

NOTES FOR MATH 535A: DIFFERENTIAL GEOMETRY

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1. REVIEW OF TOPOLOGY AND LINEAR ALGEBRA

1.1. Review of topology.

Definition 1.1. A topological space (X, \mathcal{T}) consists of a set X , together with a collection $\mathcal{T} = \{U_\alpha\}$ of subsets of X , satisfying the following:

- (1) $\emptyset \in \mathcal{T}, X \in \mathcal{T}$,
- (2) if $U_\alpha, U_\beta \in \mathcal{T}$, then $U_\alpha \cap U_\beta \in \mathcal{T}$,
- (3) if $U_\alpha \in \mathcal{T}$ for all $\alpha \in I$, then $\cup_{\alpha \in I} U_\alpha \in \mathcal{T}$. (Here I is an indexing set, and is not necessarily finite.)

\mathcal{T} is called a topology for X , and $U_\alpha \in \mathcal{T}$ is called an open set of X .

Example 1: $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ (n times) $= \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R}, i = 1, \dots, n\}$, called real n -dimensional space.

How to define \mathcal{T} , the set of open sets of \mathbb{R}^n ?

$$B_y(r) = \{x \in \mathbb{R}^n \mid |x - y| < r\}, \text{ where } |x - y| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

Is $\mathcal{T}_0 = \{B_y(r) \mid y \in \mathbb{R}^n, r \in (0, \infty)\}$ a valid topology for \mathbb{R}^n ?

No, so you must add more open sets to \mathcal{T}_0 to get a valid topology for \mathbb{R}^n .

$$\mathcal{T} = \{U \mid \forall p \in U, \exists B_p(r) \subset U\}.$$

Example 2: $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. A reasonable topology on S^1 is called the *induced topology*.

Definition 1.2. Let (X, \mathcal{T}) be a topological space and let $f : Y \rightarrow X$. Then the induced topology $f^{-1}\mathcal{T} = \{f^{-1}(U) \mid U \in \mathcal{T}\}$ is a topology on Y .

Example 3: Alternative definition of S^1 as $[0, 1]/\sim$, where $[0, 1]$ is the closed interval and the equivalence relation identifies $0 \sim 1$. A reasonable topology on S^1 is called the *quotient topology*.

Definition 1.3. Let (X, \mathcal{T}) be a topological space, \sim be an equivalence relation on X , $\overline{X} = X/\sim$ be the set of equivalence classes of X , and $\pi : X \rightarrow \overline{X}$ be the projection map which sends $x \in X$ to its equivalence class $[x]$. Then the quotient topology $\overline{\mathcal{T}}$ of \overline{X} is the set of $V \subset \overline{X}$ for which $\pi^{-1}(V)$

is open. (Alternatively, given a topological space (X, \mathcal{T}) and a surjective map $\pi : X \twoheadrightarrow Y$, the quotient topology on Y is the largest topology which makes π continuous.)

Exercise: Show that the topologies introduced in Example 2 and Example 3 are the same. (Think about what it means for them to be the same.)

Definition 1.4. A map $f : X \rightarrow Y$ between topological spaces is continuous if $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$ is open whenever $V \subset Y$ is open.

Exercise: Show that the inclusion $S^1 \subset \mathbb{R}^2$ is a continuous map. Show that the quotient map $[0, 1] \rightarrow S^1 = [0, 1] / \sim$ is a continuous map.

Definition 1.5. A map $f : X \rightarrow Y$ is a homeomorphism if there exists an inverse $f^{-1} : Y \rightarrow X$ for which f and f^{-1} are both continuous.

Zen of mathematics: Any “category” (world) in mathematics consists of spaces and maps between spaces.

Examples:

- (1) (Topological category) Topological spaces and continuous maps.
- (2) (Groups) Groups and homomorphisms.
- (3) (Linear category) Vector spaces and linear transformations.

1.2. Review of linear algebra.

Definition 1.6. A vector space V over a field $k = \mathbb{R}$ or \mathbb{C} is a set V equipped with two operations $V \times V \rightarrow V$ (called addition) and $k \times V \rightarrow V$ (called scalar multiplication) s.t.

- (1) V is an abelian group under addition.
 - (a) (Identity) There is a zero element 0 s.t. $0 + v = v + 0 = v$.
 - (b) (Inverse) Given $v \in V$ there exists an element $w \in V$ s.t. $v + w = w + v = 0$.
 - (c) (Associativity) $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$.
 - (d) (Commutativity) $v + w = w + v$.
- (2)
 - (a) $1v = v$.
 - (b) $(ab)v = a(bv)$.
 - (c) $a(v + w) = av + aw$.
 - (d) $(a + b)v = av + bv$.

Note: Keep in mind the Zen of mathematics — we have defined objects (vector spaces), and now we need to define maps between objects.

Definition 1.7. A linear map $\phi : V \rightarrow W$ between vector spaces over k satisfies $\phi(v_1 + v_2) = \phi(v_1) + \phi(v_2)$ ($v_1, v_2 \in V$) and $\phi(cv) = c \cdot \phi(v)$ ($c \in k$ and $v \in V$).

Now, what is the equivalent of *homeomorphism* in the linear category?

Definition 1.8. A linear map $\phi : V \rightarrow W$ is an isomorphism if there exists a linear map $\psi : W \rightarrow V$ such that $\phi \circ \psi = id$ and $\psi \circ \phi = id$. (We often also say ϕ is invertible.)

If V and W are finite-dimensional, then we may take bases $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_m\}$ and represent ϕ as an $m \times n$ matrix A . ϕ is then invertible if and only if $m = n$ and $\det(A) \neq 0$.

Examples of vector spaces: Let $\phi : V \rightarrow W$ be a linear map of vector spaces.

- (1) The *kernel* $\text{Ker } \phi = \{v \in V \mid \phi(v) = 0\}$ is a vector subspace of V .
- (2) The *image* $\text{Im } \phi = \{\phi(v) \mid v \in V\}$ is a vector subspace of W .
- (3) Let $V \subset W$ be a subspace. Then the *quotient* $W/V = \{w + V \mid w \in W\}$ is a vector space. Here $w + V = \{w + v \mid v \in V\}$.
- (4) The *cokernel* $\text{Coker } \phi = W/\text{Im } \phi$.

2. REVIEW OF DIFFERENTIATION

2.1. **Definitions.** Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a map. The discussion carries over to $f : U \rightarrow V$, if U and V are open sets of \mathbb{R}^n and \mathbb{R}^m .

Definition 2.1. The map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is differentiable at a point $x \in \mathbb{R}^n$ if there exists a linear map $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$ satisfying

$$(1) \quad \lim_{h \rightarrow 0} \frac{|f(x+h) - f(x) - L(h)|}{|h|} = 0,$$

where $h \in \mathbb{R}^n$.

Exercise: Show that if $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is differentiable at $x \in \mathbb{R}^n$, then there is a unique L which satisfies Equation (1).

If L exists, it is usually written as $df(x)$ and is called the *derivative of f at x* .

Lemma 2.2. If f is differentiable at x , then $df(x) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear map which satisfies

$$df(x)(v) = \lim_{t \rightarrow 0} \frac{f(x+tv) - f(x)}{t}.$$

$df(x)(v)$ is the *directional derivative* of f at x in the direction of v .

2.2. **Partial derivatives.** Let e_j be the usual basis element $(0, \dots, 1, \dots, 0)$ (1 in the j th position). Then $df(x)(e_j)$ is usually called the *partial derivative* and is written as $\frac{\partial f}{\partial x_j}(x)$ or $\partial_j f(x)$. More explicitly, if we write $f = (f_1, \dots, f_m)^T$, where $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$, then $\frac{\partial f}{\partial x_j}(x) = (\frac{\partial f_1}{\partial x_j}(x), \dots, \frac{\partial f_m}{\partial x_j}(x))^T$, and $df(x)$ can be written in matrix form as follows:

$$df(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \dots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \dots & \frac{\partial f_m}{\partial x_n}(x) \end{pmatrix}$$

The matrix is usually called the *Jacobian matrix*.

Definition 2.3. f is smooth or of class C^∞ at $x \in \mathbb{R}^n$ if all partial derivatives of all orders exist at x .

Facts:

- (1) If $\partial_i(\partial_j f)$ and $\partial_j(\partial_i f)$ are continuous on an open set $\ni x$, then $\partial_i(\partial_j f)(x) = \partial_j(\partial_i f)(x)$.
- (2) $df(x)$ exists if all $\frac{\partial f_i}{\partial x_j}(y)$, $i = 1, \dots, m$, $j = 1, \dots, n$, exist on an open set $\ni x$ and each $\frac{\partial f_i}{\partial x_j}$ is continuous at x . [Such a function f is *continuously differentiable* or of class C^1 .]

Shorthand: Assuming f is smooth, we write $\partial^\alpha f = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_n^{\alpha_n} f$ where $\alpha = (\alpha_1, \dots, \alpha_n)$.

2.3. The Chain Rule.

Theorem 2.4 (Chain Rule). *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be differentiable at x and $g : \mathbb{R}^m \rightarrow \mathbb{R}^p$ be differentiable at $f(x)$. Then $g \circ f : \mathbb{R}^n \rightarrow \mathbb{R}^p$ is differentiable at x and*

$$d(g \circ f)(x) = dg(f(x)) \circ df(x).$$

Draw a picture of the maps and derivatives.

Definition 2.5. *A map $f : U \rightarrow V$ is a C^∞ -diffeomorphism if f is a smooth map with a smooth inverse $f^{-1} : V \rightarrow U$. (C^1 -diffeomorphisms can be defined similarly.)*

One consequence of the Chain Rule is:

Proposition 2.6. *If $f : U \rightarrow V$ is a diffeomorphism, then $df(x)$ is an isomorphism for all $x \in U$.*

Proof. Let $g : V \rightarrow U$ be the inverse function. Then $g \circ f = id$. Taking derivatives, $dg \circ df = id$ as linear maps; hence $df(x)$ is an isomorphism for all x . \square

3. MANIFOLDS

3.1. Topological manifolds.

Definition 3.1. A topological manifold X of dimension n is a topological space X together with a collection $\mathcal{A} = \{\phi_\alpha : U_\alpha \rightarrow \mathbb{R}^n\}_{\alpha \in I}$ of maps (called an atlas of X) such that:

- (1) $\cup_{\alpha \in I} U_\alpha = X$,
- (2) ϕ_α is a homeomorphism onto an open subset $\phi_\alpha(U_\alpha)$ of \mathbb{R}^n .
- (3) (Technical condition 1) X is Hausdorff.
- (4) (Technical condition 2) X is second countable.

Definition 3.2. A topological space X is Hausdorff if for any $x \neq y \in X$ there exist open sets U_x and U_y containing x, y respectively such that $U_x \cap U_y = \emptyset$.

Definition 3.3. A topological space (X, \mathcal{T}) is second countable if there exists a countable sub-collection \mathcal{T}_0 of \mathcal{T} and any open set $U \in \mathcal{T}$ is a union (not necessarily finite) of open sets in \mathcal{T}_0 .

Exercise: Show that S^1 from Day 1 with either the induced topology or the quotient topology is a topological manifold.

Exercise: Give an example of a topological space X which is not a topological manifold. (You may have trouble proving that it is not a topological manifold, though.)

Observe that in the land of topological manifolds, a square and a circle are the same, i.e., they are homeomorphic! That is not the world we will explore — in other words, we seek a category (world) where squares are not the same as circles. In other words, we need derivatives!

3.2. Differentiable manifolds.

Definition 3.4. A smooth manifold is a topological manifold $(X, \mathcal{A} = \{\phi_\alpha : U_\alpha \rightarrow \mathbb{R}^n\})$ satisfying the following: For every $U_\alpha \cap U_\beta \neq \emptyset$, $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \rightarrow \phi_\beta(U_\alpha \cap U_\beta)$ is a smooth map.

Note: When we refer to a “manifold”, we mean a “smooth manifold”, unless stated otherwise.

4. EXAMPLES OF SMOOTH MANIFOLDS

Today we give some examples of smooth manifolds.

(1) \mathbb{R}^n is a smooth manifold. Atlas: $\phi : U = \mathbb{R}^n \rightarrow \mathbb{R}^n$ given by the identity map.

(2) Any open subset U of a smooth manifold M is a smooth manifold. Given an atlas $\mathcal{A} = \{\phi_\alpha : U_\alpha \rightarrow \mathbb{R}^n\}$ for M , an atlas for U is $\{\phi_\alpha : U_\alpha \cap U \rightarrow \mathbb{R}^n\}$.

(3) Let $M_n(\mathbb{R})$ be the space of $n \times n$ matrices with real entries, and let

$$GL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\}.$$

$GL(n, \mathbb{R})$ is an open subset of $M_n(\mathbb{R}) \simeq \mathbb{R}^{n^2}$, hence is a smooth n^2 -dimensional manifold. $GL(n, \mathbb{R})$ is called the *general linear group* of $n \times n$ real matrices.

(4) If M and N are smooth m - and n -dimensional manifolds, then their *product* $M \times N$ can naturally be given the structure of a smooth $(m + n)$ -dimensional manifold. Atlas: $\{\phi_\alpha \times \psi_\beta : U_\alpha \times V_\beta \rightarrow \mathbb{R}^m \times \mathbb{R}^n\}$, where $\{\phi_\alpha : U_\alpha \rightarrow \mathbb{R}^m\}$ is an atlas for M and $\{\psi_\beta : V_\beta \rightarrow \mathbb{R}^n\}$ is an atlas for N .

(5) $S^1 = \{x^2 + y^2 = 1\}$ is a smooth 1-dimensional manifold.

(i) One possible atlas: Open sets $U_1 = \{y > 0\}$, $U_2 = \{y < 0\}$, $U_3 = \{x > 0\}$, $U_4 = \{x < 0\}$, together with projections to the x -axis or the y -axis, as appropriate.

(ii) Another atlas: Open sets $U_1 = \{y \neq -1\}$ and $U_2 = \{y \neq 1\}$, together with stereographic projections from U_1 to $y = -1$ and U_2 to $y = 1$. The map $\phi_1 : U_1 \rightarrow \mathbb{R}$ is defined as follows: Take the line $L_{(x,y)}$ which passes through $(0, 1)$ and $(x, y) \in U_1$. Then let ϕ_1 be the x -coordinate of the intersection point between $L_{(x,y)}$ and $y = -1$. The map $\phi_2 : U_2 \rightarrow \mathbb{R}$ is defined similarly by projecting from $(0, -1)$ to $y = 1$. Compare the overlaps!!

(6) $S^n = \{x_1^2 + \dots + x_{n+1}^2 = 1\} \subset \mathbb{R}^{n+1}$.

(7) In dimension 2, S^2 , T^2 , genus g surface.

(8) (Real projective space) $\mathbb{R}P^n = (\mathbb{R}^{n+1} - \{(0, \dots, 0)\}) / \sim$, where

$$(x_0, x_1, \dots, x_n) \sim (tx_0, tx_1, \dots, tx_n), \quad t \in \mathbb{R} - \{0\}.$$

$\mathbb{R}P^n$ is called the *real projective space* of dimension n . The equivalence class of (x_0, \dots, x_n) is denoted by $[x_0, \dots, x_n]$.

Consider $U_0 = \{x_0 \neq 0\}$ with the coordinate chart $\phi_0 : U_0 \rightarrow \mathbb{R}^n$ is given by

$$[x_0, x_1, \dots, x_n] = \left[1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right] \mapsto \left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right).$$

Similarly, take $U_i = \{x_i \neq 0\}$ and define $\phi_i : U_i \rightarrow \mathbb{R}^n$. What about transition maps $\phi_j \circ \phi_i^{-1}$? (Explain this in detail.)

(9) (Group actions) The 2-torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. The discrete group \mathbb{Z}^2 acts on \mathbb{R}^2 by translation:

$$\mathbb{Z}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$((m, n), (x, y)) \mapsto (m + x, n + y).$$

Note that for each fixed (m, n) , we have a diffeomorphism

$$\mathbb{R}^2 \rightarrow \mathbb{R}^2,$$

$$(x, y) \mapsto (m + x, n + y).$$

$\mathbb{R}^2/\mathbb{Z}^2$ is the set of orbits of \mathbb{R}^2 under the action of \mathbb{Z}^2 . (One orbit is $(x, y) + \mathbb{Z}^2$.)

Equivalently, the 2-torus is obtained from the “fundamental domain” $[0, 1] \times [0, 1]$ by identifying $(0, y) \sim (1, y)$ and $(x, 0) \sim (x, 1)$, i.e., the sides and the top and the bottom. The assignment $\mathbb{R}^2 \rightarrow \mathbb{R}^2/\mathbb{Z}^2, x \mapsto [x]$, is injective when restricted to the interior of the fundamental domain.

The n -torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ is defined similarly.

Next time: Try to answer the question of what it means for two atlases of the same M to be “the same”.

5. SMOOTH FUNCTIONS AND SMOOTH MAPS

Today we discuss smooth functions on a manifold and smooth maps between manifolds.

5.1. Choice of atlas. Let (M, \mathcal{T}) be the underlying topological space of a manifold, and $\mathcal{A}_1 = \{(U_\alpha, \phi_\alpha)\}$, $\mathcal{A}_2 = \{(V_\beta, \psi_\beta)\}$ be two atlases.

Question: When do they represent the *same* smooth manifold?

Definition 5.1. Two atlases \mathcal{A}_1 and \mathcal{A}_2 on M are compatible if

$$\phi_\alpha(U_\alpha \cap V_\beta) \xrightarrow{\psi_\beta \circ \phi_\alpha^{-1}} \psi_\beta(U_\alpha \cap V_\beta)$$

is a smooth map for all pairs $U_\alpha \cap V_\beta \neq \emptyset$.

If \mathcal{A}_1 and \mathcal{A}_2 are compatible, then we can take $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ which is compatible with both \mathcal{A}_1 and \mathcal{A}_2 .

Definition 5.2. Given a smooth manifold (M, \mathcal{A}) , its maximal atlas $\mathcal{A}_{max} = \{(U_\alpha, \phi_\alpha)\}$ is an atlas which is compatible with \mathcal{A} and contains every atlas $\mathcal{A}' \supset \mathcal{A}$ which is compatible with \mathcal{A} .

5.2. Smooth functions.

Some more zen: You can study an object (such as a manifold) either by looking at the object itself or by looking at the space of functions on the object. In the topological category, the space of functions would be $C^0(M)$, the space of continuous functions $f : M \rightarrow \mathbb{R}$. The function space perspective has been especially fruitful in algebraic geometry.

Question: What is the appropriate space of functions for a smooth manifold (M, \mathcal{A}) ?

Definition 5.3. Given a smooth manifold (M, \mathcal{A}) , a function $f : M \rightarrow \mathbb{R}$ is smooth if

$$f \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \rightarrow \mathbb{R}$$

is smooth for each coordinate chart (U_α, ϕ_α) of \mathcal{A} .

Note that the definition of a smooth function on M depends on the atlas \mathcal{A} .

For a smooth manifold (M, \mathcal{A}) , the appropriate space of functions would be

$$C_{\mathcal{A}}^\infty(M) = \{f : M \rightarrow \mathbb{R} \text{ smooth}\}.$$

When the atlas on M is understood, we write $C^\infty(M)$ for the space of smooth functions on M .

Lemma 5.4. Two atlases \mathcal{A}_1 and \mathcal{A}_2 are compatible if and only if $C_{\mathcal{A}_1}^\infty(M) = C_{\mathcal{A}_2}^\infty(M)$.

The “only if” direction is easy; the “if” direction needs a technical lemma, called a *partition of unity*, which will be discussed later on in the course.

Pullback: Let $\phi : X \rightarrow Y$ be a continuous map between topological spaces. Then there is a naturally defined *pullback map*

$$\phi^* : C^0(Y) \rightarrow C^0(X)$$

given by $f \mapsto f \circ \phi$. Note that pullback is *contravariant*, i.e., the direction is from Y to X , which is the opposite from the original map ϕ .

Consider the smooth manifold (M, \mathcal{A}) . If $\psi : M \rightarrow M$ is a homeomorphism, then $\psi^* : C^0(M) \xrightarrow{\sim} C^0(M)$. Although $C_{\mathcal{A}}^\infty(M) \xrightarrow{\sim} \psi^*(C_{\mathcal{A}}^\infty(M))$, in general $C_{\mathcal{A}}^\infty(M) \neq \psi^*(C_{\mathcal{A}}^\infty(M))$.

Definition 5.5. Two C^∞ -structures $C_{\mathcal{A}_1}^\infty(M)$ and $C_{\mathcal{A}_2}^\infty(M)$ are equivalent if there exists a homeomorphism of M which takes $C_{\mathcal{A}_1}^\infty(M) \simeq C_{\mathcal{A}_2}^\infty(M)$.

Amazing fact: (Milnor) S^7 has several inequivalent smooth structures! (Not amazingly, S^1 has only one smooth structure.)

Major open question: (Smooth Poincaré Conjecture) How many smooth structures does S^4 have?

5.3. Smooth maps. In the category of smooth manifolds, we need to define the appropriate maps, called *smooth maps*.

Definition 5.6. A map $f : M \rightarrow N$ between smooth manifolds is smooth if $f^*(C^\infty(N)) \subset C^\infty(M)$.

Lemma 5.7. $f : M \rightarrow N$ is smooth if for any $p \in M$ there exist coordinate charts $U \ni p$ and $V \ni f(p)$ such that the composition

$$\phi(U) \xrightarrow{\phi^{-1}} U \xrightarrow{f} V \xrightarrow{\psi} \psi(V)$$

is smooth.

Upshot: Smooth maps between smooth manifolds can be “reduced” to smooth maps from \mathbb{R}^n to \mathbb{R}^m .

6. APPLICATIONS OF THE INVERSE FUNCTION THEOREM

6.1. Inverse function theorem. Recall that a smooth map $f : U \rightarrow V$ between two open sets of Euclidean space is a *diffeomorphism* if there is a smooth inverse $f^{-1} : V \rightarrow U$.

Definition 6.1. A smooth map $f : M \rightarrow N$ between two manifolds is a diffeomorphism if there is a smooth inverse $f^{-1} : N \rightarrow M$.

The inverse function theorem, given below, is the most important basic theorem in differential geometry. It says that an isomorphism in the linear category implies a local diffeomorphism in the differentiable category. Hence we can move from “infinitesimal” to “local”.

Theorem 6.2 (Inverse function theorem). *Let $f : U \rightarrow V$ be a smooth map, where U and V are open sets of \mathbb{R}^n . If $df(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an isomorphism, then df is a local diffeomorphism near x , i.e., there exist open sets U_x containing x and $V_{f(x)}$ containing $f(x)$ such that $f|_{U_x} : U_x \rightarrow V_{f(x)}$ is a diffeomorphism.*

6.2. Illustrative example. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$, $(x, y) \mapsto x^2 + y^2$. We would like to analyze the level sets $f^{-1}(a)$, where $a > 0$. To that end, we consider

$$F : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad (x, y) \mapsto (f(x, y), y).$$

Let us use coordinates (x, y) for the domain \mathbb{R}^2 and coordinates (u, v) for the range \mathbb{R}^2 . We compute:

$$dF(x, y) = \begin{pmatrix} 2x & 2y \\ 0 & 1 \end{pmatrix}.$$

We restrict our attention to the portion $x > 0$. Since $\det(dF(x, y)) = 2x > 0$, the inverse function theorem applies and there is a local diffeomorphism between a neighborhood $U_{(x,y)} \subset \mathbb{R}^2$ of a point (x, y) on the level set $f(x, y) = a$ and a neighborhood $V_{F(x,y)}$ of $F(x, y)$ on the line $u = a$. Hence $f^{-1}(a)$, restricted to $x > 0$, is a smooth manifold.

Interpret slightly differently: The above example basically says that f can be used as a coordinate function in conjunction with y , provided $x > 0$.

6.3. Rank.

Definition 6.3. The rank of a linear map $L : V \rightarrow W$ is the dimension of $\text{Im}(L)$.

Recall that the *dimension* of a vector space V is the cardinality of a basis for V . If V is finite-dimensional, then $V \simeq \mathbb{R}^n$ for some n , and $\dim V = n$.

Definition 6.4. The rank of a smooth map $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ at $x \in \mathbb{R}^n$ is the rank of $df(x) : \mathbb{R}^n \rightarrow \mathbb{R}^m$. The map f has constant rank if the rank of $df(x)$ is constant.

We can similarly define the rank of a smooth map $f : M \rightarrow N$ at a point $x \in M$ by using local coordinates.

Question: Is the rank constant under change of coordinates? Compare the ranks of $d(\psi_\alpha \circ f \circ \phi_\alpha^{-1})$ and $d(\psi_\beta \circ f \circ \phi_\beta^{-1})$, where $\phi_\alpha : U_\alpha \subset M \rightarrow \mathbb{R}^m$ and $\psi_\alpha : V_\alpha \subset N \rightarrow \mathbb{R}^n$, and ϕ_β, ψ_β are defined similarly. The invariance of rank is due to the chain rule:

$$\begin{aligned} d(\psi_\beta \circ f \circ \phi_\beta^{-1}) &= d((\psi_\beta \circ \psi_\alpha^{-1}) \circ (\psi_\alpha \circ f \circ \phi_\alpha^{-1}) \circ (\phi_\alpha^{-1} \circ \phi_\beta)) \\ &= d(\psi_\beta \circ \psi_\alpha^{-1}) \circ d(\psi_\alpha \circ f \circ \phi_\alpha^{-1}) \circ d(\phi_\alpha^{-1} \circ \phi_\beta), \end{aligned}$$

and by observing that $d(\psi_\beta \circ \psi_\alpha^{-1})$ and $d(\phi_\alpha^{-1} \circ \phi_\beta)$ are linear isomorphisms.

7. SUBMERSIONS AND REGULAR VALUES

7.1. Submersions.

Definition 7.1. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets. A smooth map $f : U \rightarrow V$ is a submersion if $df(x)$ is surjective for all $x \in U$. (Note that this means that $n \geq m$.)

Definition 7.2. Let M be a manifold with atlas $\mathcal{A} = \{\phi_\alpha : U_\alpha \rightarrow \mathbb{R}^n\}$ and let N be a manifold with atlas $\mathcal{B} = \{\psi_\beta : V_\beta \rightarrow \mathbb{R}^m\}$. Then a smooth map $f : (M, \mathcal{A}) \rightarrow (N, \mathcal{B})$ is a submersion if all $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ are submersions, where defined.

Prototype: $f : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^m, (x_1, \dots, x_{m+n}) \mapsto (x_1, \dots, x_m)$.

Theorem 7.3 (Implicit function theorem, submersion version). Let $f : U \rightarrow V$ be a submersion, where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open sets with $n \geq m$. Then for each $x \in U$ there exist $U \supset U_x \ni p$ and a diffeomorphism $F : U_x \xrightarrow{\sim} W \subset \mathbb{R}^n$ such that

$$f \circ F^{-1} : W \rightarrow \mathbb{R}^m$$

is given by

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_m).$$

Proof. Write $f = (f_1, \dots, f_m)$ where $f_i : U \rightarrow \mathbb{R}$, and define the map

$$F : U \rightarrow V \times \mathbb{R}^{n-m},$$

$$(x_1, \dots, x_n) \mapsto (f_1, \dots, f_m, x_{m+1}, \dots, x_n).$$

Here we choose the appropriate x_{m+1}, \dots, x_n so that $dF(x)$ is invertible. Then F is a local diffeomorphism by the inverse function theorem,

$$f \circ F^{-1}(f_1, \dots, f_m, x_{m+1}, \dots, x_n) = (f_1, \dots, f_m)$$

and F satisfies the conditions of the theorem. □

Carving manifolds out of other manifolds: The implicit function theorem, submersion version, has the following corollary:

Corollary 7.4. If $f : M \rightarrow N$ is a submersion, then $f^{-1}(y)$, $y \in N$, can be given the structure of a manifold.

Example: The easy way to prove that the circle $S^1 = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$ can be given the structure of a manifold is to consider the map

$$f : \mathbb{R}^2 - \{(0, 0)\} \rightarrow \mathbb{R}, \quad f(x, y) = x^2 + y^2.$$

The Jacobian is $df(x, y) = (2x, 2y)$. Since x and y are never simultaneously zero, the rank of df is 1 at all points of $\mathbb{R}^2 - \{(0, 0)\}$ and in particular on S^1 . Using the implicit function theorem, it follows that S^1 is a manifold.

7.2. Regular values and Sard's theorem.

Definition 7.5. Let $f : M \rightarrow N$ be a smooth map.

- (1) A point $y \in N$ is a regular value of f if $df(x)$ is surjective for all $x \in f^{-1}(y)$.
- (2) A point $y \in N$ is a critical value of f if $df(x)$ is not surjective for some $x \in f^{-1}(y)$.
- (3) A point $x \in M$ is a critical point of f if $df(x)$ is not surjective.

The implicit function theorem implies that $f^{-1}(y)$ can be given the structure of a manifold if y is a regular value of f .

Example: Let $M = \{x^3 + y^3 + z^3 = 1\} \subset \mathbb{R}^3$. Consider the map

$$f : \mathbb{R}^3 \rightarrow \mathbb{R}, \quad (x, y, z) \mapsto x^3 + y^3 + z^3.$$

Then $M = f^{-1}(1)$. The Jacobian is given by $df(x, y, z) = (3x^2, 3y^2, 3z^2)$ and the rank of $df(x, y, z)$ is one if and only if $(x, y, z) \neq (0, 0, 0)$. Since $(0, 0, 0) \notin M$, it follows that 1 is a regular value of f . Hence M can be given the structure of a manifold.

Exercise: Prove that $S^n \subset \mathbb{R}^{n+1}$ is a manifold.

Example: Zero sets of homogeneous polynomials in $\mathbb{R}P^n$. A polynomial $f : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ is homogeneous of degree d if $f(tx) = t^d f(x)$ for all $t \in \mathbb{R} - \{0\}$ and $x \in \mathbb{R}^{n+1}$. The zero set $Z(f)$ of f is given by $\{[x_0, \dots, x_n] \mid f(x_0, \dots, x_n) = 0\}$. By the homogeneous condition, $Z(f)$ is well-defined. We can check whether $Z(f)$ is a manifold by passing to local coordinates.

For example, consider the homogeneous polynomial $f(x_0, x_1, x_2) = x_0^3 + x_1^3 + x_2^3$ of degree 3 on $\mathbb{R}P^2$. Consider the open set $U = \{x_0 \neq 0\} \subset \mathbb{R}P^2$. If we let $x_0 = 1$, then on $U \simeq \mathbb{R}^2$ we have $f(x_1, x_2) = 1 + x_1^3 + x_2^3$. Check that 0 is a regular value of $f(x_1, x_2)$! The open sets $\{x_1 \neq 0\}$ and $\{x_2 \neq 0\}$ can be treated similarly.

More involved example: Let $SL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) = 1\}$. $SL(n, \mathbb{R})$ is called the special linear group of $n \times n$ real matrices. Consider the determinant map

$$f : \mathbb{R}^{n^2} \rightarrow \mathbb{R}, \quad A \mapsto \det(A).$$

We can rewrite f as follows:

$$f : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \rightarrow \mathbb{R}, \quad (a_1, \dots, a_n) \mapsto \det(a_1, \dots, a_n),$$

where a_i are column vectors and $A = (a_1, \dots, a_n) = (a_{ij})$.

First we need some properties of the determinant:

- (1) $f(e_1, \dots, e_n) = 1$.
- (2) $f(a_1, \dots, c_i a_i + c'_i a'_i, \dots, a_n) = c_i \cdot f(a_1, \dots, a_i, \dots, a_n) + c'_i \cdot f(a_1, \dots, a'_i, \dots, a_n)$.
- (3) $f(\dots, a_i, a_{i+1}, \dots) = -f(\dots, a_{i+1}, a_i, \dots)$.

(1) is a normalization, (2) is called *multilinearity*, and (3) is called the *alternating property*. It turns out that (1), (2), and (3) uniquely determine the determinant function.

We now compute $df(A)(B)$:

$$\begin{aligned}
 df(A)(B) &= \lim_{t \rightarrow 0} \frac{f(A + tB) - f(A)}{t} \\
 &= \lim_{t \rightarrow 0} \frac{\det(a_1 + tb_1, \dots, a_n + tb_n) - \det(a_1, \dots, a_n)}{t} \\
 &= \lim_{t \rightarrow 0} \frac{\det(a_1, \dots, a_n) + t[\det(b_1, a_2, \dots, a_n) + \det(a_1, b_2, \dots, a_n) \\
 &\quad + \dots + \det(a_1, \dots, a_{n-1}, b_n)] + t^2(\dots) - \det(a_1, \dots, a_n)}{t} \\
 &= \det(b_1, a_2, \dots, a_n) + \dots + \det(a_1, \dots, a_{n-1}, b_n)
 \end{aligned}$$

It is easy to show that 1 is a regular value of df . For example, take $b_1 = ca_1$ where $c \in \mathbb{R}$ and $b_i = 0$ for all $i \neq 1$.

Theorem 7.6 (Sard's theorem). *Let $f : U \rightarrow V$ be a smooth map. Then almost every point $y \in \mathbb{R}^n$ is a regular value.*

The notion of *almost every point* will be made precise later. But in the meantime:

Reality Check: What happens when $m < n$?

8. IMMERSIONS AND EMBEDDINGS

8.1. Some more point-set topology. We want to impose additional conditions to immersions to make them better-behaved. We first review some more point-set topology.

Let X be a topological space.

- (1) A subset $V \subset X$ is *closed* if the complement $X - V = \{x \in X, x \notin V\}$ is open.
- (2) The *closure* \overline{V} of a subset $V \subset X$ is the smallest closed set containing V .
- (3) A subset V of a topological space X is *dense* if $U \cap V \neq \emptyset$ for every open set U . In other words, $\overline{V} = X$.
- (4) A *compact set* V satisfies the following *finite covering property*: for any open cover $\{U_\alpha\}$ of V there exists a finite subcover.
- (5) A metric space is compact if and only if every sequence has a convergent subsequence.
- (6) A subset of Euclidean space is compact if and only if it is closed and bounded, i.e., is a subset of some $B_y(r)$.
- (7) A map $f : X \rightarrow Y$ is *proper* if the preimage $f^{-1}(V)$ of every compact set $V \subset Y$ is compact. (Remark: If $f : X \rightarrow Y$ is continuous, then the image of every compact set is compact.)

8.2. Immersions.

Definition 8.1. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets. A smooth map $f : U \rightarrow V$ is an immersion if $df(x)$ is injective for all $x \in U$. (Note this means $m \geq n$.) A smooth map $f : M \rightarrow N$ between manifolds is an immersion if f is an immersion with respect to all local coordinates.

Prototype: $f : \mathbb{R}^n \rightarrow \mathbb{R}^m, m \geq n, (x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0, \dots, 0)$.

Theorem 8.2 (Implicit function theorem, immersion version). Let $f : U \rightarrow V$ be an immersion, where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open sets. Then for any $x \in U$, there exist open sets $U \supset U_x \ni x$, $V \supset V_{f(x)} \ni f(x)$ and a diffeomorphism $G : V_{f(x)} \xrightarrow{\sim} W \subset \mathbb{R}^m$ such that

$$G \circ f : U_x \rightarrow \mathbb{R}^m$$

is given by

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0, \dots, 0).$$

Proof. The proof is similar to that of the submersion version. Define

$$F : U \times \mathbb{R}^{m-n} \rightarrow \mathbb{R}^m,$$

$$(x_1, \dots, x_n, y_{n+1}, \dots, y_m) \mapsto (f_1(x), \dots, f_n(x), f_{n+1}(x) + y_{n+1}, \dots, f_m(x) + y_m),$$

where $x = (x_1, \dots, x_n)$ and $f(x) = (f_1(x), \dots, f_m(x))$. We can check that dF is nonsingular after possibly reordering the f_1, \dots, f_m and that $F^{-1} \circ f : U_x \rightarrow \mathbb{R}^m$ is given by

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0, \dots, 0).$$

We then set $G = F^{-1}$. □

Zen: The implicit function theorem tells us that under a constant rank condition we may assume that locally we can straighten our manifolds and maps and pretend we are doing linear algebra.

Examples of immersions:

- (1) Circle mapped to figure 8 in \mathbb{R}^2 .
- (2) Line which wraps around unit circle in \mathbb{R}^2 infinitely many times.
- (3) Map $f : \mathbb{R} \rightarrow \mathbb{R}^2/\mathbb{Z}^2$, $t \mapsto (at, bt)$, where b/a is irrational, is an immersion. The image of f is *dense* in $\mathbb{R}^2/\mathbb{Z}^2$.

8.3. Embeddings and submanifolds. We upgrade immersions $f : M \rightarrow N$, M, N manifolds, as follows:

Definition 8.3. An embedding $f : M \rightarrow N$ is an immersion which is one-to-one and proper. The image of an embedding is a submanifold of N .

The “pathological” examples above are immersions but not embeddings. Why? (1) and (2) are not one-to-one and (3) is not proper.

Proposition 8.4. Let M and N be manifolds with topologies \mathcal{T} and \mathcal{T}' . If $f : M^m \rightarrow N^n$ is an embedding, then $f^{-1}(\mathcal{T}') = \mathcal{T}$.

Proof. Note that if f is continuous then $f^{-1}(\mathcal{T}') \subset \mathcal{T}$ and it suffices to show that $f^{-1}(\mathcal{T}') \supset \mathcal{T}$. Let $x \in M$ and U be a small open set containing x . Then by the implicit function theorem f can be written locally as $U \rightarrow \mathbb{R}^n$, $x \mapsto (x, 0)$. We want to show that there is an open set $V \subset \mathbb{R}^m$ such that $V \cap f(M) = f(U)$. Arguing by contradiction, suppose there exist $y \in f(U)$ and a sequence $\{x_i\}_{i=1}^\infty \subset M$ such that $f(x_i) \rightarrow y$ but $f(x_i) \notin f(U)$. The set $\{y\} \cup \{f(x_i)\}_{i=1}^\infty$ is compact, so $\{x\} \cup \{x_i\}_{i=1}^\infty$ is compact by properness, where we are recalling that f is one-to-one. By compactness, there is a subsequence of $\{x_i\}$ which converges to x . This implies that $x_i \in U$ and $f(x_i) \in f(U)$ for sufficiently large i , a contradiction. \square

9. TANGENT SPACES, DAY I

9.1. **Concrete example.** Consider $S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$. We compute (define?) the tangent plane $T_{(a,b,c)}S^2$. We use the fact that S^2 is the preimage of the regular value 1 of f , where

$$f : \mathbb{R}^3 \rightarrow \mathbb{R}, \quad f(x, y, z) = x^2 + y^2 + z^2 - 1.$$

The derivative of f at the point (a, b, c) is:

$$df(a, b, c)(x, y, z)^T = (2a, 2b, 2c)(x, y, z)^T.$$

The tangent directions are directions (x, y, z) where $df(a, b, c)(x, y, z)^T = 0$. Therefore, the tangent plane is the plane through (a, b, c) which is parallel to $ax + by + cz = 0$. More explicitly,

$$T_{(a,b,c)}S^2 = \{ax + by + cz = a^2 + b^2 + c^2 = 1\}.$$

Definition 9.1. Let M be a submanifold of \mathbb{R}^n . Then we can define T_pM as follows: Pick a small neighborhood $U_p \subset \mathbb{R}^n$ of p and a function $f : U_p \rightarrow \mathbb{R}^m$ such that $M \cap U_p$ is a level set of f . Then T_pM is the set of vectors $v \in \mathbb{R}^n$ such that $df(p)(v) = 0$.

Some issues with this definition:

- (1) Need to verify that T_pM does not depend on the choice of $f : U_p \rightarrow \mathbb{R}^m$. (Not so serious.)
- (2) The definition seems to depend on how M is embedded in \mathbb{R}^n . In other words, the definition is *not intrinsic*.

We will give several definitions of T_pM which are intrinsic, in increasing order of abstraction!!

9.2. **First definition.** Let M be a smooth n -dimensional manifold. If $U \subset M$ is an open set, then let $C^\infty(U)$ be the set of smooth functions $f : U \rightarrow \mathbb{R}$.

Terminology: Let $f, g : U \rightarrow \mathbb{R}$ where $U \subset \mathbb{R}$. Then $f = O(g)$ if $|f| \leq C|g|$ for some constant C . For example, $t = \cos t + O(t^3)$ near $t = 0$.

Definition 9.2 (First definition). The tangent space T_pM to M at p is the set of equivalence classes

$$T_p(M) = \{\text{smooth curves } \gamma : (-\varepsilon_\gamma, \varepsilon_\gamma) \rightarrow M, \gamma(0) = p\} / \sim,$$

where $\gamma_1 \sim \gamma_2$ if $f \circ \gamma_1(t) = f \circ \gamma_2(t) + O(t^2)$ for all pairs (f, U) where U is an open set containing p and $f \in C^\infty(U)$. Here $\varepsilon_\gamma > 0$ is a constant which depends on γ .

Let x_1, \dots, x_n be coordinate functions for an open set $U \subset M$.

Theorem 9.3 (Taylor's theorem). Let $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}$ be a smooth function and $0 \in U$. Then we can write

$$f(x) = a + \sum_i a_i x_i + \sum_{i,j} a_{ij}(x) x_i x_j,$$

on an open rectangle $(-a_1, b_1) \times \dots \times (-a_n, b_n) \subset U$ which contains 0, where a, a_i are constants and $a_{ij}(x)$ are smooth functions.

Proof. We prove the theorem for one variable x . By the Fundamental Theorem of Calculus,

$$g(1) - g(0) = \int_0^1 g'(t) dt.$$

Substituting $g(t) = f(tx)$ and integrating by parts with $u = \frac{d}{dt}f(tx)$, $v = t - 1$ we obtain:

$$\begin{aligned} f(x) - f(0) &= \int_0^1 \frac{d}{dt}f(tx) dt \\ &= x \cdot f'(tx) \cdot (t - 1) \Big|_{t=0}^{t=1} - \int_0^1 (t - 1)x^2 \cdot f''(tx) dt \\ &= -x f'(0)(-1) - x^2 \int_0^1 (t - 1) f''(tx) dt \\ &= f'(0) \cdot x + h(x) \cdot x^2. \end{aligned}$$

Here we write $h(x) = - \int_0^1 (t - 1) f''(tx) dt$. This gives us the desired result

$$f(x) = f(0) + f'(0) \cdot x + h(x) \cdot x^2$$

for one variable. □

Corollary 9.4. $\gamma_1 \sim \gamma_2$ if and only if $x_i(\gamma_1(t)) = x_i(\gamma_2(t)) + O(t^2)$ for $i = 1, \dots, n$.

If M is a submanifold of \mathbb{R}^m , $\gamma_1 \sim \gamma_2$ if and only if $\gamma_1'(0) = \gamma_2'(0)$.

Remark: At this point it's not clear whether $T_p M$ has the structure of a vector space. The vector space structure is clearer from the next definition.

10. TANGENT SPACES, DAY II

10.1. **Sheaf-theoretic notions.** We first discuss some sheaf-theoretic notions.

The set $C^\infty(U)$ of smooth functions $f : U \rightarrow \mathbb{R}$ is an *algebra over* \mathbb{R} , i.e., it has operations $c \cdot f$, $f \cdot g$, $f + g$, where $c \in \mathbb{R}$ and $f, g \in C^\infty(U)$.

Let $V \subset M$ be any set. Then we define

$$C^\infty(V) = \{(f, U) \mid U \supset V, f : U \rightarrow \mathbb{R} \text{ smooth}\} / \sim .$$

Here $(f_1, U_1) \sim (f_2, U_2)$ if there exists $V \subset U \subset U_1 \cap U_2$ for which $f_1|_U = f_2|_U$. When we refer to a *smooth function on* V , what we really mean is an element of $C^\infty(V)$, since we need open sets to define derivatives.

Given open sets $U_1 \subset U_2$, there exists a natural *restriction map* $\rho_{U_1}^{U_2} : C^\infty(U_2) \rightarrow C^\infty(U_1)$, $f \mapsto f|_{U_1}$. Then $C^\infty(V)$ is the *direct limit* of $C^\infty(U)$ for all U containing V .

Example: When $V = \{p\}$, $C^\infty(\{p\})$ (written simply as $C^\infty(p)$), is called the *stalk* at the point p or the set of *germs of smooth functions* at p .

10.2. **Second definition.**

Definition 10.1 (Second definition). A derivation is an \mathbb{R} -linear map $X : C^\infty(p) \rightarrow \mathbb{R}$ which satisfies the Leibniz rule:

$$X(fg) = X(f) \cdot g(p) + f(p) \cdot X(g).$$

The tangent space $T_p M$ is the set of derivations at p .

By the second definition, $T_p M$ is clearly a vector space over \mathbb{R} .

Remark: It does not matter whether M is a manifold — it could have been Euclidean space instead, since $C^\infty(p)$ only depends on a small neighborhood of p .

Exercise: $X(c) = 0$, $c \in \mathbb{R}$.

Examples.

- (1) $X_i = \frac{\partial}{\partial x_i}$. Take coordinates (x_1, \dots, x_n) near $p = 0$. Then let $X(f) = \frac{\partial f}{\partial x_i}(0)$. Check: this is indeed a derivation and $\frac{\partial}{\partial x_i}(x_j) = \delta_{ij}$.
- (2) Given $\gamma : (-\varepsilon, \varepsilon) \rightarrow M$ with $\gamma(0) = p$, define $X(f) = (f \circ \gamma)'(0)$. This is the directional derivative in the direction γ . It is easy to check that two $\gamma \sim \gamma'$ give rise to the same directional derivative.

Lemma 10.2. If M is an n -dimensional manifold, then $\dim T_p M = n$.

Proof. Take local coordinates x_1, \dots, x_n so that $p = 0$. Let $X_i = \frac{\partial}{\partial x_i}$. Then $X_i(x_j) = \delta_{ij}$, and clearly the X_i are linearly independent. Thus $\dim T_p M \geq n$. Now, given some derivation X , suppose $X(x_i) = b_i$. Then the derivation $Y = X - \sum_j b_j X_j$ satisfies $Y(x_i) = 0$ for all x_i . By

Taylor's Theorem, any $f \in C^\infty(p)$ can be written as $a + \sum a_i x_i + \sum a_{ij} x_i x_j$. By the derivation property, all the quadratic and higher terms vanish, and hence $Y(f) = 0$. Therefore $\dim T_p M = n$.

□

Lemma 10.3. *The first two definitions of $T_p M$ are equivalent.*

Proof. Given $\gamma : (-\varepsilon, \varepsilon) \rightarrow M$ with $\gamma(0) = p$, one simply differentiates in that direction. $X(f) = (f \circ \gamma)'(0)$. (Check this is well-defined!) Since we already calculated $\dim T_p M = n$ for the first and second definitions, we see this map is surjective and hence an isomorphism. □

10.3. Third definition. Define $\mathcal{F}_p \subset C^\infty(p)$ to be germs of functions which are 0 at p . \mathcal{F}_p is an ideal of $C^\infty(p)$. This means that if $f \in \mathcal{F}_p$ and $g \in C^\infty(p)$, then $fg \in \mathcal{F}_p$. Let $\mathcal{F}_p^2 \subset \mathcal{F}_p$ be the ideal of $C^\infty(p)$ generated by products of elements of \mathcal{F}_p , i.e., consisting of elements $\sum f_i \phi_j \phi_k$, where $\phi_j, \phi_k \in \mathcal{F}_p$.

Definition 10.4 (Third definition). $T_p M = (\mathcal{F}_p / \mathcal{F}_p^2)^*$, i.e., $T_p M$ is the dual vector space (the space of \mathbb{R} -linear functionals) of $\mathcal{F}_p / \mathcal{F}_p^2$.

Equivalence of the second and third definitions: Show that $X : \mathcal{F}_p \rightarrow \mathbb{R}$ factors through \mathcal{F}_p^2 . (Pretty easy, since it's a derivation.) Now, note that $\dim(\mathcal{F}_p / \mathcal{F}_p^2) = n$ by Taylor's Theorem.

$\mathcal{F}_p / \mathcal{F}_p^2$ is called the *cotangent space* at p , and is denoted $T_p^* M$. If $f \in C^\infty(p)$, then $f - f(p) \in \mathcal{F}_p$, and is denoted $df(p)$, when viewed as an element in $\mathcal{F}_p / \mathcal{F}_p^2$.

11. THE TANGENT BUNDLE

11.1. The tangent bundle. Let $TM = \sqcup_{p \in M} T_p M$. This is called the *tangent bundle*. We explain how to endow a smooth structure on the tangent bundle.

Consider the projection $\pi : TM \rightarrow M$ which sends any $q \in T_p M$ to p . Let $U \subset M$ be an open set with coordinates x_1, \dots, x_n . Since an element $q \in T_p M$ is written as $\sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$, we identify $\pi^{-1}(U) = U \times \mathbb{R}^n$ by sending the point $q \in TM$ to $(x_1, \dots, x_n, a_1, \dots, a_n)$.

Check the transition functions. Let $x = (x_1, \dots, x_n)$ be coordinates on U and $y = (y_1, \dots, y_n)$ be coordinates on V . Let (x, a) be coordinates on $\pi^{-1}(U)$ and (y, b) be coordinates on $\pi^{-1}(V)$. Think of y as a function of x on $U \cap V$. Write $\frac{\partial y}{\partial x} = (\frac{\partial y_i}{\partial x_j})$. In terms of y coordinates,

$$\sum a_i \frac{\partial}{\partial x_i} = \sum a_i \frac{\partial y_j}{\partial x_i} \frac{\partial}{\partial y_j}.$$

This is easily verified by thinking of evaluation on functions. Thus, $b_j = \sum_i a_i \frac{\partial y_j}{\partial x_i}$.

Then $q \in TM$ corresponds to (x, a) or $(y, \frac{\partial y}{\partial x} a)$.

Computation of the Jacobian of the transition function.

$$\begin{pmatrix} \frac{\partial y}{\partial x} & \frac{\partial y}{\partial a} \\ \frac{\partial x}{\partial x} & \frac{\partial x}{\partial a} \end{pmatrix} = \begin{pmatrix} \frac{\partial y}{\partial x} & 0 \\ \sum_k \frac{\partial y_i}{\partial x_j \partial x_k} a_k & \frac{\partial y}{\partial x} \end{pmatrix}.$$

The two terms on the bottom are obtained by differentiating $b_i = \sum_k \frac{\partial y_i}{\partial x_k} a_k$.

Thus we obtain a smooth manifold TM and a C^∞ -function $TM \xrightarrow{\pi} M$.

11.2. Examples of tangent bundles.

Example: The tangent bundle of $U \subset \mathbb{R}^n$. $TU \simeq U \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n$. A tangent vector is described by a point $(x_1, \dots, x_n) \in U$, together with a tangent direction $(a_1, \dots, a_n) \in \mathbb{R}^n$, written as $\sum_i a_i \frac{\partial}{\partial x_i}$.

Example: $S^2 \subset \mathbb{R}^3$, $S^2 = \{x_1^2 + x_2^2 + x_3^2 = 1\}$. Think of $TS^2 \subset \mathbb{R}^3 \times \mathbb{R}^3$ with coordinates $(x_1, x_2, x_3, y_1, y_2, y_3)$. At $(x_1, x_2, x_3) \in S^2$, $T_{(x_1, x_2, x_3)} S^2$ is the set of points (y_1, y_2, y_3) such that $(y_1, y_2, y_3) \cdot (x_1, x_2, x_3) = 0$. Therefore:

$$TS^2 = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid |x| = 1, x \cdot y = 0\}.$$

Think of TS^2 as follows: At each point $x \in S^2$, there is a plane (2-dimensional vector space) $T_x S^2$ sitting over it.

More abstract example: S^2 defined by gluing coordinate charts. Let $U = \mathbb{R}^2$ and $V = \mathbb{R}^2$, with coordinates $(x_1, y_1), (x_2, y_2)$, respectively. Alternatively, think of $\mathbb{R}^2 = \mathbb{C}$. Take $U \cap V = \mathbb{C} - \{0\}$.

The transition function is:

$$U - \{0\} \xrightarrow{\phi_{UV}} V - \{0\},$$

$$z \mapsto \frac{1}{z},$$

with respect to complex coordinates $z = x + iy$. With respect to real coordinates,

$$(x, y) \mapsto \left(\frac{x}{x^2 + y^2}, -\frac{y}{x^2 + y^2} \right).$$

11.3. Complex manifolds. S^2 has the structure of a *complex manifold*.

Definition 11.1. A function $\phi : \mathbb{C} \rightarrow \mathbb{C}$ is holomorphic (or complex analytic) if

$$\frac{d\phi}{dz} = \lim_{h \rightarrow 0} \frac{\phi(z+h) - \phi(z)}{h}$$

exists for all $z \in \mathbb{C}$. Here $h \in \mathbb{C} - \{0\}$.

A function $\phi : \mathbb{C}^n \rightarrow \mathbb{C}^m$ is holomorphic if

$$\frac{\partial \phi}{\partial z_i} = \lim_{h \rightarrow 0} \frac{\phi(z_1, \dots, z_i + h, \dots, z_n) - \phi(z_1, \dots, z_n)}{h}$$

exists for all $z = (z_1, \dots, z_n)$ and $i = 1, \dots, n$.

Definition 11.2. A complex manifold is a topological manifold with an atlas $\{U_\alpha, \phi_\alpha\}$, where $\phi_\alpha : U_\alpha \rightarrow \mathbb{C}^n$ and $\phi_\beta \circ \phi_\alpha^{-1} : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a holomorphic map.

Remark: A holomorphic map $f : \mathbb{C} \rightarrow \mathbb{C}$, when viewed as a map $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, is a smooth map. Hence a complex manifold is automatically a smooth manifold.

Compute the Jacobians. Rewriting as a map $\phi_{UV} : \mathbb{R}^2 - \{0\} \rightarrow \mathbb{R}^2 - \{0\}$, we compute:

$$J_{\phi_{UV}} = \left(\begin{array}{cc} \frac{\partial}{\partial x} \left(\frac{x}{x^2+y^2} \right) & \frac{\partial}{\partial y} \left(\frac{x}{x^2+y^2} \right) \\ \frac{\partial}{\partial x} \left(\frac{-y}{x^2+y^2} \right) & \frac{\partial}{\partial y} \left(\frac{-y}{x^2+y^2} \right) \end{array} \right) = \frac{1}{(x^2 + y^2)^2} \begin{pmatrix} y^2 - x^2 & -2xy \\ 2xy & y^2 - x^2 \end{pmatrix}.$$

Remark: It is not a coincidence that $a_{11} = a_{22}$ and $a_{21} = -a_{12}$.

Explain that TS^2 is obtained by gluing two copies of $(\mathbb{R}^2 - \{0\}) \times \mathbb{R}^2$ together via a map which sends $(a_1, a_2)^T$ over (x_1, y_1) to $J(a_1, a_2)^T$ over (x_2, y_2) .

12. COTANGENT BUNDLES AND 1-FORMS

12.1. The cotangent bundle. We now “topologize” the *cotangent bundle* $T^*M = \sqcup_p T_p^*M$. Again we have a projection $\pi : T^*M \rightarrow M$. Given a coordinate chart $U \subset M$, we identify $\pi^{-1}(U) \simeq U \times \mathbb{R}^n$. An element of T_p^*M is $f - f(p)$, which we usually denote by $df(p)$. In terms of the basis dx_1, \dots, dx_n , any df is written as $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$. Therefore, an element of T_p^*M is $\sum a_i dx_i$.

Transition functions. Take charts $U, V \subset M$, and coordinatize $\pi^{-1}(U)$ and $\pi^{-1}(V)$ by (x, a) , (y, b) .

Lemma 12.1.

$$dy_i = \sum_j \frac{\partial y_i}{\partial x_j} dx_j.$$

Proof. It suffices to assume that the transition map $\phi_{UV} : U \cap V \subset U \rightarrow U \cap V \subset V$ sends $0 \mapsto 0$. Then

$$dy_i = y_i(x) - y_i(0) = y_i(x) = \sum_j \frac{\partial y_i}{\partial x_j} x_j = \sum_j \frac{\partial y_i}{\partial x_j} dx_j.$$

□

We denote this more simply as $dy = \frac{\partial y}{\partial x} dx$. Then $dx = \left(\frac{\partial y}{\partial x}\right)^{-1} dy$. Hence, $\sum a_i dx_i = \sum a_i \left[\left(\frac{\partial y}{\partial x}\right)^{-1}\right]_{ij} dy_j$, i.e., $(x, a) \mapsto (y, a \left(\frac{\partial y}{\partial x}\right)^{-1})$.

Exercise: Compute the Jacobian of the transition function.

12.2. Functoriality. Let $f : M \rightarrow N$ be a smooth map between manifolds. Then we can define two natural maps.

Contravariant functor. First observe that there is a map $C^\infty(f(p)) \xrightarrow{f^*} C^\infty(p)$ and also a restriction $\mathcal{F}_{f(p)} \xrightarrow{f^*} \mathcal{F}_p$. This allows us to define:

$$\begin{aligned} f^* : T_{f(p)}^*N &\rightarrow T_p^*M, \\ dg &\mapsto dg \circ f. \end{aligned}$$

f^* is a *contravariant functor*.

Covariant functor. Next, we have:

$$f_* : T_p M \rightarrow T_{f(p)} M,$$

given by $X \mapsto X \circ f^*$. (Think of tangent spaces as derivations.) This makes sense:

$$C^\infty(f(p)) \xrightarrow{f^*} C^\infty(p) \xrightarrow{X} \mathbb{R}.$$

f_* is usually called the *derivative map*.

Exercise: Define the derivative map in terms of Definition 1 of the tangent space, and show the equivalence with the definition just given.

12.3. Properties of 1-forms.

Definition 12.2. Given the cotangent bundle $T^*M \xrightarrow{\pi} M$, a 1-form over $U \subset M$ is a smooth map $s : U \rightarrow T^*M$ such that $\pi \circ s = id$.

Note that a 1-form simply assigns, in a smooth manner, an element of T_p^*M to a given $p \in M$. The space of 1-forms on U is denoted $\Omega^1(U)$. Note that the space of 1-forms is an \mathbb{R} -vector space.

1. We often write $\Omega^0(M) = C^\infty(M)$. Then there exists a map $d : \Omega^0(M) \rightarrow \Omega^1(M)$, $f \mapsto df$.
2. Given $\phi : M \rightarrow N$, there is no natural map $T^*M \rightarrow T^*N$ unless ϕ is a diffeomorphism. However, there exists a contravariant map $\phi^* : T^*N \rightarrow T^*M$. We *pull back* forms $\theta \mapsto \phi^*\theta$.
3. Let $\phi : M \rightarrow N$ and $\psi : L \rightarrow M$ be smooth maps between smooth manifolds, and let θ be a 1-form on N . Then $(\phi \circ \psi)^*\theta = \psi^*(\phi^*\theta)$. [Exercise. Note however that the order of pulling back is reasonable.]
4. There exists a commutative diagram:

$$\begin{array}{ccc} \Omega^0(N) & \xrightarrow{\phi^*} & \Omega^0(M) \\ d \downarrow & \circlearrowleft & \downarrow d \\ \Omega^1(N) & \xrightarrow{\phi^*} & \Omega^1(M) \end{array} ,$$

i.e., $d \circ \phi^* = \phi^* \circ d$. Check this for HW by unwinding the definitions.

5. $d(fg) = fdg + gdf$. Check this for HW.

Example: $\theta = x^2dy + ydx$ on \mathbb{R}^2 . Consider $i : \mathbb{R} \rightarrow \mathbb{R}^2$, $t \mapsto (t, 0)$. Then $i^*\theta = 0$.

13. LIE GROUPS

13.1. Lie groups.

Definition 13.1. A Lie group G is a smooth manifold together with smooth maps $\mu : G \times G \rightarrow G$ (multiplication), $i : G \rightarrow G$ (inverse) which endow G the structure of a group.

Definition 13.2. A Lie subgroup $H \subset G$ is a subgroup of G which is also a submanifold of G . A Lie group homomorphism $\phi : H \rightarrow G$ is a homomorphism which is also a smooth map of the underlying manifolds.

Examples:

- (1) $GL(n, \mathbb{R}) = \{A \in M_n \mid \det(A) \neq 0\}$. We already showed that this is a manifold. The product AB is defined by a formula which is polynomial in the matrix entries of A and B , so μ is smooth. Similarly prove that i is smooth.
- (2) $SL(n, \mathbb{R}) = \{A \in M_n \mid \det(A) = 1\}$ is a Lie subgroup of $GL(n, \mathbb{R})$.
- (3) $O(n) = \{A \in M_n \mid AA^T = id\}$.
- (4) $SO(n, \mathbb{R}) = SL(n, \mathbb{R}) \cap O(n)$.

More invariantly, given a vector space V , define $GL(V)$ to be the set of isomorphisms $V \rightarrow V$.

Definition 13.3. A Lie group representation is a Lie group homomorphism $\phi : G \rightarrow GL(V)$, for some vector space V .

13.2. **Extended example:** $O(n)$. The orthogonal group is $O(n) = \{A \in M_n \mid AA^T = I\}$.

1. $AA^T = I$ implies $\det(AA^T) = \det(I) \Rightarrow \det(A) = \pm 1$. Here we are using $\det(AB) = \det(A) \cdot \det(B)$ and $\det(A^T) = \det(A)$.

2. Recall $GL(n, \mathbb{R}) = \{A \in M_n \mid \det(A) \neq 0\}$ and $SL(n, \mathbb{R}) = \{A \in M_n \mid \det(A) = 1\}$. Thus, $O(n) \subset GL(n, \mathbb{R})$ but $O(n) \not\subset SL(n, \mathbb{R})$ (not quite). $O(n)$ has two connected components $\det(A) = 1$ and $\det(A) = -1$. Examples of $\det(A) = -1$ is $(-1, 0; 0, 1)$. The connected component with $\det(A) = 1$ is called $SO(n)$. Note $SO(n) = O(n) \cap SL(n, \mathbb{R})$.

3. Show $O(n)$ is a submanifold of $GL(n, \mathbb{R})$. Consider the map $\phi : GL(n, \mathbb{R}) \rightarrow Sym(n)$ given by $A \mapsto AA^T$. The directional derivative is $d\phi(A)(B) = AB^T + BA^T = AB^T + (AB^T)^T$. Since for any matrix C there is a solution to the equation $AB^T = C$ (here A is fixed and we are solving for B), it follows that $d\phi(A)$ is surjective.

4. $\dim O(n) = \dim GL(n, \mathbb{R}) - \dim Sym(n) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

5. $SO(n)$ is the group of “rigid rotations” of $S^{n-1} \subset \mathbb{R}^n$. To see this, write $A = (a_1, \dots, a_n)^T$, where a_i are column vectors. Then $AA^T = id$ implies that $a_i^T a_j = \delta_{ij}$, and $\{a_1, \dots, a_n\}$ forms an orthonormal basis for \mathbb{R}^n with the usual inner product.

6. Show compactness. Since $O(n) = \phi^{-1}(id)$, it follows that $O(n)$ is closed. It is also bounded by the previous paragraph.

7. Exercise: $SO(2)$. Elements are of the form $(\cos \theta, \sin \theta; -\sin \theta, \cos \theta)$. Show $SO(2)$ is diffeomorphic to S^1 .

8. Consider $SO(3)$. These are the rigid rotations of $S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$. Prove that every element A of $SO(3)$ which is not the identity has a unique axis of rotation in \mathbb{R}^3 . In other words, there is a unique line through the origin in \mathbb{R}^3 which is fixed by A , and A is given by a rotation about this axis.

14. VECTOR BUNDLES

14.1. **Vector bundles.** $T^*M \xrightarrow{\pi} M$ is an example of a *vector bundle*.

Definition 14.1. A (real) vector bundle over a manifold M is a pair $(E, \pi : E \rightarrow M)$ such that:

- (1) $\pi^{-1}(p)$ has a structure of a vector space (over \mathbb{R}) of dimension n .
- (2) There exists a cover $\{U_\alpha\}$ of M such that $\pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times \mathbb{R}^n$ which restricts to a vector space isomorphism $\pi^{-1}(p) \xrightarrow{\sim} \mathbb{R}^n$

n is said to be the rank of the vector bundle. A rank 1 vector bundle is often called a line bundle.

The second condition: π admits a *local trivialization*.

Definition 14.2. A section of a vector bundle $\pi : E \rightarrow M$ over $U \subset M$ is a smooth map $s : U \rightarrow E$ such that $\pi \circ s = id$. A section over M is called a *global section*. The space of sections of E over U is often written $\Gamma(E, U)$. Also write $\Gamma(E)$ if $U = M$. $\Gamma(E, U)$ clearly has an \mathbb{R} -vector space structure.

Sections of TM are called *vector fields*. We often write $\mathfrak{X}(M) = \Gamma(TM)$. Sections of T^*M are *1-forms*. $\Omega^1(M) = \Gamma(T^*M)$.

14.2. **Transition functions, reinterpreted.** Consider $\pi : TM \rightarrow M$ and local trivializations $\pi^{-1}(U) = U \times \mathbb{R}^n$, $\pi^{-1}(V) = V \times \mathbb{R}^n$. We have transition functions $\phi_{UV} : (U \cap V) \times \mathbb{R}^n \rightarrow (U \cap V) \times \mathbb{R}^n$ (the first a subset of $U \times \mathbb{R}^n$ and the second a subset of $V \times \mathbb{R}^n$), sending $a_i \frac{\partial}{\partial x_i} \mapsto \frac{\partial y_j}{\partial x_i} a_i \frac{\partial}{\partial y_j}$. Note that $a = (a_1, \dots, a_n)^T$ is multiplied by $\frac{\partial y}{\partial x}$. Alternatively, think of $\frac{\partial y}{\partial x} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ as:

$$\Phi_{UV} : U \cap V \rightarrow GL(n, \mathbb{R}).$$

1. For double intersections $U \cap V$, we have $\Phi_{UV} \circ \Phi_{VU} = id$.
2. For triple intersections $U_\alpha \cap U_\beta \cap U_\gamma$ with coordinates x, y, z , we have $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial x}$ (chain rule), i.e.,

$$\Phi_{U_\gamma U_\alpha} = \Phi_{U_\gamma U_\beta} \circ \Phi_{U_\beta U_\alpha}.$$

This is usually called the *cocycle condition*.

What's this cocycle condition? This cocycle condition (triple intersection property) is clearly necessary if we want to construct a vector bundle by patching together $U_\alpha \times \mathbb{R}^n$. It guarantee that the gluings that we prescribe, i.e., $\Phi_{U_\alpha U_\beta}$ from U_α to U_β , etc. are *compatible*.

On the other hand, if we can find a collection $\{\Phi_{U_\alpha U_\beta}\}$ (for all U_α, U_β), which satisfies the cocycle condition, we have a legitimate vector bundle.

Consider $\pi : T^*M \rightarrow M$. Then $\Phi_{UV} : U \cap V \rightarrow GL(n, \mathbb{R})$ is given by $\left[\left(\frac{\partial y}{\partial x} \right)^{-1} \right]^T$. This makes it clear that the inverse and transpose are both necessary for the cocycle condition to be met.

15. MORE ON VECTOR BUNDLES

15.1. **Orientability.** Let $GL^+(n, \mathbb{R}) \subset GL(n, \mathbb{R})$ be the space of $n \times n$ matrices with positive determinant. Observe that $GL(n, \mathbb{R})$, like $O(n)$, is not connected, and has two connected components. M is *orientable* if there exists a subatlas $\{U_\alpha\}$ of the maximal atlas such that $\Phi_{U_\alpha U_\beta} : U_\alpha \cap U_\beta \rightarrow GL(n, \mathbb{R})$, given by the Jacobian $\frac{\partial y}{\partial x}$, factors through $GL^+(n, \mathbb{R})$. Here x is the coordinate for U_α and y is the coordinate for U_β . (In this case, “factoring through” means the image of $\Phi_{U_\alpha U_\beta}$ is in $GL^+(n, \mathbb{R})$.)

Example: Recall S^2 as a complex manifold, obtained by gluing together $U = \mathbb{C}$ and $V = \mathbb{C}$ via the map $z \mapsto \frac{1}{z}$. The transition function, written in real coordinates (x, y) , was:

$$\Phi_{UV} : U \cap V \rightarrow GL(2, \mathbb{R}),$$

$$(x, y) \mapsto \frac{1}{(x^2 + y^2)^2} \begin{pmatrix} y^2 - x^2 & -2xy \\ 2xy & y^2 - x^2 \end{pmatrix}.$$

The determinant is positive, so S^2 is oriented.

It is not easy to prove, directly from the definition, that the following examples are *not* orientable.

Example: The Möbius band B is obtained from $[0, 1] \times [0, 1]$ by identifying $(0, t) \sim (1, 1 - t)$. (Strictly speaking, B is a *manifold with boundary*, which is also locally modeled on the upper half space of \mathbb{R}^2 .) By following an oriented basis along the length of the band, we see that the orientation is reversed when we cross $\{1\} \times [0, 1]$. Hence B is not orientable.

Example: The Klein bottle K is obtained from $[0, 1] \times [0, 1]$ by identifying $(0, t) \sim (1, 1 - t)$ and $(s, 0) \sim (s, 1)$. It is not orientable.

Example: $\mathbb{RP}^2 = \mathbb{R}^3 - \{0\} / \sim$, where $x \sim tx, t \in \mathbb{R} - \{0\}$. This is the set of lines through the origin of \mathbb{R}^3 . Take the unit sphere S^2 . Then \mathbb{RP}^2 is the quotient of S^2 , obtained by identifying $x \sim -x$. By following an oriented basis from x to $-x$, we find that \mathbb{RP}^2 is not orientable. Observe that there is a two-to-one map $\pi : S^2 \rightarrow \mathbb{RP}^2$, which is called the *orientation double cover*.

Classification of compact 2-manifolds (surfaces). The oriented ones are: S^2, T^2 , surface of genus g . The nonorientable ones are: \mathbb{RP}^2 , Klein bottle, and one whose orientation double cover is an orientable surface of genus g .

15.2. **Complex manifolds.** Let $GL(n, \mathbb{C})$ be the space of $n \times n$ matrices with complex coefficients with nonzero determinant. How do you view $GL(n, \mathbb{C})$ as a Lie subgroup of $GL(2n, \mathbb{R})$? We’ll do $\rho : GL(1, \mathbb{C}) \hookrightarrow GL(2, \mathbb{R})$, and leave the general case as HW. Consider $z \in GL(1, \mathbb{C})$. If we write $z = x + iy$, then ρ maps

$$z = x + iy \mapsto \begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

It is easy to verify that ρ is a homomorphism (i.e., a representation).

Previous example: As can be seen from the transition function, for S^2 , there exists a factorization:

$$\Phi_{UV} : U \cap V \rightarrow GL(1, \mathbb{C}) \rightarrow GL(2, \mathbb{R}).$$

HW: Show that $GL(n, \mathbb{C}) \subset GL^+(2n, \mathbb{R})$. Therefore, complex manifolds are always orientable.

15.3. Constructing new vector bundles out of TM . Let M be a manifold and $\{U_\alpha\}$ an atlas for M . View TM as being constructed out of $U_\alpha \times \mathbb{R}^n$ by gluing using transition functions $\Phi_{U_\alpha U_\beta} : U_\alpha \cap U_\beta \rightarrow GL(n, \mathbb{R})$. Recall $U_\alpha \times \mathbb{R}^n$ and $U_\beta \times \mathbb{R}^n$ are glued using $\Phi_{U_\alpha U_\beta}$ along the fibers, and $\Phi_{U_\alpha U_\beta}$ satisfies the *cocycle condition*.

Consider a representation $\rho : GL(n, \mathbb{R}) \rightarrow GL(m, \mathbb{R})$.

Examples:

- (1) $\rho : GL(n, \mathbb{R}) \rightarrow GL(1, \mathbb{R}) = \mathbb{R}^\times, A \mapsto \det(A)$.
- (2) $\rho : GL(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R}), A \mapsto BAB^{-1}$.
- (3) $\rho : GL(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R}), A \mapsto (A^{-1})^T$.

Use ρ and glue $U_\alpha \times \mathbb{R}^m$ together using:

$$\rho \circ \Phi_{U_\alpha U_\beta} : U_\alpha \cap U_\beta \rightarrow GL(n, \mathbb{R}) \rightarrow GL(m, \mathbb{R}).$$

Observe that the cocycle condition is satisfied since ρ is a representation. Therefore we obtain a new vector bundle $TM \times_\rho \mathbb{R}^m$, called *TM twisted by ρ* .

Note that Example 3 above is just the cotangent bundle T^*M .

Example 1 gives rise to a *line bundle* (i.e., vector bundle of rank 1), usually denoted $\bigwedge^n TM$. The *orientability* of M is equivalent to the existence of a global section $s \in \Gamma(\bigwedge^n TM, M)$ which is never zero.

16. DAY 15

16.1. Integrating 1-forms.

Let C be an embedded arc in M , i.e., it is the image of some embedding $\gamma : [c, d] \rightarrow M$. In addition, we assume that C is *oriented*. In this case, the direction/orientation on C arises from the usual ordering on the real line. Let ω be a 1-form on M . Then we define the *integral* of ω over C to be:

$$\int_C \omega \stackrel{\text{def}}{=} \int_c^d \gamma^* \omega.$$

If t is the coordinate on $[c, d]$, then $\gamma^* \omega$ (in fact any 1-form) will have the form $f(t)dt$.

Lemma 16.1. *The definition does not depend on the particular orientation-preserving parametrization $\gamma : [c, d] \rightarrow M$.*

Proof. Take a different $\gamma_1 : [a, b] \rightarrow M$. Then there exists an orientation-preserving diffeomorphism $g : [a, b] \rightarrow [c, d]$ such that $\gamma_1 = \gamma \circ g$. (In our case, orientation-preserving means $g(a) = c$ and $g(b) = d$.) Now, $\gamma_1^* \omega = (\gamma \circ g)^* \omega = g^*(\gamma^* \omega)$, and

$$\int_c^d \gamma^* \omega = \int_c^d f(t)dt = \int_a^b f(g(s))dg(s) = \int_a^b \gamma_1^* \omega.$$

□

Now we know how to integrate 1-forms. Over the next few weeks we will define objects that we can integrate on higher-dimensional submanifolds (not just curves), called k -forms. For this we need to do quite a bit of preparation.

16.2. Linear algebra. We define some notions in linear algebra. The vector spaces we are concerned with do not need to be finite-dimensional, but you may suppose they are if you want. Let V, W be vector spaces over \mathbb{R} .

1. (Direct sum) $V \oplus W$. As a set, $V \times W$. Addition $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$. $\dim(V \oplus W) = \dim(V) + \dim(W)$.

2. $\text{Hom}(V, W) = \{\text{linear maps } \phi : V \rightarrow W\}$. In particular, we have $V^* = \text{Hom}(V, \mathbb{R})$. $\dim(\text{Hom}(V, W)) = \dim(V) \cdot \dim(W)$.

3. (Tensor product) $V \otimes W$.

Informal definition. Suppose V and W are finite-dimensional, and let $\{v_1, \dots, v_m\}, \{w_1, \dots, w_n\}$ be bases for V and W , respectively. Then $V \otimes W$ is a vector space which has $\{v_i \otimes w_j \mid i = 1, \dots, m; j = 1, \dots, n\}$ as basis. Elements of $V \otimes W$ are linear combinations $\sum_{ij} a_{ij} v_i \otimes w_j$.

Definition 16.2. Let V_1, \dots, V_k, U be vector spaces. A map $\phi : V_1 \times \dots \times V_k \rightarrow U$ is multilinear if ϕ is linear in each V_i separately, i.e., $\phi : \{v_1\} \times \dots \times V_i \times \dots \times \{v_k\} \rightarrow U$ is linear for each $v_j \in V_j, j \neq i$. If $k = 2$, we say ϕ is bilinear.

Formal definition. $V \otimes W$ is a vector space Z together with a bilinear map $i : V \times W \rightarrow Z$, which satisfies the following *universal mapping property*. Given any bilinear map $\phi : V \times W \rightarrow U$, there exists a linear map $\tilde{\phi} : Z \rightarrow U$ such that $\phi = \tilde{\phi} \circ i$.

Actual construction. Start with the free vector space $F(S)$ generated by a set S . By this we mean $F(S)$ consists of finite linear combinations $\sum_i a_i s_i$, where $s_i \in S, a_i \in \mathbb{R}$, and we are treating distinct $s_i \in S$ as linearly independent. (In other words, S is a basis for $F(S)$.) In our case we take $F(V \times W)$.

In $V \otimes W$, we want finite linear combinations of things that look like $v \otimes w$. We also would like the following:

$$\begin{aligned} (v_1 + v_2) \otimes w &= v_1 \otimes w + v_2 \otimes w, \\ v \otimes (w_1 + w_2) &= v \otimes w_1 + v \otimes w_2, \\ c(v \otimes w) &= (cv) \otimes w = v \otimes (cw). \end{aligned}$$

We therefore consider $R(V, W) \subset F(V \times W)$, the vector space generated by the “bilinear relations”

$$\begin{aligned} (v_1 + v_2, w) - (v_1, w) - (v_2, w), \\ (v, w_1 + w_2) - (v, w_1) - (v, w_2), \\ (cv, w) - c(v, w), \\ (v, cw) - c(v, w). \end{aligned}$$

Then the quotient space $F(V \times W)/R(V, W)$ is $V \otimes W$.

Verification of universal mapping property. With $V \otimes W$ defined as above, let $i : V \times W \rightarrow V \otimes W$ be $(v, w) \mapsto v \otimes w$. The bilinearity of i follows from the construction of $V \otimes W$. (For example, $i(v_1 + v_2, w) = (v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w = i(v_1, w) + i(v_2, w)$.) $\tilde{\phi}$ maps $\sum_i a_i v_i \otimes w_i \mapsto \sum_i a_i \phi(v_i, w_i)$. This map is well-defined because all the elements of $R(V, W)$ get mapped to 0.

17. TENSOR AND EXTERIOR ALGEBRA

17.1. More on tensor products. Recall the definition of the tensor product as $V \otimes W = F(V \times W)/R(V, W)$ and the universal property. The universal property is useful for the following reason: If we want to construct a linear map $V \otimes W \rightarrow U$, it is equivalent to check the existence of a bilinear map $V \times W \rightarrow U$.

Dimension of $V \otimes W$. Suppose V, W are finite-dimensional. Then we claim $\dim(V \otimes W) = \dim V \cdot \dim W$. To see this, consider the map $V^* \otimes W \rightarrow \text{Hom}(V, W)$ which sends $f \otimes w \mapsto fw$, where $fw : v \mapsto f(v)w$. The universal mapping property guarantees the well-definition of this map. $\dim \text{Hom}(V, W)$ can be easily calculated to be $\dim V \cdot \dim W$. Now, it suffices to check surjectivity and injectivity. Surjectivity: let f_i be dual to a basis $\{v_1, \dots, v_m\}$ for V , i.e., $f_i(v_j) = \delta_{ij}$; also let $\{w_1, \dots, w_n\}$ be a basis for W . Then any linear map in $\text{Hom}(V, W)$ is of the form $\sum a_{ij} f_i w_j$, i.e., comes from $\sum a_{ij} f_i \otimes w_j$. Details are left for HW.

Properties of tensor products.

- (1) $V \otimes W \simeq W \otimes V$.
- (2) $(V \otimes W) \otimes U \simeq V \otimes (W \otimes U)$.

1. Worked out. It's difficult to directly get a well-defined map $V \otimes W \rightarrow W \otimes V$, so start with a bilinear map $V \times W \rightarrow W \times V \rightarrow W \otimes V$, where $(v, w) \mapsto w \otimes v$. It then lifts to a map $V \otimes W \rightarrow W \otimes V$ which sends $v \otimes w \mapsto w \otimes v$. It is easy to verify injectivity and surjectivity.

The second property ensures us that we do not need to write parentheses when we take a tensor product of several vector spaces.

Let $A : V \rightarrow V$ and $B : W \rightarrow W$ be linear maps. Then we have

$$A \oplus B : V \oplus W \rightarrow V \oplus W$$

and

$$A \otimes B : V \otimes W \rightarrow V \otimes W.$$

We denote $V^{\otimes k}$ for the k -fold tensor product of V . Then we have a representation $\rho : GL(V) \rightarrow GL(V^{\otimes k})$, $A \mapsto A \otimes \dots \otimes A$. This gives us an associated vector bundle twisted by ρ .

The tensor algebra. $T(V) = \mathbb{R} \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} + \dots$. The multiplication is given by $(v_1 \otimes \dots \otimes v_s)(v_{s+1} \otimes \dots \otimes v_t) = v_1 \otimes \dots \otimes v_t$.

17.2. The exterior algebra. We define $\bigwedge V$ to be $T(V)/\mathcal{I}$, where \mathcal{I} is a (2-sided) ideal generated by elements of the form $v \otimes v$, i.e., elements of \mathcal{I} are finite sums of terms that look like $\eta_1 \otimes v \otimes v \otimes \eta_2$, where $\eta_1, \eta_2 \in T(V)$. Elements of $\bigwedge V$ are denoted $\sum a_{i_1 \dots i_k} v_{i_1} \wedge \dots \wedge v_{i_k}$. Then in $T(V)$ we have $v \wedge v = 0$. Also note that $(v_1 + v_2) \wedge (v_1 + v_2) = v_1 \wedge v_1 + v_1 \wedge v_2 + v_2 \wedge v_1 + v_2 \wedge v_2$. The first and last terms are zero, so $v_1 \wedge v_2 = -v_2 \wedge v_1$. Therefore, we may assume that $i_1 < \dots < i_k$ in the expression above. $\bigwedge V$ is clearly an algebra, i.e., there is a multiplication $\omega \wedge \eta$ given elements ω, η in $\bigwedge V$.

We define $\bigwedge^k V$ to be the degree k terms of $\bigwedge V$.

Alternating multilinear forms. A multilinear form $\phi : V \times \cdots \times V \rightarrow U$ is *alternating* if $\phi(v_1, \dots, v_i, v_{i+1}, \dots, v_k) = -1 \cdot \phi(v_1, \dots, v_{i+1}, v_i, \dots, v_k)$. Recall that transpositions generate the full symmetric group S_k . If $(1, \dots, k) \mapsto (i_1, \dots, i_k)$, and σ is the number of transpositions needed, then $\phi(v_1, \dots, v_k) = (-1)^\sigma \phi(v_{i_1}, \dots, v_{i_k})$.

Universal property. $\bigwedge^k V$ and $i : V \times \cdots \times V \rightarrow \bigwedge^k V$ satisfy the following. Given an alternating multilinear map $\phi : V \times \cdots \times V \rightarrow U$, there is a linear map $\tilde{\phi} : \bigwedge^k V \rightarrow U$ such that $\phi = \tilde{\phi} \circ i$.

Proposition 17.1. *Given a basis $\{e_1, \dots, e_n\}$ for V , a basis for $\bigwedge^k V$ consists of degree k monomials $e_{i_1} \wedge \cdots \wedge e_{i_k}$ with $i_1 < \cdots < i_k$. Therefore, if $k > n$, $\dim \bigwedge^k V = 0$, and if $k \leq n$, $\dim \bigwedge^k V = \binom{n}{k}$.*

If $V = \mathbb{R}^3$ with basis $\{e_1, e_2, e_3\}$, then $\bigwedge^0 V = \mathbb{R}$, $\bigwedge^1 V = \mathbb{R}\{e_1, e_2, e_3\}$, $\bigwedge^2 V = \mathbb{R}\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$, $\bigwedge^3 V = \mathbb{R}\{e_1 \wedge e_2 \wedge e_3\}$, and $\bigwedge^k V = 0$, $k > 3$.

18. DIFFERENTIAL k -FORMS

18.1. **Basis for $\bigwedge^k V$.** We'll give the proof of Proposition 17.1 in several steps. It is easy to see that $\{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1, \dots, i_k \leq n\}$ spans $\bigwedge^k V$. Moreover, by the anticommutation relations, $a \wedge e_i \wedge e_j \wedge b = -a \wedge e_j \wedge e_i \wedge b$ and $a \wedge e_i \wedge e_i \wedge b = 0$, so $\{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\}$ spans $\bigwedge^k V$.

1. If $k > n$, $\bigwedge^k V = 0$. This is clear since it is impossible to find $1 \leq i_1 < \cdots < i_k \leq n$.

2. If $k = n$, $\bigwedge^k V = \mathbb{R}$, and the basis is given by $e_1 \wedge \cdots \wedge e_n$. Since $e_1 \wedge \cdots \wedge e_n$ spans $\bigwedge^k V$, it remains to show that $e_1 \wedge \cdots \wedge e_n$ is nonzero! This is done by defining an alternating multilinear form $V \times \cdots \times V \rightarrow \mathbb{R}$ (n copies of V). Then by the universal property, $\bigwedge^n V$ cannot be zero and hence must be \mathbb{R} . Details are HW.

3. If $1 \leq k < n$, then we show that $\{e_{i_1} \wedge \cdots \wedge e_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\}$ is linearly independent. Indeed, suppose $\sum_{i_1 < \cdots < i_k} a_{i_1, \dots, i_k} e_{i_1} \wedge \cdots \wedge e_{i_k} = 0$. For each summand, there is a unique term $e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}$ which kills all the other summands and gives $\pm a_{i_1, \dots, i_k} e_1 \wedge \cdots \wedge e_n$. Hence this implies that the a_{i_1, \dots, i_k} are zero.

Remark: For any $v_1, \dots, v_k \in V$, $v_1 \wedge \cdots \wedge v_k \neq 0$ in $\bigwedge^k V$ if and only if v_1, \dots, v_k are linearly independent.

18.2. **Tensor calculus on manifolds.** We have now constructed $V^{\otimes k}$ and $\bigwedge^k V$, given a finite-dimensional vector space V . Also note that there exist natural representations $\rho_0 : GL(V) \rightarrow GL(V^{\otimes k})$ and $\rho_1 : GL(V) \rightarrow GL(\bigwedge^k V)$.

Example: $\dim V = 2$. Basis $\{v_1, v_2\}$. $\bigwedge V$ has basis $\{1, v_1, v_2, v_1 \wedge v_2\}$. If $A : V \rightarrow V$ is linear and sends $v_i \mapsto a_{1i}v_1 + a_{2i}v_2$, $i = 1, 2$, then $A(v_1 \wedge v_2) = Av_1 \wedge Av_2 = \det(A)v_1 \wedge v_2$.

Thus we can form $TM \times_{\rho_0} V^{\otimes k} = \otimes_k TM$ and $TM \times_{\rho_1} \bigwedge^k V = \bigwedge^k TM$. Also can form $\otimes_k T^*M$ and $\bigwedge^k T^*M$.

We'll focus on $\bigwedge^k T^*M$ in what follows. Sections of $\bigwedge^k T^*M$ are called k -forms and locally look like:

$$\omega = \sum_{i_1 < \cdots < i_k} f_{i_1, \dots, i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Denote by $\Omega^k(M)$ the sections of $\bigwedge^k T^*M$.

Pullback: Let $\phi : M \rightarrow N$ be a smooth map between manifolds, and ω a k -form. Then we can define the pullback $\phi^*\omega$ in a manner similar to 1-forms:

$$\phi^*\omega = \sum_{i_1 < \cdots < i_k} (f_{i_1, \dots, i_k} \circ \phi) \cdot d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k}.$$

Check: The global well-definition, i.e., independent of choice of coordinates.

18.3. The exterior derivative. We can define the extension $d_k : \Omega^k \rightarrow \Omega^{k+1}$ of $d : \Omega^0 \rightarrow \Omega^1$ as follows (in local coordinates x_1, \dots, x_n):

(1) If $f \in \Omega^0$, then $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$.

(2) If $\omega = \sum_I f_I dx_I \in \Omega^k$, then $d_k \omega = \sum_I df_I \wedge dx_I$.

Here $I = (i_1, \dots, i_k)$ is an indexing set and $dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_k}$. We will often suppress the k in d_k .

HW: Check that d_k is well-defined and independent of the choice of local coordinates.

Example on \mathbb{R}^3 . Consider \mathbb{R}^3 with coordinates (x, y, z) . Consider

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \Omega^3.$$

The first d is the *gradient*

$$d : f \mapsto df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz.$$

The second d is the *curl*

$$d : f dx + g dy + h dz \mapsto \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) dy dz + \dots$$

The last d is the *divergence*

$$d : f_1 dy \wedge dz + f_2 dz \wedge dx + f_3 dx \wedge dy \mapsto \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \right) dx \wedge dy \wedge dz.$$

Note: We will often omit the \wedge .

19. DE RHAM COHOMOLOGY

This material is nicely presented in Bott & Tu.

Lemma 19.1. *The exterior derivative d satisfies the (skew-commutative) Leibniz rule:*

$$(2) \quad d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta),$$

where $\alpha \in \Omega^k$ and $\beta \in \Omega^l$, and k or l may be zero (in which case we ignore the \wedge).

Lemma 19.2. $d_k \circ d_{k-1} = 0$.

Proof. For $d^2 : \Omega^0 \rightarrow \Omega^2$, we compute:

$$d \circ df = d \left(\sum_i \frac{\partial f}{\partial x_i} dx_i \right) = \sum_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_j \wedge dx_i = 0.$$

When $\alpha = dx_I$, we verify that $d\alpha = d(dx_I) = 0$.

Now if $\alpha = f_I dx_I$, then

$$\begin{aligned} d\alpha &= df_I \wedge dx_I + f_I d(dx_I) = df_I \wedge dx_I, \\ d^2\alpha &= (d^2 f_I) \wedge dx_I - df_I \wedge d(dx_I) = 0, \end{aligned}$$

which proves the lemma. □

Consider:

$$(3) \quad 0 \rightarrow \Omega^0 \xrightarrow{d_0} \Omega^1 \xrightarrow{d_1} \Omega^2 \xrightarrow{d_2} \dots \rightarrow \Omega^n \xrightarrow{d_n} 0,$$

where $n = \dim M$.

Example: If $M = \mathbb{R}^3$, then $d_0 = \text{grad}$, $d_1 = \text{curl}$, $d_2 = \text{div}$. Then $d^2 = 0$ is equivalent to $\text{div}(\text{curl}) = 0$, $\text{curl}(\text{grad}) = 0$.

Since $d_k \circ d_{k-1} = 0$, we have $\text{Im } d_{k-1} \subset \text{Ker } d_k$. This leads to the following definition:

Definition 19.3. *The k th de Rham cohomology group of M is given by:*

$$H_{dR}^k(M) \stackrel{\text{def}}{=} \text{Ker } d_k / \text{Im } d_{k-1}.$$

Definition 19.4. *Let $\omega \in \Omega^k(M)$. Then ω is closed if $d\omega = 0$ and is exact if $\omega = d\eta$ for some $\eta \in \Omega^{k-1}(M)$.*

Facts: The de Rham cohomology groups are diffeomorphism invariants of the manifold M , and are finite-dimensional if M is compact or admits a finite atlas.

Definition 19.5. *A sequence of vector spaces $\dots \rightarrow C^i \xrightarrow{d_i} C^{i+1} \xrightarrow{d_{i+1}} C^{i+2} \rightarrow \dots$ is said to be exact if $\text{Im } d_{i-1} = \text{Ker } d_i$ for all i .*

The de Rham cohomology groups measure the failure of Equation 3 to be *exact*.

We will often write $H^k(M)$ instead of $H_{dR}^k(M)$.

Examples.

1. $M = \{pt\}$. Then $\Omega^0(M) = \mathbb{R}$ and $\Omega^i(M) = 0, i \neq 0$. We have $H^0(pt) = \mathbb{R}$ and $H^i(pt) = 0, i > 0$.

2. $M = \mathbb{R}$. Then $\Omega^0(M) = C^\infty(\mathbb{R})$ and $\Omega^1(M) \simeq C^\infty(\mathbb{R})$ because every 1-form is of the form $f dx$. Now, $d : f \mapsto \frac{df}{dx} dx$ can be viewed as the map

$$d : C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R}), \\ f \mapsto f'.$$

It is easy to see that $\text{Ker } d = \{\text{constant functions}\}$ and hence $H^0(\mathbb{R}) = \mathbb{R}$. Next, $\text{Im } d$ is all of $C^\infty(\mathbb{R})$, since given any f we can take its antiderivative $\int_0^x f(t) dt$. Therefore, $H^1(\mathbb{R}) = 0$. Also $H^i(\mathbb{R}) = 0$ for $i > 1$ since $\Omega^i = 0$ for $i > 1$.

3. $M = S^1$. View S^1 as \mathbb{R}/\mathbb{Z} with coordinates x . Then

$$\Omega^0(S^1) = \{\text{Periodic functions on } \mathbb{R} \text{ with period } 1\}.$$

$\Omega^1(S^1)$ is also the set of periodic functions on \mathbb{R} by identifying $f(x) dx \mapsto f(x)$. $H^0(S^1) = \mathbb{R}$ as before. Now, $H^1(S^1) = \Omega^1 / \text{Im } d$ and $\text{Im } d$ is the space of all C^∞ -functions $f(x)$ with integral $\int_0^1 f(x) dx = 0$. Hence $H^1(S^1) = \mathbb{R}$. We also have an *exact sequence*:

$$0 \rightarrow \mathbb{R} \rightarrow \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{\int} \mathbb{R} \rightarrow 0.$$

Lemma 19.6. $H^0(M) \simeq \mathbb{R}$ if M is connected.

Proof. $df = 0$ if and only if f is locally constant. □

Lemma 19.7. If $M = M_1 \sqcup M_2$, then $H^k(M) \simeq H^k(M_1) \oplus H^k(M_2)$ for all $k \geq 0$.

HW: Prove the lemma.

20. DAY 19

20.1. **Pullback.** Let $\phi : M \rightarrow N$ be a smooth map between manifolds.

Lemma 20.1. $d \circ \phi^* = \phi^* \circ d$.

HW: Verify the lemma. This follows easily by computing in local coordinates.

Corollary 20.2. *There is an induced map $\phi^* : H^k(N) \rightarrow H^k(M)$ on the level of cohomology.*

Proof. Let ω be a closed k -form on N , i.e., $\omega \in \Omega^k(N)$ satisfies $d\omega = 0$. Then, $\phi^*\omega$ satisfies $d\phi^*\omega = \phi^*(d\omega) = 0$. Now, if ω is exact, i.e., $\omega = d\eta$, then $\phi^*\omega = \phi^*d\eta = d(\phi^*\eta)$ is exact as well. \square

20.2. **Mayer-Vietoris sequences.** This is a method for effectively decomposing a manifold and computing its cohomology from its components.

Let $M = U \cup V$, where U and V are open sets. Then we have natural inclusion maps

$$(4) \quad U \cap V \xrightarrow{i_U, i_V} U \sqcup V \xrightarrow{i} M.$$

Here i_U and i_V are inclusions of $U \cap V$ into U and into V .

Example: $M = S^1, U = V = \mathbb{R}, U \cap V = \mathbb{R} \sqcup \mathbb{R}$.

Theorem 20.3. *We have the following long exact sequence:*

$$\begin{aligned} 0 \rightarrow H^0(M) \xrightarrow{i^*} H^0(U) \oplus H^0(V) \xrightarrow{i_U^* - i_V^*} H^0(U \cap V) \rightarrow \\ \rightarrow H^1(M) \xrightarrow{i^*} H^1(U) \oplus H^1(V) \xrightarrow{i_U^* - i_V^*} H^1(U \cap V) \rightarrow \\ \rightarrow \dots \end{aligned}$$

Remark: $0 \rightarrow A \rightarrow B$ exact means $A \rightarrow B$ is injective. $A \rightarrow B \rightarrow 0$ exact means $A \rightarrow B$ is surjective. Hence $0 \rightarrow A \rightarrow B \rightarrow 0$ implies isomorphism.

The proof of Theorem 20.3 will be given over the next couple of lectures, but for the time being we will apply it:

Example: Compute $H^i(S^1)$ using Mayer-Vietoris.

20.3. **Poincaré lemma.** The following lemma is an important starting point when using the Mayer-Vietoris sequence to compute cohomology groups.

Lemma 20.4 (Poincaré lemma). *Let $\omega \in \Omega^k(\mathbb{R}^n)$ for $k \geq 1$. Then ω is closed if and only if it is exact.*

In other words, $H^k(\mathbb{R}^n) = 0$ for $k \geq 1$. We will give the proof later, together with some other homotopy-theoretic properties.

20.4. Partitions of unity.

Definition 20.5. Let $\{U_\alpha\}$ be an open cover of M . Then a collection of functions $\{f_\alpha \geq 0\}$ is a partition of unity subordinate to $\{U_\alpha\}$ if:

- (1) $\text{Supp}(f_\alpha) \subset U_\alpha$. Here the support $\text{Supp}(f_\alpha)$ of f_α is the closure of $\{x \in M \mid f_\alpha(x) \neq 0\}$.
- (2) At every point $x \in M$, there exists a neighborhood $N(x)$ of x such that the set

$$\{f_\alpha \mid f_\alpha|_{N(x)} \neq 0\}$$

is finite. If we write f_1, \dots, f_k for the nonzero functions, then $\sum_{i=1}^k f_i(x) = 1$.

Proposition 20.6. If $\{U_\alpha\}$ is an open cover of M , then there exists a partition of unity subordinate to $\{U_\alpha\}$.

Proof. The proof is done in stages.

Step 1. Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$:

$$f(x) = \begin{cases} e^{-1/x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

It is easy to show that $f \geq 0$ and f is smooth.

Step 2. Take $g : \mathbb{R} \rightarrow \mathbb{R}$ to be $g_{ab}(x) = f(x-a) \cdot f(b-x)$. (Suppose $a < b$.) Then $g(x)$ is a bump function.

- $g \geq 0$,
- $\text{supp}(g) = [a, b]$,
- $g > 0$ on (a, b) .

Step 3. Construct a bump function on $[a_1, b_1] \times \dots \times [a_n, b_n] \subset \mathbb{R}^n$ with coordinates (x_1, \dots, x_n) by letting $\phi(x) = g_{a_1 b_1}(x_1) \dots g_{a_n b_n}(x_n)$. Then ϕ is supported on $[a_1, b_1] \times \dots \times [a_n, b_n]$ and is positive on the interior.

Step 4. We will only treat the case where M is compact. For each $p \in M$, choose an open neighborhood U_p of p of the form $(a_1, b_1) \times \dots \times (a_n, b_n)$ whose closure is contained inside some U_α . For each U_p , construct ϕ_p as in Step 3. Now, since M is compact, there exists a finite collection of $\{p_1, \dots, p_k\}$ where $\{U_{p_i}\}$ cover M . Note that $\sum_{i=1}^k \phi_{p_i} > 0$ everywhere on M . If we let $\psi_p = \frac{\phi_p}{\sum \phi_p}$, then $\sum \psi_p = 1$. Finally, we associate to each ψ_p an open set U_α for which $U_p \subset U_\alpha$. Then ψ_α is the sum of all the ψ_p associated to U_α . \square

21. SOME HOMOLOGICAL ALGEBRA

21.1. **Short exact sequence.** Suppose $M = U \cup V$. Then we have

$$U \cap V \xrightarrow{i_U, i_V} U \sqcup V \xrightarrow{i} M.$$

i_U and i_V are two inclusions, one into U and the other into V .

Proposition 21.1. *The following is a short exact sequence.*

$$(5) \quad 0 \rightarrow \Omega^i(M) \xrightarrow{i^*} \Omega^i(U) \oplus \Omega^i(V) \xrightarrow{i_U^* - i_V^*} \Omega^i(U \cap V) \rightarrow 0.$$

Proof. We prove that $i_U^* - i_V^*$ is surjective. (The rest of the exact sequence is easy.) Use a partition of unity $\{\rho_U, \rho_V\}$ subordinate to $\{U, V\}$. Then $\rho_U + \rho_V = 1$. Given $\omega \in \Omega^i(U \cap V)$, consider $\rho_U \omega$ on U and $-\rho_V \omega$ on V . This works. \square

21.2. **Short exact sequences to long exact sequences.** Getting from the short exact sequence to the long exact sequence is a purely algebraic operation.

Define a *cochain complex* $(\mathcal{C}, d): \dots \rightarrow C^i \xrightarrow{d_i} C^{i+1} \xrightarrow{d_{i+1}} C^{i+2} \rightarrow \dots$ to be a sequence of vector spaces and maps with $d_{i+1} \circ d_i = 0$ for all i . (\mathcal{C}, d) gives rise to $H^i(\mathcal{C}) = \text{Ker } d_i / \text{Im } d_{i-1}$, the i th cohomology of the complex.

A *cochain map* $\phi : \mathcal{A} \rightarrow \mathcal{B}$ is:

$$\begin{array}{ccccccc} \xrightarrow{d_{k-2}} & A^{k-1} & \xrightarrow{d_{k-1}} & A^k & \xrightarrow{d_k} & A^{k+1} & \xrightarrow{d_{k+1}} \\ & \phi_{k-1} \downarrow & & \phi_k \downarrow & & \phi_{k+1} \downarrow & \\ \xrightarrow{d_{k-2}} & B^{k-1} & \xrightarrow{d_{k-1}} & B^k & \xrightarrow{d_k} & B^{k+1} & \xrightarrow{d_{k+1}} \end{array}$$

which satisfies $d_k \circ \phi_k = \phi_{k+1} \circ d_k$.

A cochain map $\phi : \mathcal{A} \rightarrow \mathcal{B}$ induces a map on cohomology:

$$\phi : H^k(\mathcal{A}) \rightarrow H^k(\mathcal{B}).$$

The verification is identical to that of the special case of de Rham.

Given an exact sequence $0 \rightarrow \mathcal{A} \xrightarrow{\phi} \mathcal{B} \xrightarrow{\psi} \mathcal{C} \rightarrow 0$, (i.e., we have collections of $0 \rightarrow A^i \rightarrow B^i \rightarrow C^i \rightarrow 0$ and all the maps are cochain maps),

$$\begin{array}{ccccccc}
& & d_{k+1} \uparrow & & d_{k+1} \uparrow & & d_{k+1} \uparrow \\
0 & \longrightarrow & A^{k+1} & \xrightarrow{\phi_{k+1}} & B^{k+1} & \xrightarrow{\psi_{k+1}} & C^{k+1} \longrightarrow 0 \\
& & d_k \uparrow & & d_k \uparrow & & d_k \uparrow \\
0 & \longrightarrow & A^k & \xrightarrow{\phi_k} & B^k & \xrightarrow{\psi_k} & C^k \longrightarrow 0 \\
& & d_{k-1} \uparrow & & d_{k-1} \uparrow & & d_{k-1} \uparrow
\end{array}$$

we always get a long exact sequence:

$$\begin{array}{ccccccc}
\dots & \longrightarrow & H^k(\mathcal{A}) & \xrightarrow{\phi_k} & H^k(\mathcal{B}) & \xrightarrow{\psi_k} & H^k(\mathcal{C}) \xrightarrow{\delta_k} \\
& & \longrightarrow & H^{k+1}(\mathcal{A}) & \xrightarrow{\phi_{k+1}} & H^{k+1}(\mathcal{B}) & \xrightarrow{\psi_{k+1}} & H^{k+1}(\mathcal{C}) \xrightarrow{\delta_{k+1}} \dots
\end{array}$$

Verification of $\text{Ker } \psi_k \supset \text{Im } \phi_k$. Suppose $[b] \in \text{Im } \phi_k$. Then $b = \phi_k a + db'$, where $a \in A^k$ and $b' \in B^{k-1}$. Now, $\psi_k b = \psi_k(\phi_k a) + \psi_k(db') = d(\psi_{k-1} b')$. Therefore, $[\psi_k b] = 0 \in H^k(\mathcal{C})$.

Verification of $\text{Ker } \psi_k \subset \text{Im } \phi_k$. Suppose $[b] \in \text{Ker } \psi_k$. Then $\psi_k b = dc'$, $c' \in C^{k-1}$. Next, use the fact that $B^{k-1} \rightarrow C^{k-1} \rightarrow 0$ to find $b' \in B^{k-1}$ such that $c' = \psi_{k-1} b'$. Then $\psi_k b = d(\psi_{k-1} b') = \psi_k(db')$. Hence, by the exactness, $b - db' = \phi_k(a)$ for some $a \in A^k$. (Check that $da = 0$.) Thus, $\phi_k[a] = [b]$.

Definition of $\delta_k : H^k(\mathcal{C}) \rightarrow H^{k+1}(\mathcal{A})$. Let $[c] \in H^k(\mathcal{C})$. Then $dc = 0$. Also we have $b \in B^k$ with $\psi_k b = c$ by the surjectivity of $B^k \rightarrow C^k$. Consider db . Since $\psi_{k+1}(db) = d(\psi_k b) = dc = 0$, there exists an $a \in A^{k+1}$ such that $\phi_{k+1} a = db$. Let $[a] = \delta_k[c]$. Here, $da = 0$, since $\phi_{k+2}(da) = d(\phi_{k+1} a) = d(db) = 0$, and $A^{k+2} \rightarrow B^{k+2}$ is injective. We need to show that this definition is independent of the choice of c , choice of b , and choice of a . This is left for HW.

HW: Verify the rest of the exactness.

22. INTEGRATION

Let M be an n -dimensional manifold and $\omega \in \Omega^n(M)$. We will try to make sense of $\int_M \omega$.

22.1. **Brief review of integration on \mathbb{R}^n .** For more details see Spivak.

Given a function $f : R = [a_1, b_1] \times \dots \times [a_n, b_n] \rightarrow \mathbb{R}$, take a partition $P = (P_1, \dots, P_k)$ of R into small rectangles. Consider the upper and lower bounds

$$U(f, P) = \sum_i \sup(f|_{P_i}) \cdot \text{vol}(P_i), \quad L(f, P) = \sum_i \inf(f|_{P_i}) \cdot \text{vol}(P_i).$$

Definition 22.1. *The function f is Riemann-integrable (or simply integrable) if for any $\varepsilon > 0$ there exists a partition P such that $U(f, P) - L(f, P) < \varepsilon$. If f is integrable, then we define*

$$\int_R f dx_1 \dots dx_n = \lim_P U(f, P) = \lim_P L(f, P).$$

Similarly we can define $\int_U f dx_1 \dots dx_n$, where $U \subset \mathbb{R}^n$ is an open set and $f : U \rightarrow \mathbb{R}$ is a function with compact support.

Theorem 22.2. *If f is a continuous function with support on a compact set, then f is integrable.*

Change of variables formula. If $g : [a, b] \rightarrow [c, d]$ is smooth, then

$$\int_{g(a)}^{g(b)} f = \int_a^b (f \circ g) g'.$$

This can be rewritten as:

$$\int_{g([a,b])} f = \int_{[a,b]} f \circ g |g'|.$$

Let $U, V \subset \mathbb{R}^n$ be open sets with coordinates (x_1, \dots, x_n) , (y_1, \dots, y_n) , and $\phi : U \xrightarrow{\sim} V$ a diffeomorphism. Then:

$$\int_V f(y) dy_1 \dots dy_n = \int_U f(\phi(x)) \left| \frac{\partial \phi}{\partial x} \right| dx_1 \dots dx_n.$$

In light of the change of variables formula, $\int_M \omega$ makes sense only when M is orientable, since the change of variables for an n -form does not have the absolute value. At any rate, n -forms have the wonderful property of having the correct transformation property (modulo sign) under diffeomorphisms.

22.2. **Orientation.** Recall that M is *orientable* if there exists a subatlas $\{\phi_\alpha : U_\alpha \rightarrow \mathbb{R}^n\}$ such that the Jacobians $J_{\alpha\beta} = d(\phi_\beta \circ \phi_\alpha^{-1})$ have positive determinant.

Proposition 22.3. *M is orientable if and only if there exists a nowhere zero n -form ω on M .*

Proof. Suppose M is orientable. Take a partition of unity $\{f_\alpha\}$ subordinate to U_α . Let x_1, \dots, x_n be the coordinates on U_α . Construct $\omega_\alpha = f_\alpha x_1 \wedge \dots \wedge dx_n$. ω_α is a smooth n -form on M with support contained in U_α . Let $\omega = \sum_\alpha \omega_\alpha$. This is nowhere zero, since any $(\phi_\alpha \circ \phi_\beta^{-1})^* \omega_\beta = f_\beta \circ (\phi_\alpha \circ \phi_\beta^{-1}) \det(J_{\beta\alpha}) dx_1 \wedge \dots \wedge dx_n$. The key point here is that $\det(J_{\beta\alpha})$ are positive, so $f_\beta \circ (\phi_\alpha \circ \phi_\beta^{-1}) \det(J_{\beta\alpha}) \geq 0$. At any point $p \in M$, at least one f_α is positive, and the ω_α are *additive*, so ω is nowhere zero on M .

On the other hand, suppose there exists a nowhere zero n -form ω on M . Given U_α , we choose coordinates x_1, \dots, x_n so that $dx_1 \wedge \dots \wedge dx_n$ is a positive function times ω . Once we do this, clearly $J_{\alpha\beta}$ has positive determinant. \square

Since the n -form ω is nowhere zero, what this says is that $\bigwedge^n T^*M$ is isomorphic to $M \times \mathbb{R}$ as a vector bundle, i.e., is a *trivial* vector bundle.

On a connected manifold M , any two nowhere zero n -forms ω and ω' differ by a function, i.e., \exists a positive (or negative) function f s.t. $\omega = f\omega'$. We define an equivalence relation $\omega \sim \omega'$ if $\omega = f\omega'$ and $f > 0$. Then there exist two equivalence classes of \sim and each equivalence class is called an *orientation* of M .

The standard orientation on \mathbb{R}^n is $dx_1 \wedge \dots \wedge dx_n$.

Equivalent definition of orientation. The set $Fr(V)$ of ordered bases (or *frames*) of a finite-dimensional vector space V of dimension n is diffeomorphic to $GL(V)$ (albeit not naturally): Fix an ordered basis (v_1, \dots, v_n) . Then any other basis (w_1, \dots, w_n) can be written as (Av_1, \dots, Av_n) , $A \in GL(V)$. Therefore, there is a bijection $Fr(V) \simeq GL(V)$, and we induce a smooth structure on $Fr(V)$ from $GL(V)$. (Note however that there is a distinguished point $id \in GL(V)$ but no distinguished basis in $Fr(V)$.) Since $GL(V)$ has two connected components, $Fr(V)$ has two components, and each component is called an *orientation* for V . An *orientation* for M is a choice of orientation for each $T_p M$ which is smooth in $p \in M$. [We can construct the *frame bundle* $Fr(M) = \sqcup_p Fr(T_p M)$ by topologizing as follows. Locally near p , identify its neighborhood with \mathbb{R}^n and $\sqcup_{p \in \mathbb{R}^n} Fr(T_p \mathbb{R}^n) = Fr(\mathbb{R}^n) \times \mathbb{R}^n$. The frame bundle is a *fiber bundle* over M whose fibers are diffeomorphic to $GL(V)$.]

22.3. Definition of the integral. Choose an oriented atlas $\{\phi_\alpha : U_\alpha \rightarrow \mathbb{R}^n\}$ for M . We then define:

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} (\phi_\alpha^{-1})^*(f_\alpha \omega),$$

where $\{f_\alpha\}$ is a partition of unity subordinate to $\{U_\alpha\}$. We will often be lazy and write $\int_{U_\alpha} f_\alpha \omega$ instead of $\int_{U_\alpha} (\phi_\alpha^{-1})^*(f_\alpha \omega)$ or $\int_{U_\alpha} (\phi_\alpha)_*(f_\alpha \omega)$

HW: Check that the definition of $\int_M \omega$ does not depend on the choice of oriented atlas $\{\phi_\alpha : U_\alpha \rightarrow \mathbb{R}^n\}$ as well as on the choice of $\{f_\alpha\}$ subordinate to $\{U_\alpha\}$.

23. STOKES' THEOREM

23.1. **Manifolds with boundary.** We enlarge the class of manifolds by allowing ones “with boundary”. These are locally modeled on the half-plane $\mathbb{H}^n = \{x_1 \leq 0\} \subset \mathbb{R}^n$.

Definition 23.1. A Hausdorff, second countable topological space is a manifold with boundary if there exists an atlas $\{(U_\alpha, \phi_\alpha)\}$, where $\phi_\alpha : U_\alpha \rightarrow \mathbb{H}^n$ is a homeomorphism onto an open subset of \mathbb{H}^n and the transition functions $\phi_\alpha \circ \phi_\beta^{-1}$ are smooth. The boundary of M , denoted ∂M , is the set of points of M which lie on the boundary of some half-plane \mathbb{H}^n under some map ϕ_α . Equivalently, it is the non-interior points of M . ∂M is an $(n - 1)$ -dimensional manifold.

Example: The n -dimensional unit ball $B^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \dots + x_n^2 \leq 1\}$. $\partial B^n = S^{n-1}$.

Proposition 23.2. If M is an orientable manifold with boundary, then ∂M is an orientable manifold.

Proof. Let $\{U_\alpha\}$ be an oriented atlas for M . Then we take an atlas $\{V_\alpha\}$ for ∂M as follows. Let $V_\alpha = U_\alpha \cap \{x_1 = 0\} \subset \mathbb{H}^n = \{x_1 \leq 0\}$. (Note that if any $p \in M$ is mapped to $\partial\mathbb{H}^n$ under a coordinate chart, then p cannot be mapped to the interior of \mathbb{H}^n under any other coordinate chart.) If $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n})$ is an oriented basis for M on U_α , then let $(\frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n})$ be an oriented basis for ∂M . This works because an outward normal vector $\frac{\partial}{\partial x_1}$ will go to another outward vector under a change of coordinates. \square

23.2. Stokes' Theorem.

Theorem 23.3 (Stokes' Theorem). Let ω be an $(n - 1)$ -form on a manifold with boundary M of dimension n . Then $\int_M d\omega = \int_{\partial M} \omega$.

Remark: ∂ happily switches places (jumps up or jumps down).

Zen: The significance of Stokes' Theorem is that a topological operation ∂ is related to an analytic operation d .

Proof. Take an open cover $\{U_\alpha\}$ where U_α is diffeomorphic to (i) $(0, 1) \times \dots \times (0, 1)$ (U_α does not intersect ∂M) or (ii) $(0, 1] \times (0, 1) \times \dots \times (0, 1)$ ($U_\alpha \cap \partial M = \{x_1 = 1\}$). Let $\{f_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. By linearity, it clearly suffices to compute $\int_{U_\alpha} d(f_\alpha \omega) = \int_{\partial M \cap U_\alpha} f_\alpha \omega$, i.e., assume ω is supported on one U_α .

We will treat the $n = 2$ case. The generalization is straightforward. Let ω be an $(n - 1)$ -form of type (ii). Then on $[0, 1] \times [0, 1]$ we can write $\omega = f_1 dx_1 + f_2 dx_2$.

$$\int_{\partial M} \omega = \int_{\partial M} f_1 dx_1 + f_2 dx_2 = \int_0^1 f_2(1, x_2) dx_2.$$

On the other hand,

$$\begin{aligned}
 \int_M d\omega &= \int_0^1 \int_0^1 \left(-\frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} \right) dx_1 dx_2 \\
 &= \int_0^1 \left(\int_0^1 \frac{\partial f_2}{\partial x_1} dx_1 \right) dx_2 + \int_0^1 \left(\int_0^1 -\frac{\partial f_1}{\partial x_2} dx_2 \right) dx_1 \\
 &= \int_0^1 (f_2(1, x_2) - f_2(0, x_2)) dx_2 + \int_0^1 (f_1(x_1, 0) - f_1(x_1, 1)) dx_1 \\
 &= \int_0^1 f_2(1, x_2) dx_2
 \end{aligned}$$

Try to see that $n > 2$ also work in the same way. □

Example: (Green's Theorem) Let $\Omega \subset \mathbb{R}^2$ be a compact domain with smooth boundary, i.e., Ω is a 2-dimensional manifold with boundary $\partial\Omega = \gamma$. Then

$$\int_{\gamma} f dx + g dy = \int_{\Omega} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx dy.$$

Example: Consider ω defined on $\mathbb{R}^2 - \{0\}$:

$$\omega(x, y) = \left(\frac{-y}{x^2 + y^2} \right) dx + \left(\frac{x}{x^2 + y^2} \right) dy.$$

Let $C = \{x^2 + y^2 = R^2\}$. Then $x = R \cos \theta$, $y = R \sin \theta$, and we compute

$$\int_C \omega = 2\pi.$$

It is easy to show that $d\omega = 0$.

Claim: ω is not exact! In fact, if $\omega = d\eta$, then

$$0 = \int_{\partial C} \eta = \int_C d\eta = 2\pi,$$

a contradiction.

24. APPLICATIONS OF STOKES' THEOREM

24.1. The Divergence Theorem.

Theorem 24.1. Let $\Omega \subset \mathbb{R}^3$ be a compact domain with smooth boundary. Let $F = (F_1, F_2, F_3)$ be a vector field on Ω . Then

$$\int_{\Omega} \operatorname{div} F \, dx dy dz = \int_{\partial\Omega} \langle n, F \rangle dA,$$

where $n = (n_1, n_2, n_3)$ is the unit outward normal to $\partial\Omega$,

$$dA = n_1 dy \wedge dz + n_2 dz \wedge dx + n_3 dx \wedge dy,$$

and $\langle \cdot \rangle$ is the standard inner product.

Let $\omega = F_1 dy dz + F_2 dz dx + F_3 dx dy$. Then $d\omega = (\operatorname{div} F) dx dy dz$. It remains to see why $\int_{\partial\Omega} \omega = \int_{\partial\Omega} \langle n, F \rangle dA$.

Evaluating forms. We explain what it means to take $\omega(v_1, \dots, v_k)$, where ω is a k -form and v_i are tangent vectors. Let V be a finite-dimensional vector space. There exists a map:

$$(\wedge^k V^*) \times (V \times \dots \times V) \rightarrow \mathbb{R}$$

$$(f_1 \wedge \dots \wedge f_k, (v_1, \dots, v_k)) \mapsto \sum (-1)^\sigma f_1(v_{i_1}) \dots f_k(v_{i_k}),$$

where the sum ranges over all permutations of $(1, \dots, k)$ and σ is the number of transpositions required for the transposition $(1, \dots, k) \mapsto (i_1, \dots, i_k)$. Note that this alternating sum is necessary for the well-definition of the map.

Example. Let $\omega = F_1 dy dz + F_2 dz dx + F_3 dx dy$. Then $\omega\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right) = -F_2$.

Interior product. We can define the interior product as follows: $i_v : \wedge^k V^* \rightarrow \wedge^{k-1} V^*$, $i_v \omega = \omega(v, \cdot, \dots, \cdot)$. (Insert v into the first slot to get a $(k-1)$ -form.)

Example. On \mathbb{R}^3 , let $\eta = dx dy dz$. Also let n be the unit normal vector to $\partial\Omega$. Then, along $\partial\Omega$ we can define $i_n \eta = n_1 dy dz + n_2 dz dx + n_3 dx dy$.

Why is this dA ? At any point of $p \in \partial\Omega$, take tangent vectors v_1, v_2 of $\partial\Omega$ so that n, v_1, v_2 is an oriented orthonormal basis. Then the area form dA should evaluate to 1 on v_1, v_2 . Since $\eta(n, v_1, v_2) = 1$ (since η is just the determinant), we see that $dA = i_n \eta$.

Explanation of $\langle n, F \rangle dA = F_1 dy dz + F_2 dz dx + F_3 dx dy$. Also note that $i_F \eta = F_1 dy dz + \dots$. But now, $i_F \eta(v_1, v_2) = \eta(F, v_1, v_2) = \eta(\langle n, F \rangle n, v_1, v_2) = \langle n, F \rangle dA$ (by Gram-Schmidt).

24.2. Evaluating cohomology classes. Let M be a compact, oriented manifold (without boundary) of dimension n .

Proposition 24.2. There exists a well-defined, nonzero map $\int : H^n(M) \rightarrow \mathbb{R}$.

Proof. Given $\omega \in \Omega^n(M)$, we map $\omega \mapsto \int_M \omega$. Note that every n -form ω is closed. To show the map is defined on the level of cohomology, let ω be an exact form, i.e., $\omega = d\eta$. Then

$$\int_M \omega = \int_M d\eta = \int_{\partial M} \eta = 0.$$

Next we prove the nontriviality of \int : if ω is an *orientation form* (ω is nowhere zero), then $\int_M \omega > 0$ or < 0 , since on each coordinate chart ω is some positive function times $dx_1 \dots dx_n$. \square

The proposition shows that $\dim H^n(M) \geq 1$. In fact, we have the following:

Theorem 24.3. $H^n(M) \simeq \mathbb{R}$.

We omit the proof.

Example. $M = S^n$. Then $H^i(S^n) = \mathbb{R}$ for $i = 0$ or n and $= 0$ for all other i .

25. DAY 24

25.1. Evaluating cohomology classes. Let $\phi : M^m \rightarrow N^n$ be a smooth map between compact oriented manifolds of dimensions m and n , respectively, and $\phi^* : H^k(N) \rightarrow H^k(M)$ the induced map on cohomology. Let $\omega \in \Omega^m(N)$ be a closed m -form. Suppose $\int_M \phi^* \omega \neq 0$. Then $\phi^* \omega$ represents a nonzero element in $H^m(M)$. This implies that $[\omega]$ is a nonzero cohomology class in $H^m(N)$.

Example. On $\mathbb{R}^2 - \{0\}$, consider the closed 1-form

$$\omega(x, y) = \left(\frac{-y}{x^2 + y^2} \right) dx + \left(\frac{x}{x^2 + y^2} \right) dy.$$

We computed $\int_{S^1} \phi^* \omega = 2\pi$, where $\phi : S^1 \rightarrow \mathbb{R}^2 - \{0\}$ mapped $\theta \mapsto (R \cos \theta, R \sin \theta)$. Since $[\phi^* \omega]$ is a nonzero cohomology class in $H^1(S^1)$, so is $[\omega] \in H^1(\mathbb{R}^2 - \{0\})$.

Two maps $\phi_0, \phi_1 : M \rightarrow N$ are (smoothly) homotopic if there exists a map $\Phi : M \times [0, 1] \rightarrow N$ where $\Phi(x, t) = \phi_t(x)$.

Proposition 25.1. If $\phi_0, \phi_1 : M \rightarrow N$ are homotopic and $\omega \in \Omega^k(N)$, $k = \dim M$, is closed, then $\int_M \phi_0^* \omega = \int_M \phi_1^* \omega$.

Proof.

$$\int_M \phi_1^* \omega - \int_M \phi_0^* \omega = \int_{\partial(M \times [0,1])} \Phi^* \omega = \int_M d(\Phi^* \omega) = \int_M \Phi^*(d\omega) = 0,$$

since ω is closed. □

Example, cont'd. On $N = \mathbb{R}^2 - \{0\}$. Since ω is a closed 1-form on N , $\int_C \omega = \int_{C'} \omega$ if C and C' are homotopic. That's why the integral did not depend on the radius R of the circle.

25.2. Definition of degree. This material can be found in Guillemin & Pollack.

Let $\phi : M \rightarrow N$ be a smooth map between oriented compact n -manifolds M and N . Let $y \in N$ be a regular value of ϕ . (Recall $y \in N$ is a *regular value* if, for all $x \in \phi^{-1}(y)$, $d\phi(x)$ is surjective. $y \in N$ which is not a regular value is a *critical value*.)

Claim: $\phi^{-1}(y)$ consists of a finite number of preimages x_1, \dots, x_k .

Proof. Suppose there is an infinite number of preimages. By the compactness of M , there must be an accumulation point $x = \lim_{i \rightarrow \infty} x_i$, which itself must also be in $\phi^{-1}(y)$. However, for every $x \in \phi^{-1}(y)$ there exists an open set U_x which maps diffeomorphically onto an open set around y . Therefore, x could not have been the limit of $x_i \in \phi^{-1}(y)$. □

The claim implies that for a small enough open set V_y containing y , $\phi^{-1}(V_y)$ is a finite disjoint union of open sets U_{x_1}, \dots, U_{x_k} , each of which is diffeomorphic to V_y .

Definition 25.2. *The degree of a mapping $\phi : M \rightarrow N$ is the sum of orientation numbers ± 1 for each x_i in the preimage of a regular value y . Here the sign is $+1$ if the map from a neighborhood of x_i to a neighborhood of y is orientation-preserving and -1 otherwise.*

Regular values of ϕ do exist:

Theorem 25.3 (Sard). *Let $\phi : M \rightarrow N$ be a smooth map. Then the set of critical values of ϕ has measure zero.*

A set $S \subset N$ has *measure zero* if $\{U_i\}_{i=1}^{\infty}$ is a countable atlas and, for each $U_i \subset \mathbb{R}^n$ and $\varepsilon > 0$, $U_i \cap S$ can be covered by a countable union of rectangles $[a_1, b_1] \times \cdots \times [a_n, b_n]$ with total volume ε . This actually implies that S itself can be covered by a countable union of rectangles with total volume ε : For U_i , take rectangles so that the total volume $= \varepsilon \left(\frac{1}{2}\right)^i$. Adding up over all the U_i , we get $\varepsilon \left(\frac{1}{2} + \frac{1}{4} + \dots\right) = \varepsilon$.

Consequently, the set of regular values of ϕ is dense in N .

The proof of Sard's Theorem will be given next time. We conclude with the following theorem, which will be explained in a couple of lectures.

Theorem 25.4 (Degree Theorem). *The degree of a mapping $\phi : M \rightarrow N$ is well-defined.*

26. PROOF OF SARD'S THEOREM

The proof closely follows that of Milnor, *Topology from the Differentiable Viewpoint*.

Recall the statement of Sard's Theorem.

Theorem 26.1 (Sard). *Let $f : M \rightarrow N$ be a smooth map. Then the set of critical values of f has measure zero.*

By our discussion from last time, suffices to prove Sard's Theorem in the following local situation.

Theorem 26.2. *Let $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a smooth map. If we set $C = \{x \mid \text{rank } df(x) < n\}$, then $f(C)$ has measure zero in \mathbb{R}^n .*

Remark 1. The "measure" in the term *measure zero* refers to the Lebesgue measure μ .

Remark 2. Open subsets of \mathbb{R}^n have nonzero Lebesgue measure.

Proof. We will prove the theorem for $n = 1$, i.e., $f : \mathbb{R}^m \rightarrow \mathbb{R}$. The general case is similar.

Define the following subsets of \mathbb{R}^m :

$$C_1 = \left\{ x \in \mathbb{R}^m \mid \frac{\partial f}{\partial x_i} = 0, \forall i \right\},$$

$$C_k = \{x \in \mathbb{R}^m \mid \text{all partial derivatives of } f \text{ up to and including order } k \text{ vanish at } x\}.$$

Then clearly $C = C_1 \supset C_2 \supset C_3 \dots$

Strategy.

- (1) Show $f(C_1 - C_2)$ has measure zero.
- (2) Show $f(C_k - C_{k+1})$ has measure zero.
- (3) For k large enough ($k \geq n$), $f(C_k)$ has measure zero.

Step 1. Let $x \in C_1 - C_2$. We want to show that there exists a neighborhood V of x for which $(C_1 - C_2) \cap V$ has measure zero. (This suffices because if we can cover $C_1 - C_2$ with countably many such V 's, the total measure of $C_1 - C_2$ is zero, as seen from the argument used last time, right after the statement of Theorem 25.3.) Here, $\frac{\partial f}{\partial x_1} = \dots = \frac{\partial f}{\partial x_m} = 0$, but some $\frac{\partial^2 f}{\partial x_i \partial x_j} \neq 0$.

Without loss of generality assume $\frac{\partial^2 f}{\partial x_1 \partial x_2} \neq 0$. Then consider the map $h : V \ni x \rightarrow \mathbb{R}^m$, $(x_1, \dots, x_m) \mapsto (\frac{\partial f}{\partial x_1}, x_2, \dots, x_m)$. Near x , $h : V \rightarrow V'$ is a local diffeomorphism, as can be seen easily by computing the Jacobian. Clearly, the critical values of $f : V \rightarrow \mathbb{R}$ are the same as the critical values of $f \circ h^{-1} : V' \rightarrow \mathbb{R}$, but if $(\tilde{x}_1, \dots, \tilde{x}_m)$ are coordinates of V' , then the critical values of $f \circ h^{-1}$ are the same as the critical values of $f \circ h^{-1} : \{\tilde{x}_1 = 0\} \rightarrow \mathbb{R}$. We can then induct on the dimension m .

Note. Under a diffeomorphism, the Lebesgue measure μ changes by a positive smooth function f .

Step 2. Similar to Step 1.

Step 3. Let $0 \in C_k$, $k \geq n$. Suppose $f : [-\delta, \delta] \times \dots \times [-\delta, \delta] \rightarrow \mathbb{R}$. Then Taylor's theorem (with remainder) gives us:

$$f(x+h) = f(x) + R(x; h),$$

where $|R(x; h)| \leq C|h|^{k+1}$, for all $x \in C_k \cap [-\delta, \delta]^m$ and $x+h \in [-\delta, \delta]^m$.

We subdivide $[0, 1]^m$ into cubes of length δ . Then there are roughly $\frac{1}{\delta^m}$ cubes. Consider one such cube Q which nontrivially intersects C_k . Then its volume is δ^m , whereas its image has length on the order of magnitude of δ^{k+1} from Taylor's theorem. Adding up the total volume of the image, we have $\frac{1}{\delta^m} \delta^{k+1}$, which can be made arbitrarily small by choosing δ small. \square

27. DEGREE

Recall the definition of degree: Let $\phi : M \rightarrow N$ be a smooth map between compact, oriented manifolds (without boundary) of dimension n . By Sard's Theorem, there exist (a full measure's worth of) regular values of ϕ . Let $y \in N$ be a regular value, and x_1, \dots, x_k be the preimages of y . Then $\text{deg}(\phi)$ is $\sum_{i=1}^k \pm 1$, where the contribution is $+1$ when ϕ is orientation-preserving near x_i and -1 is otherwise.

We will explain why the degree is well-defined.

27.1. Cohomological interpretation.

Theorem 27.1. *If M is an oriented, compact n -manifold (without boundary), then $\int : H^n(M) \rightarrow \mathbb{R}$ is an isomorphism.*

The proof will be given in the following section, but for the time being let us use this to reinterpret the degree. $\phi : M \rightarrow N$ induces the map $\phi^* : H^n(N) \rightarrow H^n(M)$. Then we have the commutative diagram:

$$\begin{array}{ccc} H^n(N) & \xrightarrow{\phi^*} & H^n(M) \\ f \downarrow & & f \downarrow \\ \mathbb{R} & \xrightarrow{c} & \mathbb{R} \end{array}$$

where the map $\mathbb{R} \rightarrow \mathbb{R}$ is multiplication by some real number c .

Proposition 27.2. *$\text{deg } \phi$ satisfies*

$$(6) \quad \int_M \phi^* \omega = \text{deg } \phi \int_N \omega.$$

Therefore $\text{deg } \phi$ is the constant of multiplication c .

Proof. Once we can prove Equation 6 for a suitable ω of our choice, the proposition follows. Take ω to be supported on V_y with positive integral. Then $\int_M \phi^* \omega$ will be the sum of $\int_{U_{x_i}} \phi^* \omega$. Noting that ϕ is a diffeomorphism from U_{x_i} to V_y , we have $\int_{U_{x_i}} \phi^* \omega = \pm \int_{V_y} \omega$, depending on whether the orientations agree or not. This proves Equation 6. □

27.2. Proof of Theorem 27.1. We have already shown that $\int : H^n(M) \rightarrow \mathbb{R}$ is well-defined. It suffices to show that $\ker \int$ consists of exact n -forms. Let ω be an n -form with zero integral. Let $\{U_i\}$ be a cover of M which is finite and has the property that every U_i is diffeomorphic to \mathbb{R}^n . Take a partition of unity $\{f_\alpha\}$ subordinate to a good cover. Then we can split ω into the sum $\sum_i \omega_i$, where ω_i is supported inside U_i . Note that $\int_{U_i} \omega_i$ may not be zero.

Lemma 27.3. *If ω is an n -form with compact support and zero integral inside \mathbb{R}^n , then $\omega = d\eta$, where η has compact support.*

Proof. We will prove this for $n = 2$. Then $\omega = f(x, y)dxdy$. Define $g(x) = \int_{-\infty}^{\infty} f(x, y)dy$. By Fubini's theorem and the hypothesis that $\int \omega = 0$, we have $\int_{-\infty}^{\infty} g(x)dx = 0$. Define $G(x, y) = \varepsilon(y)g(x)$, where $\varepsilon(x)$ is a bump function with total area 1. Then write:

$$\eta(x, y) = - \left(\int_{-\infty}^y [f(x, t) - G(x, t)]dt \right) dx + \left(\int_{-\infty}^x G(t, y)dt \right) dy.$$

Clearly, $d\eta = [f(x, y) - G(x, y)]dxdy + G(x, y)dxdy$ and η has compact support. □

What this means is that we can replace ω_i by a cohomologically equivalent n -form which is supported on a small neighborhood of a point $x_i \in M$, i.e., we may assume that ω_i is a bump n -form. The total volume of the ω_i is still zero. Now, engulf all the x_i in an open set $U \subset M$ which is diffeomorphic to \mathbb{R}^n so that ω is compactly supported in U and has total area zero. We use the lemma again to complete the proof of Theorem 27.1.

28. LIE DERIVATIVES

28.1. Lie derivatives. First we define the *interior product* on the linear algebra level. $i_v : \bigwedge^k V^* \rightarrow \bigwedge^{k-1} V^*$, $v \in V$, is given by:

$$f_1 \wedge \cdots \wedge f_k \mapsto \sum_l (-1)^{l+1} f_1 \wedge \cdots \wedge f_l(v) \cdots \wedge f_k.$$

Check this is well-defined!

Let X be a vector field on M . Then the *interior product* $i_X : \Omega^k(M) \rightarrow \Omega^{k-1}(M)$ satisfies the following properties:

- (1) For 1-forms ω , $i_X(\omega) = \omega(X)$.
- (2) In general, we obtain the relation:

$$i_X(\alpha \wedge \beta) = (i_X\alpha) \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge i_X\beta.$$

Note: we define $i_X : \Omega^0(M) \rightarrow \Omega^{-1}(M)$ as the zero map.

Now define $\mathcal{L}_X = d \circ i_X + i_X \circ d : \Omega^k(M) \rightarrow \Omega^k(M)$. \mathcal{L}_X is called the *Lie derivative* with respect to X .

Proposition 28.1.

- (1) If $f \in \Omega^0(M)$, then $\mathcal{L}_X f = d(i_X f) + i_X(df) = df(X) = X(f)$. Hence, $\mathcal{L} : \Omega^0(M) \rightarrow \Omega^0(M)$ satisfies the Leibniz rule.
- (2) $\mathcal{L}_X(d\omega) = d(\mathcal{L}_X\omega)$.
- (3) $\mathcal{L}_X : \Omega^k(M) \rightarrow \Omega^k(M)$ satisfies $\mathcal{L}_X(\alpha \wedge \beta) = \mathcal{L}_X(\alpha) \wedge \beta + \alpha \wedge \mathcal{L}_X(\beta)$, i.e., the Leibniz rule.

The proof is a simple computation, and is left for HW.

Hence, $\mathcal{L} : \Omega^k(M) \rightarrow \Omega^k(M)$ naturally extends the derivation $X : \Omega^0(M) \rightarrow \Omega^0(M)$. (We will usually call anything that satisfies the Leibniz rule a “derivation”.)

The following is also a source of derivations $\Omega^k \rightarrow \Omega^k$: Let $\phi_t : M \rightarrow M$ be a *1-parameter family of diffeomorphisms*, i.e., there exists $\Phi : M \times [0, 1] \rightarrow M$ smooth such that $\phi_t(\cdot) \stackrel{\text{def}}{=} \Phi(\cdot, t)$, $t \in [0, 1]$, is a diffeomorphism. Assume in addition that $\phi_0 = id$. Then

$$\frac{d}{dt} \phi_t^* \omega|_{t=0}$$

is a derivation (verification is easy). If $f \in \Omega^0(M)$, then

$$\frac{d}{dt} \phi_t^* f|_{t=0} = \frac{d}{dt} f(\phi_t)|_{t=0} = df(X_0) = X_0(f),$$

where X_0 is the vector field which corresponds to ϕ_t (think in terms of the first definition of the tangent space: at every $x \in M$, we have an arc $\phi_t(x)$, $t \in [-\varepsilon, \varepsilon]$).

We have the following proposition:

Proposition 28.2 (Cartan formula). *Every $\frac{d}{dt}\phi_t^*|_{t=0} : \Omega^k \rightarrow \Omega^k$ is given by $d \circ i_X + i_X \circ d$.*

Proof. It suffices to check the following:

- $\frac{d}{dt}\phi_t^*$ and \mathcal{L}_X both satisfy the Leibniz rule. (Already verified!)
- $\frac{d}{dt}\phi_t^*$ and \mathcal{L}_X agree on $\Omega^0(M)$. (Yes, they are both vector fields.)
- d commutes with $\frac{d}{dt}\phi_t^*$ and with \mathcal{L}_X .

The above three properties allow us to do an induction on degree. □

29. HOMOTOPY PROPERTIES

29.1. Homotopy properties of de Rham cohomology.

Proposition 29.1. *Let $\phi_t : M \rightarrow M$, $t \in [0, 1]$, be a 1-parameter family of diffeomorphisms. Then ϕ_t^* induce the same map $H^k(M) \rightarrow H^k(M)$ for all $t \in [0, 1]$.*

Note. If $\phi_0 = id$, and we write X_0 as the vector field on M given by $\phi_t(x) : [-\varepsilon, \varepsilon] \rightarrow M$ at the point x , then $\frac{d}{dt}\phi_t^*\omega|_{t=0} = (d \circ i_{X_0} + i_{X_0} \circ d)\omega$. We can generalize this as follows: Let X_{t_0} be the vector field on M where the arc $\phi_t(x) : [t_0 - \varepsilon, t_0 + \varepsilon] \rightarrow M$ is assigned at the point $\phi_{t_0}(x)$ (note NOT at x). Then

$$\frac{d}{dt}\phi_t^*\omega|_{t=t_0} = \phi_{t_0}^*(d \circ i_{X_{t_0}} + i_{X_{t_0}} \circ d)\omega.$$

Proof. Consider a closed k -form ω on M . Then $\frac{d}{dt}\phi_t^*\omega|_{t=t_0} = \phi_{t_0}^*(d \circ i_{X_{t_0}} + i_{X_{t_0}} \circ d)\omega = d(\phi_{t_0}^*i_{X_{t_0}}\omega)$. Therefore it is exact. Now, $\phi_t^*\omega - \omega = \int_0^t \frac{d}{ds}\phi_s^*\omega|_{t=s}dt$, and the difference is exact as well. (This is evident by thinking of the integral as a limit of Riemann sums.) \square

Next, we say two maps $\phi_0, \phi_1 : M \rightarrow N$ are (smoothly) homotopic if there exists a smooth map $\Phi : M \times [0, 1] \rightarrow N$ with $\phi_t(\cdot) = \Phi(\cdot, t)$, $t = 0, 1$. ϕ_t is said to be the homotopy from ϕ_0 to ϕ_1 .

Proposition 29.2 (Homotopy invariance). *Suppose $\phi_t : M \rightarrow N$ is a homotopy, $t \in [0, 1]$. Then $\phi_t^* : H^k(N) \rightarrow H^k(M)$ is independent of t .*

Proof. Consider $\Phi : M \times \mathbb{R} \rightarrow N$. (It is easy to extend $\Phi : M \times [0, 1] \rightarrow N$ to $\Phi : M \times \mathbb{R} \rightarrow N$.) Then for $\omega \in \Omega^k(M)$ closed, consider $\Omega = \Phi^*\omega$. We have inclusions $i_t : M \rightarrow M \times \mathbb{R}$, $x \mapsto (x, t)$, and clearly $\phi_t^*\omega = i_t^*\Omega$. Now take a diffeomorphism $\Psi_t : M \times \mathbb{R} \rightarrow M \times \mathbb{R}$, $(x, s) \mapsto (x, s + t)$. Since $i_t = \Psi_t \circ i_0$, $i_t^* = i_0^* \circ \Psi_t^*$. By the previous proposition, Ψ_t^* is independent of t . Hence so are i_t and ultimately ϕ_t . \square

29.2. Homotopy equivalence. We say $\phi : M \rightarrow N$ is a homotopy equivalence if there exists $\psi : N \rightarrow M$ such that $\phi \circ \psi : N \rightarrow N$ and $\psi \circ \phi : M \rightarrow M$ are homotopic to $id : N \rightarrow N$ and $id : M \rightarrow M$. Using Proposition 29.2, it is easy to show:

Proposition 29.3 (Homotopy equivalence). *A homotopy equivalence $\phi : M \rightarrow N$ induces an isomorphism $\phi^* : H^k(N) \rightarrow H^k(M)$.*

Proof. This is because $\phi^* \circ \psi^* = id$ (by homotopy invariance) and $\psi^* \circ \phi^* = id$. This proves that ϕ^* and ψ^* are left and right inverses (as linear maps) and are isomorphisms. \square

Corollary 29.4 (Poincaré lemma). *$H_{dR}^k(\mathbb{R}^n) = 0$ if $k > 0$.*

Proof. We will show that \mathbb{R}^n is homotopy equivalent to $\mathbb{R}^0 = \{pt\}$. Consider maps $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^0$, $(x_1, \dots, x_n) \mapsto 0$, and $\psi : \mathbb{R}^0 \rightarrow \mathbb{R}^n$, $0 \mapsto (0, \dots, 0)$. Clearly, $\phi \circ \psi : \mathbb{R}^0 \rightarrow \mathbb{R}^0$, $0 \mapsto 0$, is the identity map. Next, $\psi \circ \phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $(x_1, \dots, x_n) \mapsto 0$ is homotopic to the identity map. In fact, consider $F : \mathbb{R}^n \times [0, 1] \rightarrow \mathbb{R}^n$, $((x_1, \dots, x_n), t) \mapsto (tx_1, \dots, tx_n)$. \square

Example. Consider a band $S^1 \times (-1, 1)$. It is homotopy equivalent to S^1 . We have maps $\phi : S^1 \times (-1, 1) \rightarrow S^1$, $(\theta, t) \mapsto \theta$ and $\psi : S^1 \rightarrow S^1 \times (-1, 1)$, $\theta \mapsto (\theta, 0)$. $\phi \circ \psi : S^1 \rightarrow S^1$ is *id*. $\psi \circ \phi : S^1 \times (-1, 1) \rightarrow S^1 \times (-1, 1)$ is $(\theta, t) \mapsto (\theta, 0)$ is homotopic to *id*. In fact, take $F : S^1 \times (-1, 1) \times [0, 1] \rightarrow S^1 \times (-1, 1)$, $(\theta, t, s) \mapsto (\theta, ts)$. Therefore, we have:

$$H^k(S^1 \times (-1, 1)) \simeq H^k(S^1)$$

Example. Similarly, $H^k(M \times \mathbb{R}^n) \simeq H^k(M)$. More generally, if E is a vector bundle over M , then $H^k(E) \simeq H^k(M)$.

29.3. Extended example: surface of genus g . Consider a surface Σ of genus g . If you remove a disk from Σ , you are left with a bouquet of $2g$ bands. You can now use Mayer-Vietoris with U a disk and V a bouquet of $2g$ bands.

29.4. Euler characteristic. Let M be an n -dimensional manifold. Then we define the Euler characteristic of M to be:

$$\chi(M) = \sum_{i=0}^n (-1)^i \dim H_{dR}^i(M).$$

Examples.

- (1) $\chi(\mathbb{R}^n) = 1$.
- (2) $\chi(S^2) = 1 + 0 + 1 = 2$.
- (3) $\chi(T^2) = 1 - 2 + 1 = 0$.
- (4) $\chi(\text{genus } g \text{ surface}) = 2 - 2g$.

Note. For compact surfaces, the Euler characteristic is given by the classical formula $V - E + F$, where V is the number of vertices, E is the number of edges, and F is the number of faces of a polyhedron representing the surface.

HW: Prove that if $0 \rightarrow C_1 \rightarrow C_2 \rightarrow \cdots \rightarrow C_k \rightarrow 0$ is an exact sequence, then

$$\sum_{i=1}^k (-1)^i \dim C_i = 0.$$

Proposition 29.5. If $M = U \cup V$, then $\chi(M) = \chi(U) + \chi(V) - \chi(U \cap V)$.

Proof. Use the Mayer-Vietoris sequence and add up the dimensions, using the above HW. □

30. VECTOR FIELDS

Recall a *vector field* on $U \subset M$ is a section of TM defined over U .

30.1. Lie brackets. Given two vector fields X and Y on M viewed as derivations, we can define its Lie bracket $[X, Y] = XY - YX$, i.e., for $f \in C^\infty(M)$, $[X, Y](f) = X(Yf) - Y(Xf)$.

Proposition 30.1. $[X, Y]$ is also a derivation, hence is a vector field.

Proof. This is a local computation. Take $X = \sum_i a_i \frac{\partial}{\partial x_i}$ and $Y = \sum_j b_j \frac{\partial}{\partial x_j}$. Then:

$$\begin{aligned} [X, Y](f) &= \sum_i a_i \frac{\partial}{\partial x_i} \left(\sum_j b_j \frac{\partial f}{\partial x_j} \right) - \sum_j b_j \frac{\partial}{\partial x_j} \left(\sum_i a_i \frac{\partial f}{\partial x_i} \right) \\ &= \sum_{ij} a_i \frac{\partial b_j}{\partial x_i} \frac{\partial f}{\partial x_j} - \sum_{ij} b_j \frac{\partial a_i}{\partial x_j} \frac{\partial f}{\partial x_i} \end{aligned}$$

In other words,

$$\left[\sum_i a_i \frac{\partial}{\partial x_i}, \sum_j b_j \frac{\partial}{\partial x_j} \right] = \sum_{ij} \left(a_j \frac{\partial b_i}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j} \right) \frac{\partial}{\partial x_i}$$

□

Properties of Lie brackets.

- (1) (Anticommutativity) $[X, Y] = -[Y, X]$.
- (2) (Jacobi identity) $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$.
- (3) $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$.

These properties are easy to verify, and are left as exercises.

30.2. Fundamental Theorem of Ordinary Differential Equations.

Theorem 30.2. Given a vector field X on a manifold M and $p \in M$, there exist an open set $U \ni p$, $\varepsilon > 0$, and a smooth map $\Phi : U \times (-\varepsilon, \varepsilon) \rightarrow M$ such that if we set $\gamma_x(t) = \Phi(x, t)$, $x \in U$, then $\gamma_x(0) = x$ and $\gamma_x(t)$ is an arc through x whose tangent vector at t is $X(\gamma_x(t))$.

Locally, take coordinates x_1, \dots, x_n . If $X = \sum_{i=1}^n a_i(x) \frac{\partial}{\partial x_i}$ and we write $x(t) = \gamma_x(t)$, then

$$\frac{dx}{dt}(t) = (a_1(x(t)), \dots, a_n(x(t))).$$

We omit the proof of this theorem.

Definition 30.3. A curve $\gamma : (a, b) \rightarrow M$ is an integral curve of X if $\frac{d\gamma}{dt} = X(\gamma(t))$.

Remarks.

- (1) By definition, $\gamma_x(t)$ are integral curves of X . If $\gamma : (-\delta, \delta) \rightarrow M$ is another integral curve of X with $\gamma(0) = x$, then $\gamma(t) = \gamma_x(t)$ on $(-\varepsilon, \varepsilon) \cap (-\delta, \delta)$. Therefore, the flow $\Phi : U \times (-\varepsilon, \varepsilon) \rightarrow M$ is unique on the domain of definition.
- (2) If M is compact (without boundary) and X is a vector field on M , then there exists a global flow $\Phi : M \times \mathbb{R} \rightarrow M$ with $\phi_0 = id$. Since M is compact, the finite covering property ensures that we may choose ε to work for all the open sets U . If we know that there is a flow for a short time ε , we can repeat the flow and obtain a flow for an arbitrarily long time.
- (3) However, if M is not compact, then there are vector fields X which do not admit global flows. (See example below.)
- (4) $\phi_s \circ \phi_t = \phi_{s+t}$, and $\phi_t^{-1} = \phi_{-t}$. In particular, on M compact, $\phi_t, t \in \mathbb{R}$, forms a 1-parameter group of diffeomorphisms.

Example. On $\mathbb{R} - \{0\}$ consider $X = \frac{\partial}{\partial x}$. X , when integrated on \mathbb{R} , clearly gives $\Phi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, $(x, t) \mapsto x + t$. However, when $\{0\}$ is removed, no matter how small an ε you take, there is no $\Phi : (\mathbb{R} - \{0\}) \times (-\varepsilon, \varepsilon) \rightarrow \mathbb{R} - \{0\}$.

Corollary 30.4. *Suppose $X(p) \neq 0$. Then there exists a coordinate system near p such that $X = \frac{\partial}{\partial x}$.*

Proof. If M is n -dimensional, choose an $(n - 1)$ -manifold Σ (defined in a neighborhood of p) which is transverse to X . (Here Σ is *transverse* to X if $T_x \Sigma$ and $X(x)$ span $T_x M$ at all $x \in \Sigma$.) (Why does such a Σ exist?) Now take $\psi : \Sigma \times (-\varepsilon, \varepsilon) \rightarrow M$ given by Φ restricted to Σ . Since Σ is transverse to X , ψ is a diffeomorphism near p by the inverse function theorem. In the coordinate system $\Sigma \times (-\varepsilon, \varepsilon)$, X is clearly $\frac{\partial}{\partial t}$. \square

31. VECTOR FIELDS AND LIE DERIVATIVES

31.1. Pullback.

Proposition 31.1. *Let $f : M \rightarrow N$ be a smooth map and $\omega \in \Omega^k(N)$. Then we have:*

$$f^*\omega(x)(X_1, \dots, X_k) = \omega(f(x))(f_*X_1, \dots, f_*X_k).$$

Proof. It suffices to show this for 1-forms dg . Then:

$$(f^*dg)(X) = d(g \circ f)(X) = X(g \circ f) = f_*X(g),$$

by definition of the pushforward of X . If we write this in coordinates, then

$$dg = \sum_i \frac{\partial g}{\partial y_i} dy_i$$

and

$$f^*dg = \sum_{ij} \frac{\partial g}{\partial y_i} \frac{\partial y_i}{\partial x_j} dx_j,$$

so

$$f^*dg\left(\frac{\partial}{\partial x_j}\right) = \sum_i \frac{\partial g}{\partial y_i} \frac{\partial y_i}{\partial x_j},$$

whereas

$$dg\left(f_*\frac{\partial}{\partial x_j}\right) = \sum_i \frac{\partial g}{\partial y_i} dy_i\left(\sum_l \frac{\partial y_l}{\partial x_j} \frac{\partial}{\partial y_l}\right) = \sum_i \frac{\partial g}{\partial y_i} \frac{\partial y_i}{\partial x_j}.$$

□

31.2. Lie derivatives. Let X be a vector field on M . Then there exist a local or global flow $\Phi : M \times (-\varepsilon, \varepsilon) \rightarrow M$, $\phi_t(x) = \Phi(x, t)$, such that $\phi_0(x) = x$. We defined the *Lie derivative* \mathcal{L}_X on forms ω as:

$$\mathcal{L}_X\omega = \frac{d}{dt}\phi_t^*\omega|_{t=0}.$$

Lie derivatives can be defined on vector fields Y as well:

$$\mathcal{L}_XY = \frac{d}{dt}(\phi_{-t})_*Y|_{t=0}.$$

Here, vector fields cannot usually be pulled back, but for a diffeomorphism ϕ , there is a suitable substitute, namely $(\phi^{-1})_*$.

Ultimately, it is easy to see that \mathcal{L}_X can be defined on any tensor of the type $\bigwedge^k T^*M \otimes \bigwedge^l TM$.

Properties of \mathcal{L}_X .

- (1) $\mathcal{L}_Xf = Xf$.
- (2) $\mathcal{L}_X\omega = (d \circ i_X + i_X \circ d)\omega$
- (3) $\mathcal{L}_X(\omega(X_1, \dots, X_k)) = (\mathcal{L}_X\omega)(X_1, \dots, X_k) + \sum_i \omega(X_1, \dots, \mathcal{L}_X X_i, \dots, X_k)$.
- (4) $\mathcal{L}_XY = [X, Y]$.

(1), (2) are already proven. (3) is left for homework. Use Proposition 31.1 above. We will do (4), assuming (1), (2), (3). We compute:

$$\begin{aligned}
X(Y(f)) &= \mathcal{L}_X(Yf) \\
&= \mathcal{L}_X(df(Y)) \\
&= (\mathcal{L}_X df)(Y) + df(\mathcal{L}_X Y) \\
&= (d \circ \mathcal{L}_X f)Y + df(\mathcal{L}_X Y) \\
&= d(X(f))Y + (\mathcal{L}_X Y)(f) \\
&= Y(X(f)) + (\mathcal{L}_X Y)(f).
\end{aligned}$$

Therefore, $(\mathcal{L}_X Y)f = X(Y(f)) - Y(X(f)) = [X, Y](f)$.

31.3. Interpretation of $\mathcal{L}_X Y = [X, Y]$. As before, X, Y may not have global flows, but for simplicity let us assume they do. Let $\phi_s : M \rightarrow M, s \in \mathbb{R}$, be the 1-parameter group of diffeomorphisms generated by X and $\psi_t : M \rightarrow M, t \in \mathbb{R}$, be the 1-parameter group of diffeomorphisms generated by Y . Noting that $Y(x) = \lim_{t \rightarrow 0} \frac{\psi_t(x) - x}{t}$, we have

$$\begin{aligned}
\mathcal{L}_X Y(x) &= \lim_{s \rightarrow 0} \frac{((\phi_{-s})_* Y)(x) - Y(x)}{s} \\
&= \lim_{s, t \rightarrow 0} \frac{(\phi_{-s} \circ \psi_t \circ \phi_s(x) - x) - (\psi_t(x) - x)}{st} \\
&= \lim_{s, t \rightarrow 0} \frac{\phi_{-s} \circ \psi_t \circ \phi_s(x) - \psi_t(x)}{st} \\
&= \lim_{s, t \rightarrow 0} \psi_t \left(\frac{\psi_t^{-1} \circ \phi_s^{-1} \circ \psi_t \circ \phi_s(x) - x}{st} \right) \\
&= \lim_{s, t \rightarrow 0} \frac{\psi_t^{-1} \circ \phi_s^{-1} \circ \psi_t \circ \phi_s(x) - x}{st}
\end{aligned}$$

Hence, the Lie bracket $[X, Y]$ measures the infinitesimal discrepancy when you flow s units along X , t units along Y , $-s$ units along X and finally $-t$ units along Y .

32. DAY 31

32.1. Relationship between d and $[\cdot, \cdot]$.

Proposition 32.1. Consider $\theta \in \Omega^1(M)$ and $X, Y \in \mathfrak{X}(M)$. Then $d\theta(X, Y) = X\theta(Y) - Y\theta(X) - \theta([X, Y])$.

Proof.

$$\begin{aligned} d\theta(X, Y) &= i_Y i_X d\theta = i_Y(\mathcal{L}_X - d \circ i_X)\theta \\ &= i_Y(\mathcal{L}_X\theta - d(\theta(X))) \\ &= (\mathcal{L}_X\theta)Y - Y\theta(X) \\ &= X\theta(Y) - Y\theta(X) - \theta([X, Y]), \end{aligned}$$

by using

$$\mathcal{L}_X(\theta(Y)) = (\mathcal{L}_X\theta)Y + \theta([X, Y]).$$

□

More generally we have, for $\Omega^k(M)$, $X_1, \dots, X_{k+1} \in \mathfrak{X}(M)$:

$$\begin{aligned} d\omega(X_1, \dots, X_{k+1}) &= \sum_i (-1)^{i+1} X_i(\omega(X_1, \dots, \widehat{X}_i, \dots, X_{k+1})) \\ &\quad + \sum_{i < j} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_{k+1}). \end{aligned}$$

Here \widehat{X}_i means omit the term with X_i . The proof is for HW.

32.2. Distributions. Recall that if X is a vector field with $X(p) \neq 0$, then locally near p there exists an open set with coordinates (x_1, \dots, x_n) where $X = \frac{\partial}{\partial x_1}$. Can we generalize this? If X, Y are two vector fields such that $X(p), Y(p)$ span a 2-dimensional subspace of T_pM , then the span of $X(x)$ and $Y(x)$ is a 2-plane field for every x in a neighborhood of p .

Definition 32.2. Let M be an n -dimensional manifold.

- (1) A k -dimensional distribution \mathcal{D} is a smooth choice of a k -dimensional subspace of T_pM at every point $p \in M$. By a smooth choice we mean there exist k linearly independent vector fields X_1, \dots, X_k which span \mathcal{D}_x locally near p .
- (2) An integral submanifold N of M is a submanifold where $T_pN \subset \mathcal{D}_x$ at every $p \in N$. $\dim N$ is not necessarily $\dim \mathcal{D}$, but $\dim N \leq \dim \mathcal{D}$.
- (3) \mathcal{D} is an integrable distribution if there is a coordinate system $\{x_1, \dots, x_n\}$ near every $p \in M$ such that $\mathcal{D} = \text{Span}\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\}$. Equivalently, \mathcal{D} is integrable if there locally exist functions f_1, \dots, f_{n-k} such that $\{f_1 = \text{const}, \dots, f_{n-k} = \text{const}\}$ are integral submanifolds of \mathcal{D} and the f_i are independent, i.e., $df_1 \wedge \dots \wedge df_{n-k} \neq 0$.

Dually, we can define a k -dimensional distribution on M (of dimension n) locally by prescribing $n - k$ linearly independent 1-forms $\omega_1, \dots, \omega_{n-k}$.

Example: On \mathbb{R}^3 , let $\omega = dz$. Then $\mathcal{D} = \ker \omega = \text{Span}\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$. The *integral surfaces* (surfaces everywhere tangent to \mathcal{D}) are $z = \text{const}$. \mathcal{D} is an integrable 2-plane field distribution.

Example: On \mathbb{R}^3 , consider $\omega = dz + (xdy - ydx)$. Then $\mathcal{D} = \text{Ker } \omega = \text{Span}\{x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\}$. \mathcal{D} is called a *contact distribution*, and is not integrable.

For 2-plane fields in \mathbb{R}^3 integrability amounts to: Can you find a function f such that $f = \text{const}$ are everywhere tangent to \mathcal{D} ?

First calculate $\omega \wedge d\omega = 2dxdydz \neq 0$. Then \mathcal{D} is not integrable for the following reason: If $\mathcal{D} = \mathbb{R}\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\}$, then ω is of the form $f dx_3$. Now, $d\omega = df dx_3$ and $\omega \wedge d\omega = f dx_3 \wedge df \wedge dx_3 = 0$. This gives a contradiction. Therefore, the contact 2-plane field distribution is not integrable.

33. FROBENIUS' THEOREM

Let M be an n -dimensional manifold. A distribution \mathcal{D} of rank k is a rank k subbundle of TM . Locally, \mathcal{D} is defined as the span of independent vector fields X_1, \dots, X_k or as the kernel of independent 1-forms $\omega_1, \dots, \omega_{n-k}$.

Theorem 33.1 (Frobenius' Theorem). *A distribution $\mathcal{D} \subset TM$ of rank k is integrable if and only if for all $X, Y \in \Gamma(\mathcal{D})$, $[X, Y] \in \Gamma(\mathcal{D})$.*

33.1. Proof of Frobenius' Theorem. Suppose $\mathcal{D} \subset TM$ is integrable. Then there exist coordinates x_1, \dots, x_n so that $\mathcal{D} = \text{Span}\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\}$. Hence $X = \sum_{i=1}^k a_i(x) \frac{\partial}{\partial x_i}$ and $Y = \sum_{j=1}^k b_j(x) \frac{\partial}{\partial x_j}$, and

$$[X, Y] = \sum_{i=1}^k \sum_{j=1}^k \left(a_i \frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j} \frac{\partial}{\partial x_i} \right) \in \Gamma(\mathcal{D}).$$

Suppose for all $X, Y \in \Gamma(\mathcal{D})$, $[X, Y] \in \Gamma(\mathcal{D})$. We will find coordinates x_1, \dots, x_n so that $\mathcal{D} = \text{Span}\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}\}$. Note that all our computations are local, so we restrict to $M = \mathbb{R}^n$. We will first do a slightly easier situation.

Proposition 33.2. *Let X_1, \dots, X_k be independent vector fields with $\mathcal{D} = \text{Span}\{X_1, \dots, X_k\}$. If $[X_i, X_j] = 0$ for all i, j , then \mathcal{D} is integrable.*

Proof. We will deal with the case where $\dim \mathcal{D} = 2$ and $M = \mathbb{R}^3$. Suppose $[X, Y] = 0$. Using the fundamental theorem of ODE's, we can write $X = \frac{\partial}{\partial x_1}$. Then $Y = \sum_{i=1}^3 b_i \frac{\partial}{\partial x_i}$, and $[X, Y] = 0$ implies that $\frac{\partial b_i}{\partial x_1} = 0$, i.e., $b_i = b_i(x_2, x_3)$ (there is no dependence on x_1). Now take $Y' = Y - b_1 X = b_2(x_2, x_3) \frac{\partial}{\partial x_2} + b_3(x_2, x_3) \frac{\partial}{\partial x_3}$. If we project to \mathbb{R}^2 with coordinates x_2, x_3 , then Y' can be integrated to $\frac{\partial}{\partial x'_2}$, after a possible change of coordinates. Therefore, $\mathcal{D} = \text{Span}\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x'_2}\}$. \square

Still assuming $\dim \mathcal{D} = 2$ and $M = \mathbb{R}^3$, suppose $[X, Y] = AX + BY$. Without loss of generality, $X = \frac{\partial}{\partial x_1}$ and $Y = b_2 \frac{\partial}{\partial x_2} + b_3 \frac{\partial}{\partial x_3}$. Then,

$$[X, Y] = \frac{\partial b_2}{\partial x_1} \frac{\partial}{\partial x_2} + \frac{\partial b_3}{\partial x_1} \frac{\partial}{\partial x_3} = A \frac{\partial}{\partial x_1} + B b_2 \frac{\partial}{\partial x_2} + B b_3 \frac{\partial}{\partial x_3}.$$

This implies: $A = 0$, $\frac{\partial b_2}{\partial x_1} = B b_2$, $\frac{\partial b_3}{\partial x_1} = B b_3$. Hence,

$$b_2 = f(x_2, x_3) e^{\int_{t=0}^{t=x_1} B(t, x_2, x_3) dt}, \quad b_3 = g(x_2, x_3) e^{\int_{t=0}^{t=x_1} B(t, x_2, x_3) dt}.$$

Therefore, $Y = e^{\int B} (f(x_2, x_3) \frac{\partial}{\partial x_2} + g(x_2, x_3) \frac{\partial}{\partial x_3})$, and by rescaling Y we get $Y' = f(x_2, x_3) \frac{\partial}{\partial x_2} + g(x_2, x_3) \frac{\partial}{\partial x_3}$. As before, now Y' can be integrated to give $\frac{\partial}{\partial x'_2}$.

HW: Write out a general proof.

33.2. Restatement in terms of forms. If \mathcal{D} has rank k on M of dimension n , then dually there exist 1-forms $\omega_1, \dots, \omega_{n-k}$ such that $\mathcal{D} = \{\omega_1 = \dots = \omega_{n-k} = 0\}$.

Proposition 33.3. \mathcal{D} is integrable if and only if $d\omega_i = \sum_{j=1}^{n-k} \theta_{ij} \wedge \omega_j$, where θ_{ij} are 1-forms.

Proof. We use the identity

$$(7) \quad d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]).$$

First suppose $d\omega_i = \sum_{j=1}^{n-k} \theta_{ij} \wedge \omega_j$. Then for sections X, Y of \mathcal{D} ,

$$d\omega_i(X, Y) = X\omega_i(Y) - Y\omega_i(X) - \omega_i([X, Y]) = -\omega_i([X, Y]).$$

On the other hand, $d\omega_i(X, Y) = 0$. Hence $\omega_i([X, Y]) = 0$ for all i , which implies that $[X, Y]$ is a section of \mathcal{D} .

Next, suppose \mathcal{D} is integrable. Complete $\omega_1, \dots, \omega_{n-k}$ into a basis by adding η_1, \dots, η_k . Then

$$d\omega_i = \sum_{i < j} a_{ij} \omega_i \wedge \omega_j + \sum_{i=1}^k \sum_{j=1}^{n-k} b_{ij} \eta_i \wedge \omega_j + \sum_{i < j} c_{ij} \eta_i \wedge \eta_j.$$

Using Equation 7, for X, Y sections of \mathcal{D} , $d\omega_i(X, Y) = 0$. Taking X_1, \dots, X_k dual to η_1, \dots, η_k , we find that $d\omega_i(X_r, X_s) = c_{rs}$ (or $-c_{sr}$). This proves that all the c_{ij} are zero. \square

34. CONNECTIONS

34.1. **Definition.** Let E be a rank k vector bundle over M and let s be a section of E . s may be local (i.e., in $\Gamma(E, U)$) or global (i.e., in $\Gamma(E, M)$). Also let X be a vector field. We want to differentiate s at $p \in M$ in the direction of $X(p) \in T_p M$.

Definition 34.1. A connection or covariant derivative ∇ assigns to every vector field $X \in \mathfrak{X}(M)$ a differential operator $\nabla_X : \Gamma(E) \rightarrow \Gamma(E)$ which satisfies:

- (1) $\nabla_X s$ is \mathbb{R} -linear in s , i.e., $\nabla_X(c_1 s_1 + c_2 s_2) = c_1 \nabla_X s_1 + c_2 \nabla_X s_2$ if $c_1, c_2 \in \mathbb{R}$.
- (2) $\nabla_X s$ is $C^\infty(M)$ -linear in X , i.e., $\nabla_{fX+gY}s = f\nabla_X s + g\nabla_Y s$.
- (3) (Leibniz rule) $\nabla_X(fs) = (Xf)s + f\nabla_X s$.

Note: The definition of connection is tensorial in X (condition (2)), so $(\nabla_X s)(p)$ depends on s near p but only on X at p .

34.2. **Flat connections.** We will now present the first example of a connection.

A vector bundle E of rank k is said to be *trivial* or *parallelizable* if there exist sections $s_1, \dots, s_k \in \Gamma(E, M)$ which span E_p at every $p \in M$. Although not every vector bundle is parallelizable, locally every vector bundle is trivial since $E|_U \simeq U \times \mathbb{R}^k$. We will now construct connections on trivial bundles.

Write any section s as $s = \sum_i f_i s_i$, where $f_i \in C^\infty(M)$. Then define

$$\nabla_X s = \sum_i (Xf_i) s_i = (Xf_1)s_1 + \dots + (Xf_k)s_k \in \Gamma(E).$$

This connection is usually called a *flat connection*.

HW: Check that this satisfies the axioms of a connection.

Note that $\nabla_X s_i = 0$ for all $X \in \mathfrak{X}(M)$. Sections s satisfying such a property are said to be *covariant constant*.

Important remark: We can define a connection ∇ for each trivialization $E|_U = U \times \mathbb{R}^k$, and there is nothing canonical about the connection ∇ above. (It depends on the choice of trivialization.) The space of connections is a large space (to be made more precise later).

Proposition 34.2. Any two covariant constant frames s_1, \dots, s_k and $\overline{s}_1, \dots, \overline{s}_k$ differ by an element of $GL(k, \mathbb{R})$.

Proof. Let $\overline{s}_1, \dots, \overline{s}_k$ be another covariant constant frame, i.e., $\nabla_X \overline{s}_i = 0$. Since we can write

$$\overline{s}_i = \sum_j f_{ij} s_j,$$

with $f_{ij} \in C^\infty(M)$, we have:

$$\begin{aligned}
 0 = \nabla_X \bar{s}_i &= \sum_j \nabla_X (f_{ij} s_j) \\
 &= \sum_j [(X f_{ij}) s_j + f_{ij} \nabla_X s_j] \\
 &= \sum_j (X f_{ij}) s_j.
 \end{aligned}$$

This proves that $X f_{ij} = 0$ for all X and hence $f_{ij} = \text{const.}$ □

Therefore, a flat connection determines a covariant constant frame $\{s_1, \dots, s_k\}$ up to an element of $GL(k, \mathbb{R})$.

35. MORE ON CONNECTIONS

35.1. Preliminaries on vector bundles. Let E be a vector bundle over M and $\phi : N \rightarrow M$ be a smooth map. Then we can define the *pullback bundle* $\phi^{-1}E$ over N as follows:

- (1) The fiber $(\phi^{-1}E)_n$ over $n \in N$ is the fiber $E_{\phi(n)}$ over $\phi(n) \in M$.
- (2) There exist sufficiently small open sets $V \subset N$, so that $\phi(V) \subset U$ and $\varphi_U : E|_U \xrightarrow{\sim} U \times \mathbb{R}^k$. The trivialization $\phi^{-1}E|_V \simeq V \times \mathbb{R}^k$ is induced from this.

Next, if E and F are vector bundles over M , then we can define $E \oplus F$ as follows:

- (1) The fiber $(E \oplus F)_m$ over $m \in M$ is $E_m \oplus F_m$.
- (2) Take $U \subset M$ small enough so that $E|_U \xrightarrow{\sim} U \times \mathbb{R}^k$ and $F|_U \xrightarrow{\sim} U \times \mathbb{R}^l$. Then we get $(E \oplus F)|_U \simeq U \times (\mathbb{R}^k \oplus \mathbb{R}^l)$.

$E \otimes F$ is defined similarly.

35.2. Existence. Let M be an n -dimensional manifold and E be a rank k vector bundle over M . Recall a connection ∇ is a way of differentiating sections of E in the direction of a vector field X .

$$\begin{aligned}\nabla_X : \Gamma(E) &\rightarrow \Gamma(E), \\ \nabla_X(fs) &= (Xf)s + f\nabla_X s.\end{aligned}$$

Definition 35.1. A connection ∇ on E is flat if there exists an open cover $\{U_\alpha\}$ of M such that $E|_{U_\alpha}$ admits a covariant constant frame s_1, \dots, s_k .

Proposition 35.2. Connections exist on any vector bundle E over M .

Note that if E is parallelizable we have already defined connections globally on E . The key point is to pass from local to global when E is not globally trivial.

Let ∇' and ∇'' be two connections on $E|_U$. Let us see whether $\nabla' + \nabla''$ is a connection.

$$\begin{aligned}(\nabla'_X + \nabla''_X)(fs) &= \nabla'_X(fs) + \nabla''_X(fs) \\ &= (Xf)s + f\nabla'_X s + (Xf)s + f\nabla''_X s \\ &= 2(Xf)s + f(\nabla'_X + \nabla''_X)s.\end{aligned}$$

This is not quite a connection, since $2(Xf)$ should be Xf instead. However, a simple modification presents itself:

Lemma 35.3. Suppose $\lambda_1, \lambda_2 \in C^\infty(U)$ satisfies $\lambda_1 + \lambda_2 = 1$. Then $\lambda_1\nabla' + \lambda_2\nabla''$ is a connection on $E|_U$.

Proof. HW. □

Proof of Proposition 35.2. Let $\{U_\alpha\}$ be an open cover such that $E|_{U_\alpha}$ is trivial. Let ∇^α be a flat connection on $E|_{U_\alpha}$ associated to some trivialization. Next let $\{f_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. Then form $\sum_\alpha f_\alpha\nabla^\alpha$. By the previous lemma, the Leibniz rule is satisfied. □

Remark: Although each of the pieces ∇^α is flat before patching, the patching destroys flatness. There is no guarantee that (even locally) there exist sections s_1, \dots, s_k which are covariant constant. In fact, for a generic connection, there is not even a single covariant constant section. One way of measuring the failure of the existence of covariant constant sections is the *curvature*.

35.3. The space of connections. Given two connections ∇ and ∇' , we compute their difference:

$$(\nabla_X - \nabla'_X)(fs) = f(\nabla_X - \nabla'_X)s.$$

Therefore, the difference of two connections is *tensorial* in s .

Locally, take sections s_1, \dots, s_k (not necessarily covariant constant). Then $(\nabla_X - \nabla'_X)s_i = \sum_j a_{ij}s_j$, where (a_{ij}) is a $k \times k$ matrix of functions. In other words, $\nabla - \nabla'$ can be represented by a matrix $A = (A_{ij})$ of 1-forms A_{ij} . Here $a_{ij} = A_{ij}(X)$. Hence, locally it makes sense to write:

$$\nabla = d + A.$$

Here $s = \sum f_i s_i$ corresponds to $(f_1, \dots, f_k)^T$ and more precisely

$$\nabla(f_1, \dots, f_k)^T = d(f_1, \dots, f_k)^T + A(f_1, \dots, f_k)^T.$$

Globally, $\nabla - \nabla'$ is a section of $T^*M \otimes \text{End}(E)$. Here $\text{End}(E) = \text{Hom}(E, E)$. The space of such sections is often written as $\Omega^1(\text{End}(E))$ and a section is called a “1-form with values in $\text{End}(E)$ ”. This proves:

Proposition 35.4. *The space of connections on E is an affine space $\Omega^1(\text{End}(E))$.*

Remark: We view $\Omega^1(\text{End}(E))$ not as a vector space (which has a preferred zero element) but rather as an affine space, which is the same thing except for the lack of a preferred zero element.

36. CURVATURE

Let $E \rightarrow M$ be a rank r vector bundle and ∇ be a connection on E .

Definition 36.1. The curvature R_∇ (or simply R) of a connection ∇ is given by:

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]} = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]},$$

or

$$R(X, Y)s = [\nabla_X, \nabla_Y]s - \nabla_{[X, Y]}s.$$

Proposition 36.2.

- (1) $R(X, Y)s$ is tensorial, i.e., $C^\infty(M)$ -linear, in each of X, Y , and s .
- (2) $R(X, Y) = -R(Y, X)$.

Proof. (2) is easy. For (1), we will prove that $R(X, Y)$ is tensorial in s and leave the verification for X and Y as HW.

$$\begin{aligned} R(X, Y)(fs) &= (\nabla_X \nabla_Y - \nabla_Y \nabla_X)(fs) - \nabla_{[X, Y]}(fs) \\ &= \nabla_X((Yf)s + f\nabla_Y s) - \nabla_Y((Xf)s + f\nabla_X s) - (([X, Y]f)s + f\nabla_{[X, Y]}s) \\ &= fR(X, Y)s \end{aligned}$$

□

Proposition 36.3. The flat connection $\nabla_X s = \sum (Xf_k)s_k$ has $R = 0$. (Here s_1, \dots, s_r trivializes $E|_U$ and $s = \sum f_k s_k$.)

Proof. We use $X = \frac{\partial}{\partial x_i}, Y = \frac{\partial}{\partial x_j}$. Since $R(X, Y)$ is tensorial, it suffices to compute it for our choices.

$$\begin{aligned} R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)s &= \sum_k \left(\nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{\partial}{\partial x_j}} - \nabla_{\frac{\partial}{\partial x_j}} \nabla_{\frac{\partial}{\partial x_i}} \right) f_k s_k \\ &= \sum_l \left[\nabla_{\frac{\partial}{\partial x_i}} \left(\frac{\partial f_k}{\partial x_j} s_k \right) - \nabla_{\frac{\partial}{\partial x_j}} \left(\frac{\partial f_k}{\partial x_i} s_k \right) \right] \\ &= \sum_k \left(\frac{\partial^2 f_k}{\partial x_i \partial x_j} - \frac{\partial^2 f_k}{\partial x_j \partial x_i} \right) s_k = 0. \end{aligned}$$

□

36.1. Interpretations of curvature. Think of ∇ as $d + A$ in local coordinates if necessary. We have a sequence:

$$\Omega^0(E) \xrightarrow{\nabla} \Omega^1(E) \xrightarrow{\nabla} \Omega^2(E) \rightarrow \dots$$

The first map is covariant differentiation (interpreted slightly differently). It turns out that this sequence is not a chain complex, i.e., $\nabla \circ \nabla \neq 0$ usually. In fact the obstruction to this being a chain complex is the curvature. Let us locally write:

$$\nabla \circ \nabla s = (d + A)(d + A)s = (d^2 + Ad + dA + A \wedge A)s = (dA + A \wedge A)s.$$

Proposition 36.4. $R = dA + A \wedge A$, i.e., $R(X, Y)s = (dA + A \wedge A)(X, Y)s$.

Proof. It suffices to prove the proposition for $X = \frac{\partial}{\partial x_i}$, $Y = \frac{\partial}{\partial x_j}$, and $s = s_k$, where s_1, \dots, s_r is a local frame for $E|_U$. A is an $r \times r$ matrix of 1-forms $(A_{ij}^t dx_t)$. (We will use the Einstein summation convention – if the same index appears twice we assume it is summed over this index.) Then we compute:

$$(8) \quad \nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{\partial}{\partial x_j}} s_k = \nabla_{\frac{\partial}{\partial x_i}} (s_m A_{mk}^j) = s_m \frac{\partial A_{mk}^j}{\partial x_i} + s_n A_{nm}^i A_{mk}^j$$

The computation of the rest is left for HW. □

37. RIEMANNIAN METRICS, LEVI-CIVITA

37.1. **Leftovers from last time.** Last time we defined the curvature R_∇ of a connection ∇ . Locally, if ∇ is given by $d + A$, then $R = dA + A \wedge A$.

Theorem 37.1. ∇ is a flat connection if and only if $R_\nabla \equiv 0$.

We have already shown the easy direction: If ∇ is flat, then $R_\nabla \equiv 0$. The other direction will be omitted for now (probably will be given next semester), since a “good proof” will take us a bit far afield. Our only comment is that $R = dA + A \wedge A = 0$ or $dA = -A \wedge A$ looks a lot like the Frobenius integrability condition given in terms of forms....

Corollary 37.2. Let E be a rank r vector bundle over \mathbb{R} and ∇ be a connection on E . Then ∇ is flat.

Proof. This is because all 2-forms on \mathbb{R} are zero. □

Remark: There are lots of connections which are not flat, since it is easy to find A so that $dA + A \wedge A \neq 0$.

37.2. Riemannian metrics.

Definition 37.3. A Riemannian metric $\langle \cdot, \cdot \rangle$ or g on M is a positive definite symmetric bilinear form (or inner product) $g(x) : T_x M \times T_x M \rightarrow \mathbb{R}$ which is smooth in $x \in M$.

Recall: A bilinear form $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$ is positive definite if $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ if and only if $v = 0$. $\langle \cdot, \cdot \rangle$ is symmetric if $\langle v, w \rangle = \langle w, v \rangle$.

Example: On \mathbb{R}^n take the standard Euclidean metric $g \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = \delta_{ij}$. This is usually written as $g = \sum_i dx_i \otimes dx_i$. Any other Riemannian metric on \mathbb{R}^n can be written as $g(x) = \sum_{ij} g_{ij}(x) dx_i \otimes dx_j$, where $g_{ij}(x) = g_{ji}(x)$.

Proposition 37.4. Every manifold M admits a Riemannian metric.

Proof. Let $\{U_\alpha\}$ be an open cover so that $U_\alpha \simeq \mathbb{R}^n$. On each U_α , we take the standard Euclidean metric g_α . Now let $\{f_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. Then $\sum_\alpha f_\alpha g_\alpha$ is the desired metric. □

The pair (M, g) of a manifold M together with a Riemannian metric g on M is called a Riemannian manifold.

Let $i : N \rightarrow (M, g)$ be an embedding or immersion. Then the induced Riemannian metric i^*g on N is defined as follows:

$$i^*g(x)(v, w) = g(i(x))(i_*v, i_*w),$$

where $v, w \in T_x N$. The injectivity of i_* is required for the positive definiteness.

37.3. Levi-Civita connections. Connections on $TM \rightarrow M$ have extra structure because X and Y are the same type of object in the expression $\nabla_X Y$. In fact, we can define the *torsion*:

$$\mathcal{T}_\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

Proposition 37.5. $\mathcal{T}_\nabla(X, Y)$ is tensorial in X and Y .

This is an easy exercise and is left for HW. (Note that the notion of torsion does not depend at all on the Riemannian metric.) We say ∇ is *torsion-free* if $\mathcal{T}_\nabla = 0$.

Definition 37.6. ∇ is compatible with g if $X\langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$. Here $X, Y, Z \in \mathfrak{X}(M)$.

Theorem 37.7. Let (M, g) be a Riemannian manifold. Then there exists a unique torsion-free connection which is compatible with g .

Proof. For any vector fields X, Y, Z , we have:

$$(9) \quad X\langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle,$$

$$(10) \quad Y\langle X, Z \rangle = \langle \nabla_Y X, Z \rangle + \langle X, \nabla_Y Z \rangle,$$

$$(11) \quad Z\langle X, Y \rangle = \langle \nabla_Z X, Y \rangle + \langle X, \nabla_Z Y \rangle.$$

Taking (9) + (10) – (11), we get:

$$(12) \quad 2\langle \nabla_X Y, Z \rangle + \langle [Y, X], Z \rangle + \langle [X, Z], Y \rangle + \langle [Y, Z], X \rangle = X\langle Y, Z \rangle + Y\langle X, Z \rangle - Z\langle X, Y \rangle,$$

and solving for $\langle \nabla_X Y, Z \rangle$, we get:

$$(13) \quad \langle \nabla_X Y, Z \rangle = \frac{1}{2}(\langle [X, Y], Z \rangle + \langle [Z, X], Y \rangle + \langle [Z, Y], X \rangle + X\langle Y, Z \rangle + Y\langle X, Z \rangle - Z\langle X, Y \rangle).$$

It is clear that the values of Equation 13 determine ∇ . It remains to show that Equation 13 indeed defines a connection which satisfies the torsion-free and compatibility conditions. It is clear that $\langle \nabla_X Y, Z \rangle = \langle \nabla_Y X, Z \rangle$ when X, Y, Z are of the form $\frac{\partial}{\partial x_i}$, and that Equation 9 can be recovered from Equation 13. The details are left for HW. \square

The unique torsion-free, compatible connection is called the *Levi-Civita connection* for (M, g) .

38. SHAPE OPERATOR

Let Σ be a surface embedded in the standard Euclidean (\mathbb{R}^3, g) , and let \bar{g} be the induced metric on Σ . We will denote the Levi-Civita connection on (\mathbb{R}^3, g) by ∇ and the Levi-Civita connection on (Σ, \bar{g}) by $\bar{\nabla}$.

Claim: ∇ satisfies $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = 0$.

The verification is easy. The claim implies that $\nabla_X Y$ is simply $\frac{d}{dt} Y(\gamma(t))|_{t=0}$, where $\gamma(t)$ is the arc representing X at a given point.

What we will do today is valid for hypersurfaces ($(n - 1)$ -dimensional submanifolds) M inside (N, g) of dimension n , but we will restrict our attention to $N = \mathbb{R}^3$ for simplicity.

Definition 38.1. Let X, Y be vector fields of \mathbb{R}^3 which are tangent to Σ , and let N be the unit normal vector field to Σ inside \mathbb{R}^3 . The shape operator is $S(X, Y) = \langle \nabla_X Y, N \rangle$. In other words, $S(X, Y)$ is the projection in the N -direction of $\nabla_X Y$.

Proposition 38.2. $S(X, Y)$ is tensorial in X, Y and is symmetric.

Proof. $S(X, Y) = S(Y, X)$ follows from the torsion-free condition and the fact that $[X, Y]$ is tangent to Σ . Now,

$$S(fX, Y) = \langle \nabla_{fX} Y, N \rangle = \langle f \nabla_X Y, N \rangle = f S(X, Y).$$

Tensoriality in Y is immediate from the symmetric condition. □

Remark: The shape operator is usually called the *second fundamental form* in classical differential geometry and measures how curved a surface is. (In case you are curious what the *first fundamental form* is, it's simply the induced Riemannian metric.)

Also observe that $S(X, Y) = \langle \nabla_X Y, N \rangle = \langle \nabla_X N, Y \rangle$, by using the fact that $\langle Y, N \rangle = 0$ (since N is a normal vector and Y is tangent to Σ).

38.1. Induced connection vs. Levi-Civita. If $X, Y \in \mathfrak{X}(M)$, we can write:

$$\nabla_X Y = \nabla_X^h Y + S(X, Y)N,$$

where $\nabla_X^h Y$ denotes the projection of $\nabla_X Y$ onto $T\Sigma$.

Proposition 38.3. $\nabla^h = \bar{\nabla}$, i.e., ∇^h is the Levi-Civita connection of (Σ, \bar{g}) .

Proof. We have defined $\nabla_X^h Y = \nabla_X Y - S(X, Y)N$. It is easy to verify that ∇^h satisfies the properties of a connection on Σ .

∇^h is torsion-free:

$$\begin{aligned} \nabla_X^h Y - \nabla_Y^h X &= (\nabla_X Y - S(X, Y)N) - (\nabla_Y X - S(Y, X)N) \\ &= \nabla_X Y - \nabla_Y X \\ &= [X, Y] \end{aligned}$$

∇^h is compatible with \bar{g} :

$$\begin{aligned} X\langle Y, Z \rangle &= \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle \\ &= \langle \nabla_X^h Y, Z \rangle + \langle Y, \nabla_X^h Z \rangle, \end{aligned}$$

since $\langle N, X \rangle = 0$ for any vector field N on Σ . □

It seems miraculous that somehow the induced connection is a Levi-Civita connection. Classically, the induced covariant derivative came first, and Levi-Civita came as an abstraction of the covariant derivative.

39. GAUSS' THEOREMA EGREGIUM

Let (Σ, \bar{g}) be a 2-dimensional Riemannian submanifold of the standard Euclidean (\mathbb{R}^3, g) . The shape operator is a symmetric bilinear form:

$$S : T_x \Sigma \times T_x \Sigma \rightarrow \mathbb{R},$$

$$S(X, Y) = \langle \nabla_X Y, N \rangle$$

where N is a unit normal to Σ , X, Y are vectors in $T_x \Sigma$ which are extended to an arbitrary vector field tangent to Σ , and ∇ is the Levi-Civita connection for (\mathbb{R}^3, g) . We can represent $S(x)$, $x \in \Sigma$, as a matrix by taking an orthonormal basis $\{e_1, e_2\}$ at $T_x \Sigma$ and taking the entries $S(e_i, e_j)$. The trace of this matrix is called the *mean curvature* and the determinant is called the *scalar curvature* or the *Gaussian curvature*.

Denote by ∇ the Levi-Civita connection for g and $\bar{\nabla}$ the Levi-Civita connection for \bar{g} . Also write $R = R_{\nabla}$ and \bar{R} for $R_{\bar{\nabla}}$.

Theorem 39.1 (Gauß' Theorema Egregium). *If X, Y are vector fields on Σ , then*

$$\langle \bar{R}(X, Y)Y, X \rangle = S(X, X)S(Y, Y) - S(X, Y)^2.$$

What this says is that the right-hand side, an extrinsic quantity (depends on the embedding into 3-space) is equal to the left-hand side, an intrinsic quantity (only depends on the Riemannian metric \bar{g} and not on the particular embedding into \mathbb{R}^3). Therefore, the scalar curvature is expressed purely in terms of the curvature of the induced metric.

Proof. Let N be the unit normal vector to Σ .

$$\begin{aligned} \langle \bar{\nabla}_X \bar{\nabla}_Y Y, X \rangle &= X \langle \bar{\nabla}_Y Y, X \rangle - \langle \bar{\nabla}_Y Y, \bar{\nabla}_X X \rangle \\ &= X \langle \nabla_Y Y - S(Y, Y)N, X \rangle - \langle \nabla_Y Y - S(Y, Y)N, \nabla_X X - S(X, X)N \rangle \\ &= X \langle \nabla_Y Y, X \rangle - \langle \nabla_Y Y, \nabla_X X \rangle + \langle S(X, X)N, \nabla_Y Y \rangle \\ &= \langle \nabla_X \nabla_Y Y, X \rangle + S(X, X)S(Y, Y). \end{aligned}$$

Similarly,

$$\begin{aligned} \langle \bar{\nabla}_Y \bar{\nabla}_X Y, X \rangle &= \langle \nabla_Y \nabla_X Y, X \rangle - S(X, Y)^2, \\ \langle \bar{\nabla}_{[X, Y]} Y, X \rangle &= \langle \nabla_{[X, Y]} Y, X \rangle. \end{aligned}$$

Finally,

$$\begin{aligned} \langle \bar{R}(X, Y)Y, X \rangle &= \langle R(X, Y)Y, X \rangle + S(X, X)S(Y, Y) - S(X, Y)^2 \\ &= S(X, X)S(Y, Y) - S(X, Y)^2. \end{aligned}$$

□

40. EULER CLASS

40.1. Compatible connections. Let E be a rank k vector bundle over a manifold M . A *fiber metric* is a family of positive definite inner products $\langle \cdot, \cdot \rangle_x : E_x \times E_x \rightarrow \mathbb{R}$ which varies smoothly with respect to $x \in M$. A connection ∇ is *compatible* with $\langle \cdot, \cdot \rangle$ if $X\langle s_1, s_2 \rangle = \langle \nabla_X s_1, s_2 \rangle + \langle s_1, \nabla_X s_2 \rangle$, for all vector fields X and $s_1, s_2 \in \Gamma(E)$.

Remark: We can view the Riemannian metric g on M as a fiber metric of $TM \rightarrow M$. When we think of TM as a vector bundle over M , we forget the fact that TM was derived from M .

Let $U \subset M$ be an open set over which E is trivialisable, and let $\{s_1, s_2, \dots, s_k\}$ be an orthonormal frame of E over U . An orthonormal frame can be obtained by starting from some frame of E over U and applying the Gram-Schmidt orthogonalization process.

With respect to $\{s_1, \dots, s_k\}$ we can write $\nabla = d + A$, where A is a $k \times k$ matrix with entries which are 1-forms on U .

Lemma 40.1. A is a skew-symmetric matrix, i.e., $A^T = -A$.

Proof. If we write $A = (A_{ij}^k dx_k)$, then we have $\nabla_{\frac{\partial}{\partial x_k}} s_j = s_i A_{ij}^k$.

$$\frac{\partial}{\partial x_k} \langle s_i, s_j \rangle = \langle \nabla_{\frac{\partial}{\partial x_k}} s_i, s_j \rangle + \langle s_i, \nabla_{\frac{\partial}{\partial x_k}} s_j \rangle,$$

so we have

$$A_{ij}^k = -A_{ji}^k.$$

□

Lemma 40.2. Let $\{s'_1, \dots, s'_k\}$ be another orthonormal frame for E over U . If $g : U \rightarrow SO(k)$ is the transformation sending coordinates with respect to s_i to coordinates with respect to s'_i (by left multiplication), then the connection matrix transforms as: $A \mapsto g^{-1}dg + g^{-1}Ag$.

Proof.

$$\begin{aligned} g^{-1}(d + A)g &= g^{-1}dg + g^{-1}gd + g^{-1}Ag \\ &= d + (g^{-1}dg + g^{-1}Ag). \end{aligned}$$

You may want to check that if A is skew-symmetric and g is orthogonal, then $g^{-1}dg + g^{-1}Ag$ is also skew-symmetric. □

40.2. Rank 2 case. Suppose from now on that E has rank 2 over M of arbitrary dimension. Then A_U (the connection matrix over U with respect to some trivialization) is given by

$$A_U = \begin{pmatrix} 0 & A_{21} \\ -A_{21} & 0 \end{pmatrix}.$$

Then the curvature matrix R_U is

$$R_U = dA_U + A_U \wedge A_U = \begin{pmatrix} 0 & \omega_U \\ -\omega_U & 0 \end{pmatrix},$$

where ω_U is the 2-form dA_{21} .

Theorem 40.3. *There is a global closed 2-form ω which coincides with ω_U on each open set U . Hence a connection ∇ on E gives rise to an element $[\omega] \in H_{dR}^2(M)$. This cohomology class is independent of the choice of connection ∇ compatible with \langle, \rangle , and hence is an invariant of the vector bundle E . It is called the Euler class of E and is denoted $e(E)$.*

Proof. We need to show that on overlaps $U \cap V$, $\omega_U = \omega_V$. If $g : U \cap V \rightarrow SO(2)$ is the orthogonal transformation taking from U to V , then we compute R with respect to the connection 1-form $g^{-1}dg + g^{-1}A_Ug$. It is not hard to see that we still get

$$R = \begin{pmatrix} 0 & \omega_U \\ -\omega_U & 0 \end{pmatrix}.$$

Now, two different connections ∇ and ∇' have difference in $\Omega^1(End(E))$. (Moreover, they have values in 2×2 skew-symmetric matrices.) It is not hard to see that if we pick out the upper right hand corner of the matrix on each local coordinate chart U , then they coincide and yield a global 1-form α , and the difference between R_{∇} and $R_{\nabla'}$ will be the exact form $d\alpha$. \square

Example: For the Levi-Civita connection $\bar{\nabla}$ on a surface $(\Sigma, \bar{g}) \hookrightarrow (\mathbb{R}^3, g)$, we have, locally,

$$R_U = \begin{pmatrix} 0 & \kappa\theta_1 \wedge \theta_2 \\ -\kappa\theta_1 \wedge \theta_2 & 0 \end{pmatrix},$$

where κ is the scalar curvature, $\{e_1, e_2\}$ is an orthonormal frame, and $\{\theta_1, \theta_2\}$ is dual to the frame (called the *dual coframe*). (The fact that κ is the scalar curvature is the content of the Theorema Egregium!)

40.3. The Gauß-Bonnet Theorem. Let (M, g) be an oriented Riemannian manifold of dimension n . Then there exists a naturally defined volume form ω which has the following property: At $x \in M$, let e_1, \dots, e_n be an oriented orthonormal basis for T_xM . Then $\omega(x)(e_1, \dots, e_n) = 1$. If we change the choice of orthonormal basis by multiplying by $A \in SO(n)$, then we have a change of $\det(A)$, which is still 1. Therefore, ω is well-defined.

For surfaces (Σ, g) , we have an area form dA .

Theorem 40.4 (Gauß-Bonnet). *Let Σ be a compact submanifold of Euclidean space (\mathbb{R}^3, g) . Then, for one of the orientations of Σ ,*

$$\int_{\Sigma} \kappa dA = 2\pi\chi(\Sigma).$$

Here κ is the scalar curvature, dA is the area form for \bar{g} induced from (\mathbb{R}^3, g) , and $\chi(\Sigma)$ is the Euler characteristic of Σ .

The *Euler characteristic* of a compact manifold M of dimension n is:

$$\chi(M) = \sum_{i=0}^n (-1)^i \dim H_{dR}^i(M).$$

Note that a compact surface Σ (without boundary) of genus g has $\chi(\Sigma) = 2 - 2g$.

Proof. Notice that κdA is simply $\kappa\theta_1 \wedge \theta_2$ above in the Example, and hence the Euler class is $e(TM) = [\kappa dA]$. In order to evaluate $\int_{\Sigma} \kappa dA$, we therefore need to compute $\int_{\Sigma} \omega$ for the connection of our choice on $T\Sigma$ compatible with g , by using Theorem 40.3.

In what follows we will frequently identify $SO(2)$ with the unit circle $S^1 = \{e^{i\theta} | \theta \in [0, 2\pi]\}$ in \mathbb{C} via

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \leftrightarrow e^{i\theta}.$$

We will do a sample computation in the case of the sphere S^2 . Let S^2 be the union of two regions $U = \{|z| \leq 1\}$ and $V = \{|w| \leq 1\}$ identified via $z = 1/w$ along their boundaries. Here z, w are complex coordinates. (Note that U and V are not open sets, but it doesn't really matter...) If we trivialize $T\Sigma$ on U and V using the natural trivialization from $T\mathbb{C}$, then the gluing map $g : U \cap V \rightarrow SO(2)$ is given by $\theta \mapsto e^{2i\theta}$. If we set A_V to be identically zero, then $A_U = g^{-1}dg + g^{-1}A_Vg = g^{-1}dg = \begin{pmatrix} 0 & 2d\theta \\ -2d\theta & 0 \end{pmatrix}$ along ∂U (after transforming via g). No matter how we extend A_U to the interior of U , we have the following by Stokes' Theorem:

$$\int_U \omega_U = \int_{\partial U} 2d\theta = 4\pi = 2\pi\chi(S^2).$$

Now let Σ be a compact surface of genus g (without boundary). Then we can remove g annuli $S^1 \times [0, 1]$ from Σ so that Σ becomes a disk Σ' with $2g - 1$ holes. We make A flat on the annuli, and see what this induces on Σ' . A computation similar to the one above gives the desired formula. (Check this!!) □