rho}^v(t)$ be the solution of the initial value problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\tilde{\rho}_j = \lambda_v^* G_j^v(\tilde{\rho}^v) - \mu_j^\delta(\tilde{\rho}_j) \\ \tilde{\rho}_j(0) = \rho_j^{\text{eq}}\,. \end{cases}$$

Observe that, thanks to Assumption **(A4)**, for every $\rho^v \neq \rho^{\text{eq}} \in \mathbb{R}_+^{\mathcal{E}_v^+}$ such that

$$\rho_j = \rho_j^{\text{eq}} \quad \text{for some } j \in \mathcal{E}_v^+, \qquad \rho_e \geq \rho_e^{\text{eq}}, \quad \forall e \neq j \in \mathcal{E}_v^+,$$

one has $G_j^v(\rho^v) > G_j^v(\rho^{\text{eq}})$, and then, for all $t \geq 0$,

$$\lambda^* G_j(\rho^v(t)) - \mu_j^\delta(\rho_j) > (\sum_e f_e^{eq}) G_j(\rho^*) - \mu_j^\delta(\rho_j^{\text{eq}}) \geq 0\,.$$

Therefore, if we consider the region

$$\mathcal{R}^{eq} := \{\rho^v :\ \rho_j \geq \rho_j^{\text{eq}}\,,\ \forall j \in \mathcal{E}\}\,,$$

and denote by n the outward-pointing normal vector to $\partial\mathcal{R}^*$, one has that

$$\left(\lambda(t) G_j^v(\rho^v) - \mu_j^\delta(\rho_j)\right) \cdot n < 0\,, \quad \forall \rho^v \in \partial\mathcal{R}^*\,,\ t \geq 0\,.$$

It follows that

$$\tilde{\rho}_e(t) \geq \rho_e^{\text{eq}}\,, \qquad \forall e \in \mathcal{E},\ \forall t \geq 0\,. \tag{14}$$

On the other hand, thanks to Lemma 4.1, there exists ρ^* such that

$$(\sum_e f_e^{eq}) G_j^v(\rho^*) = \mu_j^\delta(\rho_j^*)\,, \qquad \forall j \in \mathcal{E}_v^+\,.$$

Then, for all $\rho^v \neq \rho^*$ such that

$$\rho_j = \rho_j^* \quad \text{for some } j \in \mathcal{E}, \qquad \rho_e \leq \rho_e^*, \quad \forall e \neq j \in \mathcal{E}\,.$$

Assumption **(A4)** implies that $G_j^v(\rho^v) < G_j^v(\rho^*)$, and then

$$\lambda(t) G_j^v(\rho^v) - \mu_j^\delta(\rho_j) < \lambda^* G_j^v(\rho^*) - \mu_j^\delta(\rho_j^*) = 0\,.$$

Therefore,

$$\left(\lambda^* G_j^v(\rho^v) - \mu_j^\delta(\rho_j)\right) \cdot n < 0\,, \quad \forall \rho^v \in \partial\mathcal{R}^*\,,$$

where

$$\mathcal{R}^* := \{\rho^v :\ \rho_j \leq \rho_j^*\,,\ \forall j \in \mathcal{E}_v^+\}\,,$$

and n is the outward-pointing normal vector to its boundary. This proves that

$$\tilde{\rho}_e(t) \leq \rho_e^*, \qquad \forall e \in \mathcal{E},\ \forall t \geq 0,$$

from which, in particular, it follows that

$$\sum_{e\in\mathcal{E}} \mu_e^\delta(\tilde{\rho}_e(t)) \leq \sum_{e\in\mathcal{E}} \mu_e^\delta(\rho_e^*) = \lambda^*. \tag{15}$$

Now, combining (15) with (14), one gets that, for all $t \geq 0$

$$\begin{array}{rcl}\sum_{j\in J}\mu_j^\delta(\tilde{\rho}_j(t)) & \leq & \lambda_v^* - \sum\limits_{j\notin J}\mu_j^\delta(\tilde{\rho}_j(t)) \\ & \leq & \lambda_v^* - \sum\limits_{j\notin J}\mu_j^\delta(\rho_j^{\mathrm{eq}}) \\ & = & \delta_\lambda + \sum\limits_{j\in J} f_j^{\mathrm{eq}} + \sum_{j\notin\mathcal{J}}\left(f_j^{\mathrm{eq}} - \mu_j^\delta(\rho_j^{\mathrm{eq}})\right) \\ & \leq & \delta_\lambda + \sum\limits_{j\in J} f_j^{\mathrm{eq}} + \delta_\mu\,,\end{array}$$

where $\delta_\lambda = \lambda_v^* - \lambda_v^-(f^{\mathrm{eq}})$ and $\delta_\mu = \sum_{j\in\mathcal{E}_v^+}\delta_j$. To complete the proof, it remains to show that

$$\tilde{\rho}_j(t) \geq \rho_j(t), \qquad t \geq 0. \tag{16}$$

for all $j \in \mathcal{E}_v^+$. In order to see this, first observe that $\tilde{\rho}_j(0) = \rho_j(0)$. Moreover, if $\tilde{\rho}_j(t) = \rho_j(t)$ for some j, and $\tilde{\rho}_e(t) \geq \rho_e(t)$ for all other e, then Assumption (A4) guarantees that $G_j^v(\tilde{\rho}^v) \geq G_j^v(\rho^v)$, and, as a consequence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\rho}_j = \lambda^* G_j^v(\tilde{\rho}^v) - \mu_j^\delta(\tilde{\rho}_j) \geq \lambda(t) G_j^v(\rho^v) - \mu_j^\delta(\rho_j) = \frac{\mathrm{d}}{\mathrm{d}t}\rho_j\,,$$

which in turn can be shown to imply (16). ■

Remark 4.3: Lemma 4.2 implies that for a $G \in \overline{\Upsilon}_1$, the effect of a disturbance diffuses away from the location where it is applied. Specifically, the increase in the flow on all the edges downstream from a node whose outgoing edges are affected by a disturbance, is less than the magnitude of the disturbance.

One can exploit the local stability and diffusivity property of G in $\overline{\Upsilon}_1$ from Lemma 4.2 along with a standard induction associated with a topological ordering of an acyclic graph to prove the following theorem.

Theorem 4.4: Given a graph $\mathcal{G}$ satisfying (A1), equilibrium flow $f^{\mathrm{eq}} \in \mathbb{F}(\mathcal{G}, f^{\max})$, we have that

$$\gamma(\mathcal{G}, G, f^{\mathrm{eq}}) \geq \min_{u\in\mathcal{V}\setminus\{d\}} \sum_{e\in\mathcal{E}_u^+}\left(f_{\mathrm{e}}^{\max} - f_e^{\mathrm{eq}}\right) \quad \forall G \in \overline{\Upsilon}_1,$$

and hence,

$$\Gamma(\mathcal{G}, \Upsilon_1, f^{\mathrm{eq}}) = \min_{u\in\mathcal{V}\setminus\{d\}} \sum_{e\in\mathcal{E}_u^+}\left(f_{\mathrm{e}}^{\max} - f_e^{\mathrm{eq}}\right) \quad \forall G \in \overline{\Upsilon}_1. \tag{17}$$

Note that the second part of the Theorem 4.4 follows by combining the first part and Theorem 3.2.

Remark 4.5: Theorem 4.4 implies that, for a given f^{eq}, the route choice functions G in $\overline{\Upsilon}_1$ give the maximum possible margin of stability.

V. Robust equilibrium selection and optimal toll selection

In the previous sections, we studied robustness properties of a transportation network around a given equilibrium point. We now return to our secondary objective of identifying the most robust equilibrium operating point for the network.

A. Robust equilibrium selection as an optimization problem

The robust equilibrium selection problem can be posed as an optimization problem as follows:

$$\begin{array}{rl}\text{maximize} & \Gamma(\mathcal{G}, \Upsilon_1, f^{\mathrm{eq}}), \\ \text{subj. to} & f^{\mathrm{eq}} \in \mathbb{F}(\mathcal{G}, f^{\max}).\end{array} \tag{18}$$

The solution to this optimization problem can help a system planner evaluate the distribution of traffic flow that is most robust to disruptions and can implement this distribution using, for example, using tolls τ, e.g., see [5]. Similar optimization problems and their solution methodologies have been widely studied in context of traffic assignment in [1].

Equation (17) shows that, under these conditions, Γ^* is a minimum of a set of functions linear in f^{eq} and hence is concave in f^{eq}. Therefore the optimization problem stated in Equation (18) is equivalent to minimizing a convex function over a convex polytope. However, note that the objective function, $\Gamma(\mathcal{G}, \Upsilon_1, f^{\mathrm{eq}})$ is non-smooth and one needs to use non-smooth convex optimization techniques, e.g., see [12], to solve this problem.

B. The robustness price of anarchy

Conventionally, transportation networks have been viewed as static flow networks, where a given equilibrium traffic flow is an outcome of driver's selfish behavior in response to the delays associated with various paths and the incentive mechanisms in place. The price of anarchy [13] has been suggested as a metric to measure how sub-optimal a given equilibrium is with respect to the societal optimal equilibrium, where the societal optimality is related to the average delay faced by the agents. In the context of robustness analysis of transportation networks, it is natural to consider societal optimality from the robustness point of view, thereby motivating a notion of the robustness price of anarchy. Formally, it can be defined as

$$P(\mathcal{G}, \Upsilon, f^{\mathrm{eq}}) = \Gamma^*(\mathcal{G}, \Upsilon) - \Gamma(\mathcal{G}, \Upsilon, f^{\mathrm{eq}}).$$

It is worth noting that, for a parallel topology, we have that $\Gamma^*(\mathcal{G}, \Upsilon_1, f^{\mathrm{eq}}) = \Gamma^*(\mathcal{G}, \Upsilon, f^{\mathrm{eq}}) = \sum_{e\in\mathcal{E}} f_{\mathrm{e}}^{\max} - 1$ for all f^{eq}. That is, the margin of stability is independent of the equilibrium operating condition and hence, for a parallel topology, $P(\mathcal{G}, \Upsilon, f^{\mathrm{eq}}) = 0$ for all f^{eq}. However, for a general topology and a general equilibrium, this quantity is non-zero. In the next section, we discuss the use of tolls to yield a robust equilibrium point for a given topology, i.e., the one for which the robustness price of anarchy is zero.

C. Tolls for the robust equilibrium point

In this section, we discuss the use of tolls τ to yield a desired equilibrium operating condition, $f^{\text{eq}}(\tau)$, for the network.

Proposition 5.1: Given a graph $\mathcal{G}$ satisfying (A1), $\mu \in \Psi$, and a desired equilibrium $f^* \in \mathring{\mathbb{F}}(\mathcal{G}, f^{\max})$, a set of tolls $\tau^* \in \mathbb{R}_+^{\mathcal{E}}$ that yield this desired equilibrium is given by

$$\tau^* = \max_{e \in \mathcal{E}} \frac{T_e(1)}{T_e(f^{\text{eq}}(0))} \cdot T(f^{\text{eq}}(0)) - T(f^*),$$

where $f^{\text{eq}}(0)$ is the Wardrop equilibrium for tolls set to zero.

Proof Let Π be a simplex of dimension $\mathcal{P}$ (number of paths in $\mathcal{G}$ between o and d). Consider the function $V : \Pi \to \mathbb{R}$ that serves as a potential function for the congestion game at hand:

$$\begin{aligned} V(\pi) =& \sum_{e \in \mathcal{E}} \int_0^{f_e} (\tau_e + T_e(z))\, dz \\ =& \tau^T f + \sum_{e \in \mathcal{E}} \int_0^{f_e} T_e(z) dz, \end{aligned} \tag{19}$$

where $f = A^T \pi$, with $A \in \{0,1\}^{\mathcal{P} \times \mathcal{E}}$ being the path-link incidence matrix, i.e., for all $e \in \mathcal{E}$ and $p \in \mathcal{P}$, $A_{p,e} = 1$ if e lies in path p and zero otherwise. Equation (19) can be rewritten as

$$V(\pi) = (A\tau)^T \pi + \sum_{e \in \mathcal{E}} \int_0^{(A^T\pi)_e} T_e(z) dz,$$

Following assumption (A2) and the discussion thereafter, it is easy to see that $V(\pi)$ is convex in π. It is known, e.g., see Theorem 2.1 in [1], that the (unique) Wardrop equilibrium $f^{\text{eq}}(\tau)$ is equivalent to the first order optimality condition of the following optimization problem:

$$\begin{aligned} &\text{minimize } \; V(\pi), \\ &\text{subj. to } \; \pi \in \Pi. \end{aligned} \tag{20}$$

Let $\zeta \in \mathbb{R}$ be the Lagrange multiplier corresponding to the constraint in (20). The Lagrangian function can then be written as

$$L(\pi, \zeta) := (A\tau)^T \pi + \sum_{e \in \mathcal{E}} \int_0^{(A^T\pi)_e} T_e(z) dz + \zeta \left(1 - \mathbf{1}_{\mathcal{P}}^T \pi\right).$$

Considering the first order optimality conditions, the necessary and sufficient condition for $f^* \in \mathring{\mathbb{F}}(\mathcal{G}, f^{\max})$ to be a Wardrop equilibrium is the existence of $\tau^* \in \mathbb{R}_+^{\mathcal{E}}$ and $\zeta^* \in \mathbb{R}$ that satisfy the following condition:

$$A\left(\tau^* + T(f^*)\right) = \zeta^* \mathbf{1}_{\mathcal{P}}. \tag{21}$$

Since $f^{\text{eq}}(0)$ is a Wardrop equilibrium for $\tau = 0$, the first order optimality conditions imply that there exists $\hat{\zeta} \in \mathbb{R}$ such that

$$AT(f^{\text{eq}}(0)) = \hat{\zeta} \mathbf{1}_{\mathcal{P}}. \tag{22}$$

Using Equation (22) and simple algebra, one can verify that Equation (21) is satisfied for $\tau^* = \max_{e \in \mathcal{E}} \frac{T_e(1)}{T_e(f^{\text{eq}}(0))} \cdot T(f^{\text{eq}}(0)) - T(f^*)$ and $\zeta^* = \max_{e \in \mathcal{E}} \frac{T_e(1)}{T_e(f^{\text{eq}}(0))} \cdot \hat{\zeta}$. ■

Remark 5.2: The set of tolls that yield a desired equilibrium operating condition is not unique. In fact, any toll of the form $\tau^* = \eta T(f^{\text{eq}}(0)) - T(f^*)$, with $\eta \geq \max_{e \in \mathcal{E}} \frac{T_e(f^*)}{T_e(f^{\text{eq}}(0))}$ would yield f^* as the equilibrium condition. Proposition 5.1 gives just one such set of tolls.

VI. Conclusion

In this paper, we studied robustness properties of transportation networks with respect to its pre-disturbance equilibrium operating condition and the agents' response to the disturbance. We considered disturbances that reduce the maximum flow carrying capacities of the edges by affecting their congestion properties. We define the margin of stability of the network to be the maximum sum of capacity losses, under which the traffic densities on all the edges remain bounded over time. We characterized the class of route choice functions that yield the maximum margin of stability for a given equilibrium operating condition and also the formulated an optimization problem to find the most robust equilibrium point. Finally, we discussed the use of tolls in yielding a desired equilibrium operating condition.

In future, we plan to extend the research in several directions. First, we plan to study the dependence of the margin on stability on the amount of information available to the agents. We also plan to perform robustness analysis in a probabilistic framework versus the min-max framework of this paper, possibly considering other general models for disturbances. Finally, we also plan to consider more general graph topologies, e.g., graphs have cycles, multiple origin-destination pairs etc.

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