ROBUST ADAPTIVE CONTROL FOR A CLASS OF MIMO NONLINEAR SYSTEMS

by

Haojian Xu and Petros A. Ioannou

Department of Electrical Engineering
University of Southern California
Los Angeles, CA 90089
Phone: (213)740-4452
FAX: (213)821-1109
Email: ioannou@usc.edu

Abstract

The design of stabilizing controllers for nonlinear plants with unknown nonlinearities is a challenging problem. The inability to identify the nonlinearities on-line or off-line accurately motivates the design of stabilizing controllers based on approximations or on approximate estimates of the plant nonlinearities. The price paid in such case, could be lack of theoretical guarantees for global stability, and/or nonzero tracking or regulation error at steady state. In this paper a nonlinear robust adaptive control algorithm is designed and analyzed for a class of multi-input multi-output nonlinear systems with unknown nonlinearities. The controller guarantees closed loop semi-global stability and convergence of the tracking error to a small residual set. The size of the residual set for the tracking error depends solely on design parameters, which can be chosen to meet desired upper bounds for the tracking error. Consequently the proposed methodology provides a design procedure to meet a priori specified performance guarantees for the steady state tracking error.

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1. INTRODUCTION

The traditional way of designing feedback control systems is based on the use of Linear Time Invariant (LTI) models for the plant. Off-line frequency domain techniques could be used to fit such an LTI model to experimental data and identify its parameters. In the case, where the parameters of the LTI model change with time, gain scheduling, on-line parameter identification, adaptive control, robust control techniques etc. are developed over the years to address such situations [1]. The reliance on LTI models for control design purposes often puts limitations on the performance improvement that could be achieved for the plant under consideration. For example if the plant consists of strong nonlinearities, its approximation by an LTI model, may considerably reduce the region of attraction in the presence of disturbances and other modeling uncertainties. During the recent years, considerable research efforts have been made to deal with the design of stabilizing controllers for classes of nonlinear plants. These efforts are described in detail in a recent survey paper [2], where a very elegant and informative historical perspective of the evolution of nonlinear control design is presented and discussed. Most of the recent efforts (e.g., surveyed in [2], and [3]-[4]) on nonlinear control design assumed that the plant nonlinearities are known. The case where the plant nonlinearities are products of unknown constant parameters with known nonlinearities gave rise to a number of adaptive control techniques [1], [5]-[10]. Adaptive nonlinear control techniques based on universal function approximators have been found to be particularly useful for controlling more general unknown nonlinearities [11]-[19]. However there are no systematic analytical methods of ensuring the stability, robustness, and performance of the closed loop system. In [20] we designed and analyzed a robust adaptive control scheme for a class of SISO nonlinear linearizable systems with unknown nonlinearities. We developed upper bounds for the tracking error that can be computed apriori. In fact given any desired nonzero upper bound for the tracking error at steady state, a number of design parameters can be chosen apriori to meet such bound.
The purpose of this paper is to develop similar results as in [20] for a wider class of nonlinear systems, namely nonlinear linearizable MIMO systems. We develop a robust adaptive control scheme for a MIMO nonlinear linearizable system with unknown nonlinearities. The only assumptions made are that the unknown nonlinear functions are smooth, and a sufficient condition for controllability is satisfied. The proposed scheme guarantees semi-global stability and convergence of the tracking error to a residual set whose size depends on design parameters that can be chosen apriori. For any given desired upper bound for the tracking error at steady state, our approach provides a procedure for choosing the design parameters to meet the tracking error bound.

This paper is organized as follows: In section 2 the problem statement and preliminaries are presented. In section 3 we present the proposed adaptive control scheme is discussed. In section 4, the theoretical results are applied to the control of a two-link robot. Finally, section 5 includes the conclusions. Throughout this paper, $|\cdot|$ indicates the absolute value, and $\|\cdot\|$ indicates the Euclidean vector norm.

## 2. PROBLEM STATEMENT AND PRELIMINARIES

Consider the following multi-input multi-output nonlinear system:

$$
\begin{align*}
X^{(1)}_i &= f_i(x) + b_{i1}(x)u_i + \cdots + b_{im}(x)u_m \\
& \vdots \\
X^{(m)}_i &= f_m(x) + b_{m1}(x)u_i + \cdots + b_{mm}(x)u_m \\
Y_i &= X_1, \ldots, Y_m = X_m
\end{align*}
$$

(1)

where, $X_i^{(\nu)} \overset{\text{def}}{=} \frac{d\nu}{dt} X_i$, $X = [x_1, \ldots, x_i^{(\nu-1)}, \ldots, x_m, \ldots, x_{m}^{(\nu-1)}]^T \in \mathbb{R}^r$, $r = r_1 + \cdots + r_m$, is the overall state vector, $u_i \in \mathbb{R}$, $i = 1, 2, \ldots, m$, are the inputs and $y_i \in \mathbb{R}$, $i = 1, 2, \ldots, m$, are the outputs of the system. The nonlinear functions $f_i, \ldots, f_m$ and $b_{ij}, i, j = 1, \ldots, m$ are assumed to be smooth functions. The problem is to
design the control input $u = [u_1, \cdots, u_m]^T \in \mathbb{R}^n$ such that the outputs of the system $y_1, \cdots, y_m$ track the desired trajectories $y_{d_1}(t), \cdots, y_{d_m}(t)$ respectively.

The system (1) can also be written in the compact form

$$[x_1^{(r_1)} \ x_2^{(r_2)} \ \cdots \ x_m^{(r_m)}]^T = f(x) + B(x)u$$

where,

$$f(x) = [f_1(x) \ \cdots \ f_m(x)]^T \in \mathbb{R}^n$$

$$B(x) = \begin{bmatrix} b_{11}(x) & \cdots & b_{1m}(x) \\ \vdots & \ddots & \vdots \\ b_{m1}(x) & \cdots & b_{mm}(x) \end{bmatrix} \in \mathbb{R}^{m\times n}$$

We make the following assumptions:

**Assumption 1**: The matrix $\frac{1}{2} \left( B(x) + B^T(x) \right)$ is known to be either uniformly positive definite or uniformly negative definite for all $x \in \Omega$ where $\Omega \subset \mathbb{R}^r$ is a compact set, i.e.,

$$\sigma \left( \frac{B(x) + B^T(x)}{2} \right) \geq \bar{\sigma} > 0, \ \forall \ x \in \Omega$$

(3)

where $\sigma(\cdot)$ represents the smallest singular value of the matrix inside the bracket and $\bar{\sigma}$ is its lower bound.

Assumption 1 guarantees that the nonlinear system (1) is uniformly strongly controllable.

**Assumption 2**: The desired trajectories $y_{d_i}(t), \ i=1,2,\ldots,m$ are known bounded functions of time with bounded known derivatives and $y_d \stackrel{\text{def}}{=} [y_{d_1}, \ \cdots, y_{d_1}^{(r_1-1)}, \ \cdots, y_{d_m}, \ \cdots, y_{d_m}^{(r_m-1)}]^T, \ y_d \in \Omega_{y_d} \subset \mathbb{R}^r$, where $\Omega_{y_d}$ is a known compact set.

**Assumption 3**: The state $x$ of the system is available for measurement.
**Assumption 4:** The functions $f_i(x)$ and $b_{ij}(x)$, $i, j = 1, 2, \cdots, m$ are smooth functions but otherwise completely unknown.

Let us first consider the case where $f_i$ and $b_{ij}$, $i, j = 1, 2, \cdots, m$, are completely known and examine whether we can meet the control objective. This is a reasonable step to take since if we cannot meet the control objective in the case of known nonlinearities, it is unlikely that we will do so in the case of unknown nonlinearities.

We define the following error metric, $S_i$, that describes the desired dynamics of the error system:

$$S_i(t) = \left( \frac{d}{dt} + \lambda_i \right)^{i-1} e_i(t), \quad e_i(t) = y_i - y_{d_i}$$

$$S_n(t) = \left( \frac{d}{dt} + \lambda_n \right)^{n-1} e_n(t), \quad e_n(t) = y_n - y_{d_n}$$

where $\lambda_1, \cdots, \lambda_m$ are positive constants to be selected. It follows from (4) that for $S_i(t) = 0$, $i = 1, 2, \cdots, m$, we have a set of linear differential equations whose solutions imply that $e_i(t)$ converges to zero with time constant $(r_i - 1)/\lambda_i$. In addition all the derivatives of $e_i(t)$ up to $r_i - 1$ also converge to zero [3]-[4].

It follows from (4) that

$$\begin{bmatrix}
\dot{S}_1 \\
\vdots \\
\dot{S}_n
\end{bmatrix} = \begin{bmatrix}
f_1(x) \\
\vdots \\
f_m(x)
\end{bmatrix} + \begin{bmatrix}
v_1(t) \\
\vdots \\
v_m(t)
\end{bmatrix} + \begin{bmatrix}
h_{11}(x) & \cdots & h_{1m}(x) \\
\vdots & \ddots & \vdots \\
h_{m1}(x) & \cdots & h_{mm}(x)
\end{bmatrix} \begin{bmatrix}
u_{11} \\
\vdots \\
u_{mm}
\end{bmatrix}$$

where,

$$v_1(t) = -y_{d1}^{(r_1)}(t) + \alpha_{11} e_{1}^{(r_1-1)}(t) + \cdots + \alpha_{11} \dot{e}_1(t)$$

$$\vdots$$

$$v_m(t) = -y_{d_m}^{(r_m)}(t) + \alpha_{m1} e_{m}^{(r_m-1)}(t) + \cdots + \alpha_{m1} \dot{e}_m(t)$$

\[ (5) \]
and \( \alpha_{i_1,i_0-1}, \ldots, \alpha_{i_1} \), \( i=1, \ldots, m \) are coefficients of the binomial expansion of the corresponding error metric in (4).

Equation (5) can be written in the compact form:

\[
\dot{S} = f(x) + v(t) + B(x)u
\]

where,

\[
S(x, y_x) = \begin{bmatrix} S_1 & \cdots & S_m \end{bmatrix}^\top \in \mathbb{R}^m
\]

\[
v(t) = \begin{bmatrix} v_1(t) & \cdots & v_m(t) \end{bmatrix}^\top \in \mathbb{R}^m
\]

Let us consider the Lyapunov-like function

\[
V(t) = \frac{1}{2} S^\top S
\]

Then

\[
\dot{V} = S^\top \dot{S} = S^\top f(x) + S^\top v(t) + S^\top B(x)u
\]

If we now choose the control law \( u \) so that

\[
\dot{V} = S^\top \dot{S} < 0
\]

then it can be shown that \( \lim_{t \to \infty} ||S|| = 0 \).

If \( B(x) \) is invertible, then the control law [3]

\[
u = B^{-1}(x)[-f(x) - v(t) - KS] \]

where \( K = \text{diag}(k_1, \cdots, k_m) \), and \( k_i > 0, \ i = 1, \ldots, m \), guarantees

\[
\dot{V} = -\sum_{i=1}^{m} k_i S_i^2 \leq 0
\]

which implies that \( V(t) \) and therefore \( S_i(t) \) converges to zero exponentially fast.

The inversion calculation of \( B(x) \) in control law (11) can be avoided if we use assumption 1 and the following lemma to modify the control law (11).
**Lemma 1:** For any square matrix $B(x)$ satisfying *Assumption 1*, there exists a smooth scalar function $\mu(x,y_d)$ such that for any vector $S(x,y_d)$, we have $S^T(x,y_d)B(x)S(x,y_d) = \mu(x,y_d)\|S(x,y_d)\|^2$, where $\|\mu(x,y_d)\| > \overline{\mu}$.

**Proof:** The proof is presented in Appendix A.

Instead of (11) let us now consider the control law:

$$u = \frac{S}{S^T B(x)S} \left[ -k_s \|S\|^2 - S^T f(x) - S^T v(t) \right]$$  \hspace{1cm} (13)

where $k_s > 0$ is a design constant. Using lemma 1 we can rewrite (13) as

$$u = \frac{S_x}{\mu(x,y_d)} \left[ -k_s \|S_x\|^2 - S_x^T f(x) - S_x^T v(t) \right]$$  \hspace{1cm} (14)

where $S_x = S / \|S\|$ is the unit vector. The control law (14) guarantees that

$$\dot{\|S\|^2} = S^T \dot{S} = -k_s \|S\|^2 \leq 0$$  \hspace{1cm} (15)

which in turn implies that $\lim_{t \to \infty} S(t) = 0$ and therefore $e_i$ and all its derivatives up to $r_i - 1$ converge to zero.

In the case where $f(x)$ and $B(x)$ are unknown nonlinear functions, the control law (14) can no longer be used. In the following section we will modify the control law (14) to guarantee stability and meet the control objective when $f(x)$ and $B(x)$ are unknown.

3. **ROBUST ADAPTIVE CONTROL SCHEME**

Let us consider the system (1) and control problem solved in section 2 for the case of known nonlinearities. In this section we assume that the nonlinear functions in (1) are unknown functions and design a control law to meet the control objective.
We assume that the nonlinear functions \( f(x) \) and \( \mu(x, y_d) \) can be approximated by the general one layer neural network [12]-[13], [15]-[19] as

\[
f_i(x) \approx f_i^a(x) = \sum_{j=1}^{l_i} \theta_{ij} g_{ij}(x), \quad i = 1, 2, \cdots, m \tag{16a}
\]

\[
\mu(x, y_d) \approx \mu^a(x, y_d) = \sum_{j=1}^{l_{y_d}} \theta_{ij} g_{ij}(x, y_d) \tag{16b}
\]

where \( g_{ij}(x, y_d), \ g_{ij}(x), \ i = 1, 2, \cdots, m \) are some known basis functions and \( \theta_{ij} \) are constant parameters and \( j = 1, 2, \cdots, l_i \). This assumption is made precise as follows:

**Assumption 5**: There exists a set of constant parameters \( \theta_{ij}, \ i=0,1,\ldots, m \), referred to as optimal output weights such that the continuous functions \( f_i(x) \) and \( \mu(x, y_d) \) can be approximated by (16a-b) with any desired accuracy \( \varepsilon_f > 0 \) over any compact set \( x \in \Omega \subset \mathbb{R}^r \) and \( \varepsilon_\mu > 0 \) over any compact set \( \Omega \times \Omega_{y_d} \), where \( y_d \in \Omega_{y_d} \subset \mathbb{R}^{r'} \), i.e.,

\[
\left| d_i^f(x) \right|_{\text{max}} = \left| f_i(x) - f_i^a(x) \right|_{\text{max}} \leq \varepsilon_f, \quad i = 1, \ldots, m, \quad \forall \ x \in \Omega \tag{17a}
\]

\[
\left| d_\mu(x, y_d) \right|_{\text{max}} = \left| \mu(x, y_d) - \mu^a(x, y_d) \right|_{\text{max}} \leq \varepsilon_\mu, \quad \forall \ x, y_d \in \Omega \times \Omega_{y_d} \tag{17b}
\]

where \( \Omega \subset \mathbb{R}^r \) is a compact set whose size depends on the number of nodes characterized by the integers \( l_i, \ i=0,1,\ldots, m \). It is also assumed that the basis functions, number of nodes \( l_i, \ i=0,1,\ldots, m \) are specified by the designer, and the only unknowns are the output weights. As shown in [12], [15], [18], [21]-[22] and the references therein, different basis functions can be used to satisfy assumption 5. If we denote the estimate of the unknown functions, \( f_i^a(x) \) and \( \mu^a(x, y_d) \), by \( \hat{f}_i^a(x, t) \) and \( \hat{\mu}^a(x, y_d, t) \) respectively, we can write

\[
\hat{f}_i^a(x, t) = \sum_{j=1}^{l_i} \hat{\theta}_{ij}(t) g_{ij}(x), \quad i = 1, \ldots, m \tag{18a}
\]

\[
\hat{\mu}^a(x, y_d, t) = \sum_{j=1}^{l_{y_d}} \hat{\theta}_{ij}(t) g_{ij}(x, y_d) \tag{18b}
\]
where $\hat{\theta}_j(t)$ and $\hat{\theta}_{0j}(t)$ are the estimates of $\theta_j$ and $\theta_{0j}$ respectively.

The difference between the estimated and actual parameter values results in the estimation errors

\[
\tilde{f}^a_i(x,t) = \sum_{j=1}^{l_i} \hat{\theta}_j(t) g_{ij}(x) 
\]

\[
\tilde{\mu}^a_i(x,y_d,t) = \sum_{j=1}^{l_j} \hat{\theta}_{0j}(t) g_{0ij}(x,y_d)
\]

where,

\[
\tilde{\theta}_j(t) = \hat{\theta}_j(t) - \theta_j, \quad i=0,1,\ldots,m
\]

are the parameter errors. Given the estimates $\hat{f}_i^a(x,t)$ and $\hat{\mu}_i^a(x,y_d,t)$ we can use the Certainty Equivalence (CE) principle [1] to come up with an initial guess for the control law, i.e.,

\[
u = \frac{S_s}{\tilde{\mu}^a_i(x,y_d,t)} \left[ -k_s \|S\| - S_i^T \tilde{f}^a_i(x,t) - S_i^T v(t) \right]
\]

where $\tilde{f}^a_i(x,t) = [\hat{f}_1^a(x,t) \ldots \hat{f}_m^a(x,t)]^T$ and $k_s > 0$ is a constant chosen by the designer, and design an adaptive law for generating the parameter estimates $\hat{\theta}_j(t)$ and therefore $\hat{f}^a_i(x,t)$ and $\hat{\mu}^a_i(x,y_d,t)$ so that the overall system is stable and the tracking error goes to zero with time. However, the CE control law (21) cannot be used to stabilize the closed loop system for a couple of reasons. First $\hat{f}_i^a(x,t), \hat{\mu}_i^a(x,y_d,t)$ are estimates of the approximate nonlinear functions $f_i^a, \mu_i^a$, and may deviate from the actual functions $f_i, \mu$. Second, it is well known in adaptive control that the estimate $\hat{\mu}_i^a(x,y_d,t)$ cannot be guaranteed to be away from zero for any given time $t$. This implies that $u$ cannot be guaranteed to be bounded uniformly with time. Therefore instead of (21) we propose

\[
u = \frac{\tilde{\mu}^a_i(x,y_d,t) S_i}{(\tilde{\mu}^a_i(x,y_d,t))^2 + \delta_\mu} \left[ -k_s \|S\| - S_i^T \tilde{f}^a_i(x,t) - S_i^T v(t) - \sigma_i \|v(t)\| - \sigma_f \|\tilde{f}^a_i(x,t)\| \right]
\]

\[
S_i = \begin{cases} S / \|S\| & \text{if } \|S\| > \Phi \\ S / \Phi & \text{if } \|S\| \leq \Phi \end{cases}
\]
In (22) and (23) \( \delta_\mu > 0, \sigma_\sigma > 0, \sigma_f > 0, \Phi > 0 \) are small design constants, \( k_s > 0 \) is a constant chosen by the designer. The small linear boundary layer \( \Phi \) is used to smooth out the control discontinuity and avoid possible singularities in calculating \( S_i \). By design, the control law in (22) cannot become singular since \( (\hat{\mu}^a(x, y_d, t))^2 + \delta_\mu \geq \delta_\mu > 0, \forall x, y_d, t \). Therefore, the proposed controller overcomes the difficulty encountered in many adaptive control laws where the identified model becomes uncontrollable at some points of time. It is also interesting to note that \( u \rightarrow 0 \) with the same speed as \( \hat{\mu}^a(x, y_d, t) \rightarrow 0 \). Thus, when the estimate \( \hat{\mu}^a(x, y_d, t) \) approaches zero, the control input remains bounded and also reduces to zero. In other words in such case it is pointless to control what appears to the controller as uncontrollable plant. This design is critical since the potential loss of controllability has been the main drawback of many nonlinear adaptive laws that are based on inverse dynamics. The nonlinear terms \( \sigma_v \|v(t)\|, \sigma_f \|\hat{\mu}^a(x, t)\| \) are used to compensate for the effect of the estimation errors \( e_f, e_\mu \) and effect of the design constant \( \delta_\mu \).

The control law (22) rewritten in the compact form

\[
\mathbf{u} = S_i u_0
\]  

where

\[
u_0 = \frac{\hat{\mu}^a(x, y_d, t)}{(\hat{\mu}^a(x, y_d, t))^2 + \delta_\mu} \left[ -k_s \|\mathbf{S}_i\|^{-1} S_i^T \hat{\mu}^a(x, t) - S_i^T v(t) - \sigma_v \|v(t)\| - \sigma_f \|\hat{\mu}^a(x, t)\| \right]
\]

provides some insight into the action of the controller. The control vector \( \mathbf{u} \) is expressed as a normalized directional vector \( S_i \) scaled by the control effort \( u_0 \). \( S_i \) represents the direction of the error metric \( S \). The control vector \( \mathbf{u} \) can be viewed as apportioning the total control effort \( u_0 \) in different directions. The components, corresponding to large values in \( S_i \), have relatively larger control energy than those components with smaller values. Intuitively, this suggests that the bigger control energy is directed to those \( S_i(t) \) with higher values.
The adaptive laws for generating the estimates \( \hat{\theta}_j(t) \) are as follows:

\[
\dot{\hat{\theta}}_j(t) = k_{f_j} S_i \frac{S_{\phi}}{S_{\phi} + \Phi} g_{ij}(x), \quad i = 1, \ldots, m \tag{26a}
\]

\[
\dot{\hat{\theta}}_j(t) = k_{\mu} S_{\phi} u_{ai} g_{ii}(x, y_d) + \rho k_{\mu} \sigma_{\mu} S_{\phi} \text{sgn}(\mu(x, y_d))[\|u_{\mu}\| + \|\mu\|]g_{ij}(x, y_d) \tag{26b}
\]

where

\[
u' = \frac{1}{(\hat{\mu}^u(x, y_d, t))^2 + \delta_{\mu}} \left[ -k_{\delta} \|S\| - S_1^T \hat{f}^u(x, t) - S_1^T v(t) - \sigma_{\mu} \|v(t)\| - \sigma_{\delta} \|\hat{f}^u(x, t)\| \right] \tag{27}
\]

\[
S_{\phi} = \|S\| - \Phi \text{sat}(\|S\|/\Phi) \tag{28}
\]

\[
\text{sat}(\|S\|/\Phi) = \begin{cases} 1, & \text{if } \|S\| > \Phi \\ \|S\|/\Phi, & \text{if } \|S\| \leq \Phi \end{cases} \tag{29}
\]

sgn(\cdot) is the sign function (\text{sgn}(x)=1, \text{if } x \geq 0 \text{ and sgn}(x)=-1, \text{otherwise}). Then sgn(\mu)=1 \text{ if } \frac{1}{2}(B(x) + B^T(x)) \text{ is positive definite and sgn}(\mu)=-1 \text{ if } \frac{1}{2}(B(x) + B^T(x)) \text{ is negative definite.}

\( \rho \) is a switching function defined as:

\[
\rho = \begin{cases} 1, & \text{if } |\hat{\mu}^u| \leq |\hat{\theta} - \varepsilon_{\mu} - \Delta \\ (|\hat{\theta} - \varepsilon_{\mu} - \Delta - |\hat{\mu}^u|)/\Delta, & \text{if } (|\hat{\theta} - \varepsilon_{\mu} - \Delta| < |\hat{\mu}^u| < (|\hat{\theta} - \varepsilon_{\mu} - \Delta|) \\ 0, & \text{if } |\hat{\mu}^u| \geq |\hat{\theta} - \varepsilon_{\mu} \tag{30}
\)

where \( k_{f_i} > 0, i = 1, \ldots, m \) and \( k_{\mu} > 0 \) are adaptive gains chosen by the designer, \( \sigma_{\mu} > 0 \) is a small design parameter, \( u_{\mu} \) is defined in (25), and \( \Delta > 0 \) is a design parameter used to avoid discontinuity in \( \rho \). A continuous switching function \( \rho \), instead of a discontinuous one, is used to guarantee that the resulting differential equation representing the closed loop system satisfies the conditions for existence and uniqueness of solutions [23]. The constant \( \varepsilon_{\mu} \) is defined in (17b) and represents the upper bound in the approximation of \( \mu(x, y_d) \) with \( \mu^u(x, y_d) \).
Lemma 2: The function $S_\phi$ defined in (28) has the following properties:

$$\dot{S}_\phi = 0, \text{ if } \|s\| \leq \Phi$$

$$\dot{S}_\phi = S_i^T \dot{s}, \text{ if } \|s\| > \Phi$$

Proof: If $\|s\| \leq \Phi$, $S_\phi = 0$, then $\dot{S}_\phi = 0$. If $\|s\| > \Phi$, $S_\phi = \|s\| - \Phi$, then $\dot{S}_\phi = d(\|s\|)/dt = \frac{S_i^T \dot{s}}{\|s\|} = S_i^T \dot{s}$. ■

The properties of the overall control law (22), (26a-b) are described by the following theorem.

Theorem: Consider the system (1), the control law (22) and the adaptive laws (26a-b). Assume that assumptions 1-5 hold. If the lower bound $\bar{b}$ of $\mu(x, y_d)$ satisfies the condition $\bar{b} > \sqrt{\delta_{\mu} + 3\varepsilon_{\mu} + \Delta}$, then for all $y_d \in \Omega_{y_d} \subset \mathbb{R}^r$, $x(0) \in \Omega_x$, where $\Omega_x \subset \Omega \subset \mathbb{R}^r$, and $\tilde{\theta}_0(0) \in \Omega_\theta \subset \mathbb{R}^l$, where $l = l_0 + l_1 + \cdots + l_m$, there exist positive constants $\delta_1^*, \delta_2^*, \delta_3^* < 1$ such that for $k_s > \frac{\sqrt{m\varepsilon_f}}{1 - \delta_i^*}$, $\sigma_v \geq \frac{\delta_i^*}{1 - \delta_i^*}$, $\sigma_f \geq \frac{\delta_i^*}{1 - \delta_i^*}$, and $\sigma_{\mu} \geq \max(\delta_1^*, \delta_2^*)$, all signals in the closed-loop system (1) are bounded and the tracking errors $e_i(t)$ converge to the residual set $\Omega_{e_i} = \{e_i \mid |e_i(t)| \leq \lambda_i^{n_i+1} \Phi \}$, $i=1, \ldots, m$, that can be made small by selecting small $\Phi$.

Proof: Let us consider the following Lyapunov-like function:

$$V(t) = \frac{1}{2} S_{\phi}^2 + \left\{ \frac{1}{2k_f} \sum_{j=1}^{l_f} (\tilde{\theta}_j(t))^2 + \cdots + \frac{1}{2k_m} \sum_{j=1}^{l_m} (\tilde{\theta}_m(t))^2 \right\} + \frac{1}{2k_{\phi}} \sum_{j=1}^{l_1} (\tilde{\theta}_{j1}(t))^2 (32)$$

Using the adaptive laws (26a-b), we can establish that

$$\dot{V}(t) = 0, \text{ if } \|s\| \leq \Phi$$

$$\dot{V}(t) = S_{\phi} \dot{S}_{\phi} + \left\{ \frac{1}{k_f} \sum_{j=1}^{l_f} \tilde{\theta}_j(t) \dot{\tilde{\theta}}_j(t) + \cdots + \frac{1}{k_m} \sum_{j=1}^{l_m} \tilde{\theta}_m(t) \dot{\tilde{\theta}}_m(t) \right\} + \frac{1}{k_{\phi}} \sum_{j=1}^{l_1} \tilde{\theta}_{j1}(t) \dot{\tilde{\theta}}_{j1}(t), \text{ if } \|s\| > \Phi$$

In the following proof, we only consider the region $\|s\| > \Phi$. Rewrite $u_0$ in (25) as:
\[ u_0 = \frac{\hat{\mu}^a(x, y_d, t)}{(\hat{\mu}^a(x, y_d, t))^2 + \delta \mu} \] (35)

\[ \bar{u} = -k_s \|S\| - S_t^T \hat{f}^a(x, t) - S_t^T \nu(t) - \sigma_f \| \hat{f}^a(x, t) \| \] (36)

In view of (35), (36) and using Lemma 2,

\[ \dot{S}_\theta = S_t^T \dot{S} = S_t^T f(x) + S_t^T \nu(t) + S_t^T B(x) \] 

\[ = S_t^T f(x) + S_t^T \nu(t) + \mu(x, y_d) u_0 \] 

\[ = S_t^T f(x) + \frac{\hat{\mu}^a(x, y_d, t)}{(\hat{\mu}^a(x, y_d, t))^2 + \delta \mu} \bar{u} + \{\mu(x, y_d) - \hat{\mu}^a(x, y_d, t)\} u_0 \] 

\[ = S_t^T f(x) + \frac{\delta \mu}{(\hat{\mu}^a(x, y_d, t))^2 + \delta \mu} \bar{u} + \{\mu(x, y_d) - \hat{\mu}^a(x, y_d, t)\} u_0 \] 

\[ = -k_s \|S\| - \sigma_f \| \nu(t) \| - \sigma_f \| \hat{f}^a(x, t) \| + S_t^T \{ f(x) - \hat{f}^a(x, t) \} + \{\mu(x, y_d) - \hat{\mu}^a(x, y_d, t)\} u_0 - \delta \mu u' \] (37)

Using the identities,

\[ f(x) - \hat{f}^a(x, t) = (f(x) - f^a(x)) - (\hat{f}^a(x, t) - f^a(x)) \] 

\[ = d_f(x) - \bar{f}^a(x, t) \] (38a)

\[ \mu(x, y_d) - \hat{\mu}^a(x, y_d, t) = (\mu(x, y_d) - \mu^a(x, y_d)) - (\hat{\mu}^a(x, y_d, t) - \mu^a(x, y_d)) \] 

\[ = d_\mu(x, y_d) - \bar{\mu}^a(x, y_d, t) \] (38b)

where, \( \bar{f}^a(x, t) = [\bar{f}_1^a(x, t) \cdots \bar{f}_m^a(x, t)] \).

It follows that

\[ \dot{S}_\theta = -k_s \|S\| - \sigma_f \| \nu(t) \| - \sigma_f \| \hat{f}^a(x, t) \| - S_t^T \bar{f}^a(x, t) - \bar{\mu}^a(x, y_d, t) u_0 + S_t^T d_f(x) + \{d_\mu(x, y_d) u_0 - \delta \mu u' \} \] (39)

The last term in (39) depends on the approximation error \( d_\mu(x, y_d) \) and the design constant \( \delta \mu \).

Define:

\[ \delta_1 \overset{\text{def}}{=} \frac{\epsilon_\mu}{b - \epsilon_\mu - \Delta} + \frac{\delta \mu}{(b - \epsilon_\mu - \Delta)^2 + \delta \mu} \] (40a)

\[ \delta_2 \overset{\text{def}}{=} \frac{2 \delta \mu}{b - \epsilon_\mu} \] (40b)

\[ \delta_3 \overset{\text{def}}{=} \frac{\epsilon_\mu}{b - \epsilon_\mu} \] (40c)
As shown in Appendix B, the absolute value of the last term in \( \dot{S}_\phi \) can be expressed as:

\[
|d_j(x, y_j)u_0 - \delta_j u_i| \leq \delta_j |\tilde{v}| + \rho \delta_j |\tilde{\mu}^a(x, y_j, t)|u_i| + \rho \delta_j |\tilde{\mu}^a(x, y_j, t)|u_0|
\]  

(41)

Since \( \|s\| = S_\phi + \Phi \), one has

\[
|\tilde{v}| \leq k_5 S_\phi + k_5 \Phi + (1 + \sigma_i) \|v(t)\| + (1 + \sigma_j) \|f^a(x, t)\|
\]  

(42)

Then (41) can be rewritten as

\[
|d_j(x, y_j)u_0 - \delta_j u_i| \leq \delta_j k_5 S_\phi + \delta_j k_5 \Phi + \delta_j (1 + \sigma_i) \|v(t)\| + \delta_j (1 + \sigma_j) \|f^a(x, t)\| + \rho \delta_j |\tilde{\mu}^a(x, y_j, t)|u_i| + \rho \delta_j |\tilde{\mu}^a(x, y_j, t)|u_0|
\]  

(43)

Using (39), the first term in \( \dot{v} \) is,

\[
S_\phi \dot{S}_\phi = -k_5 \|s\| S_\phi - \sigma_i \|v(t)\| S_\phi - \sigma_j \|f^a(x, t)\| S_\phi - S_i^T \tilde{f}^a(x, t) S_\phi \]

\[
+ \tilde{\mu}^a(x, y_j, t)u_0 S_\phi + S_i^T d_j(x) S_\phi + (d_i(x, y_j)u_0 - \delta_j u_i) S_\phi
\]  

(44)

By substituting \( \|s\| = S_\phi + \Phi \) and (43) into (44), we obtain:

\[
S_\phi \dot{S}_\phi \leq -(1 - \delta_i) k_5 S_\phi^2 + \{(1 - \delta_i) k_5 \Phi - \sqrt{m \epsilon} \} S_\phi - (\sigma_i - \delta_i (1 + \sigma_i)) \|v(t)\| S_\phi - (\sigma_j - \delta_j (1 + \sigma_j)) \|f^a(x, t)\| S_\phi
\]

\[
- S_i^T \tilde{f}^a(x, t) S_\phi - \tilde{\mu}^a(x, y_j, t)u_0 S_\phi + \rho \delta_j |\tilde{\mu}^a(x, y_j, t)|u_i| + \rho \delta_j |\tilde{\mu}^a(x, y_j, t)|u_0|
\]  

(45)

In reaching (45), the following inequalities were used:

\[
S_i^T d_j(x) S_\phi \leq \|d_j\| S_\phi \leq \sqrt{m \epsilon} S_\phi
\]  

(46)

Let us now consider the second term of \( \dot{v} \) in (34). Using the adaptive law (26a) we have

\[
\frac{1}{k_{f_i}} \sum_{j=1}^{l_i} \dot{\theta}_{i,j} + \cdots + \frac{1}{k_{f_m}} \sum_{j=1}^{l_m} \dot{\theta}_{m,j} = S_i \tilde{f}^a_i(x, t) S_\phi S_\phi + \Phi + \cdots + S_m \tilde{f}^a_m(x, t) S_\phi S_\phi + \Phi
\]

\[
= S_i^T \tilde{f}^a(x, t) S_\phi S_\phi + \Phi
\]

\[
= S_i^T \tilde{f}^a(x, t) S_\phi
\]  

(47)

where \( \tilde{f}^a_i(x, t) = \tilde{f}^a_i(x, t) - f^a_i(x) \).

Finally, in view of the adaptive law (26b), the last term of \( \dot{v} \) in (34) can be written as:
\[
\frac{1}{k} \sum_{j=1}^k \nabla \delta_j \nabla \delta_j = \frac{1}{k} \sum_{j=1}^k \{ k \mu S_{\delta} u_0 g_{\delta j}(x, y_j) + \rho k \mu \sigma_{\delta} S_{\phi} \text{sgn}(\mu(x, y_j)) ||u_0||_1 \}; \]

\[
= \mu^a(x, y_d) S_{\delta} u_0 - \rho \sigma_{\delta} \mu^a(x, y_d, t) ; \]}

Now, \( \mu^a(x, y_d) = \mu^a(x, y_d, t) - \mu^a(x, y_d) = \{ \mu^a(x, y_d, t) + d_\mu(x, y_d) \} - \mu(x, y_d) \). This together with the fact that \( \rho \neq 0 \) only for \( ||\mu^a(x, y_d, t)|| < b - \varepsilon \mu \) and \( ||\mu^a(x, y_d, t) + d_\mu(x, y_d) || \leq ||\mu^a(x, y_d, t)|| + ||d_\mu(x, y_d)|| \leq b \) implies that for \( \rho \neq 0 \), the sign of \( \rho \mu^a(x, y_d, t) \) is always the opposite sign of \( \mu(x, y_d) \), \( \forall \ t \geq 0 \).

Combining (45), (47), and (48), \( \dot{V} \) can be expressed as:

\[
\dot{V} \leq -(1 - \delta_1) k_\delta S_{\delta}^2 - (1 - \delta_1) k_\delta \Phi - \sqrt{m \varepsilon f} S_{\phi} \|v(t)||S_{\phi} - (1 - \delta_1) \sigma_{\varepsilon} - \delta_1 \|v(t)||S_{\phi} \\
- (1 - \delta_1) \sigma_{\varepsilon} - \delta_1 \|v(t)||S_{\phi} - \rho(\sigma_{\mu} - \delta_2) \|\mu^a(x, y_d, t)||u_0||S_{\phi} - \rho(\sigma_{\mu} - \delta_3) \|\mu^a(x, y_d, t)||u_0||S_{\phi} 
\]

Let us now choose the design constants \( k_\delta \), \( \Phi \), \( \sigma_{\varepsilon} \), \( \sigma_{\mu} \) so that the following inequalities are satisfied.

\[
\delta_1 < 1 \quad (50a) \\
(1 - \delta_1) k_\delta \Phi \geq \sqrt{m \varepsilon f} \quad (50b) \\
(1 - \delta_1) \sigma_{\varepsilon} \geq \delta_1 \quad (50c) \\
(1 - \delta_1) \sigma_{\mu} \geq \delta_1 \quad (50d) \\
\delta_\mu \geq \delta_2 \quad (50e) \\
\delta_\mu \geq \delta_3 \quad (50f)
\]

It follows that

\[
\dot{V} \leq -(1 - \delta_1) k_\delta S_{\phi}^2 < 0 \quad (51)
\]

We now need to establish that such design constants exist to satisfy the inequality (50a-f). In (50a)

\[
\delta_1 = \frac{\varepsilon_{\mu}}{b - \varepsilon_{\mu} - \Delta} + \frac{\delta_{\mu}}{(b - \varepsilon_{\mu} - \Delta)^2 + \delta_{\mu}^2} < 1
\]

Assume that

\[
b > \sqrt{\delta_{\mu} + 3 \varepsilon_{\mu} + \Delta}
\]

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then (50a) is satisfied. This inequality depends on the value of $\bar{b}$ which is a characteristic of the nonlinear system and it is unknown in general. The inequality however suggests that the design constant $\delta_\mu$ has to be chosen relatively small and a sufficient number of nodes in $\mu^a(x, y_j)$ have to be used in order to make $\varepsilon_\mu$ small.

Given (53) we design $k_\delta$, $\Phi$, $\sigma_\varepsilon$, $\sigma_f$, $\sigma_\mu$ to satisfy

$$k_\delta \Phi \geq \sqrt{m \epsilon_f} \frac{1}{1-\delta_1} \quad (54a)$$

$$\sigma_\varepsilon, \sigma_f \geq \frac{\delta_1}{1-\delta_1} \quad (54b)$$

$$\sigma_\mu \geq \max\{\delta_2, \delta_3\} \quad (54c)$$

Given that the above inequalities are satisfied we have

$$\dot{V} = 0, \text{ if } \|S\| \leq \Phi \quad (55a)$$

$$\dot{V} \leq -(1-\delta_1)k_\delta S_\Phi^{2} < 0, \text{ if } \|S\| > \Phi \quad (55b)$$

The results (55a-b) are arrived under assumptions 1 and 5. Since assumptions 1 and 5 only hold on a compact set, i.e., $x \in \Omega$, all states need to remain in this compact set for all $t \geq 0$. Consider the set

$$M = \{x, \tilde{\theta}_g \mid V(t) \leq V_0\} \quad (56)$$

where $V_0 > V(0)$ and $V_0 > \Phi$ is chosen as the largest constant for which $M = \Omega_x \times \Omega_\theta$, where $\Omega_x \subset \Omega$, $\Omega_\theta \subset \mathbb{R}^l$, $l = l_0 + l_1 + \cdots + l_m$. Then for $x(0) \in \Omega_x$ and $\tilde{\theta}_g(0) \in \Omega_\theta$, it follows from (32) and (55a-b) that $V(t)$ is bounded from above by $V_0$ for all $t \geq 0$, which implies that $x \in \Omega_x \subset \Omega$, $\forall t \geq 0$. Hence assumptions 1 and 5 will never be violated.

This condition together with $\dot{V} \leq 0$ implies that $V(t)$ and therefore $S_\phi, \tilde{\theta}_g(t)$ are bounded for all $t \geq 0$, i.e., $S_\phi, \tilde{\theta}_g(t) \in L_\infty$. This in turn implies that $x, u$ are bounded and $V(t)$ has a limit, i.e. $\lim_{t \to \infty} V(t) = V_\infty$. Using the fact that $S_\phi = 0$ for $\|S\| \leq \Phi$ and (55b), we have $\lim_{t \to \infty} \int_0^t S_\phi^2 dt = \frac{V(0) - V_\infty}{(1-\delta_1)k_\delta} < \infty$ which implies that $S_\phi \in L_2$. 


From $S_\alpha, \tilde{S}_\theta(t) \in L_\infty$, it follows that all signals are bounded which implies that $\dot{S}_\alpha \in L_\infty$. From $\dot{S}_\alpha \in L_\infty$ and $S_\alpha \in L_2$ we have $S_\alpha \to 0$ as $t \to \infty$ which implies that $\|S\|$ converges to the residual set $\Omega_S = \{S \mid \|S\| \leq \Phi \}$ [1]. Inside the residual set we also have $|S_i| \leq \Phi$ which in turn implies that $|e_i(t)| \leq \lambda_i^{-\gamma_i} \Phi$, $i = 1, \cdots, m$ [3], [9].

As discussed in the SISO case in [20], the trade-off for incorporating the design constant $\delta_\mu$ and the approximation errors, $\epsilon_f$, $\epsilon_\mu$, into the control law is that the norm of the error metric $\|S\|$, or equivalently the tracking error, converges to a small residual set defined by $\|S\| \leq \Phi$ instead of converging to zero.

Remark 1: In view of (54a), the dead zone width $\Phi$ is an important feature of the proposed robust adaptive controller. Since the tracking error converges to the residual set $|e_i(t)| \leq \lambda_i^{-\gamma_i} \Phi$, $i = 1, \cdots, m$, the accuracy of the tracking depends on the width $\Phi$ of the dead zone. However, from (54a), one has

$$\Phi > \frac{\sqrt{m} \epsilon_f}{k_s (1 - \delta_1)}.$$  

The width $\Phi$ will depend on $\epsilon_f$, $\delta_1$ and $k_s$. For small tracking errors, one needs a large gain, $k_s$, and small $\epsilon_f$, $\delta_1$. Define $\delta_4 = \frac{\Delta}{b - \epsilon_\mu}$, the value of $\delta_1$, rewritten as

$$\delta_1 = \frac{\delta_3 + \delta_2}{1 - \delta_4 + \frac{\delta_2}{2(b - \epsilon_\mu)(1 - \delta_4)^2 + \delta_2}},$$

is a function of $\delta_2$, $\delta_3$, $\delta_4$. Since (50a) requires $\delta_1 < 1$, it follows that $\delta_3 < 1$, $\delta_4 < 1$. A small value of $\delta_1$ can be obtained by keeping $\delta_2$, $\delta_3$, $\delta_4$ small. Furthermore,

$$\delta_2 = \frac{2(\delta_\mu / \bar{b})}{1 - (\epsilon_\mu / \bar{b})}, \delta_3 = \frac{(\epsilon_\mu / \bar{b})}{1 - (\epsilon_\mu / \bar{b})}, \delta_4 = \frac{(\Delta / \bar{b})}{1 - (\epsilon_\mu / \bar{b})},$$

the values of $\delta_2$, $\delta_3$, $\delta_4$ depend on the ratios $\delta_\mu / \bar{b}$, $\epsilon_\mu / \bar{b}$, $\Delta / \bar{b}$. However, small values of $\epsilon_f$, $\epsilon_\mu / \bar{b}$ require a larger number of nodes, which implies higher order neural network. A small value $\delta_\mu / \bar{b}$ may imply a larger control input when $\hat{\mu}^a(x, y_d, t)$ is
small. Another way to reduce the tracking error is to increase the gain, \( k_s \). However, a large gain is undesirable, since it will require a large control input. Thus, there is a tradeoff between the tracking error, the control gain, \( k_s \), the approximation error, \( \varepsilon_f \), and the ratios \( \delta \mu / \tilde{b} \), \( \varepsilon \mu / \tilde{b} \), \( \Delta / \tilde{b} \).

**Remark 2:** The \( \sigma \)-modification term \( \rho k_\mu \sigma \mu S_\sigma \text{sgn}(\mu(x, y_d))(|\mu|+|p|)g_\mu(x, y_d) \) has been incorporated in the adaptive law (26b) to ensure stability and robustness. It is activated only when the estimate \( \dot{\mu}^a \) approaches zero. In fact, it guarantees the convergence of the tracking error even if \( \dot{\mu}^a \) approaches zero since it appears as a negative term in the derivative of Lyapunov-like function. Thereby this special \( \sigma \)-modification ensures robustness whereas the classical \( \sigma \)-modification is to avoid the estimate of parameters to drift to infinity [1].

**Remark 3:** The class of systems described by (1) can be derived from a more general class of nonlinear systems described by:

\[
\begin{align*}
\dot{x} &= p(x(t)) + \sum_{i=1}^{m} g_i(x(t))u_i \\
y_i &= h_i(x) \\
i &= 1, \ldots, m
\end{align*}
\]

in which \( x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n \) and \( p, g, h \) are smooth functions. If the above system is feedback linearizable, it can be reduced to the system (1) as described in [3]. Let:

\[
\begin{align*}
f(x) &= \left[ L_p^i h_1 \cdots L_p^i h_m \right]^T \\
B(x) &= \begin{bmatrix} L_p \left( L_p^{i-1}(h_1) \right) \cdots L_p \left( L_p^{i-1}(h_i) \right) \\
&\vdots \\
&L_p \left( L_p^{i-1}(h_m) \right) \cdots L_p \left( L_p^{i-1}(h_n) \right) \end{bmatrix}
\end{align*}
\]

where, the Lie Derivative expressions \( L_p \) and \( L_p^i \) are defined as:

\[
L_p(h) = \langle dh, p \rangle = \frac{\partial h}{\partial x_1} p_1(x) + \cdots + \frac{\partial h}{\partial x_n} p_n(x)
\]
Here, \( r_i \) is the equivalent linearizability index for output \( y_i \), i.e., one needs to differentiate the output, \( y_i \), \( r_i \) times until one of the control inputs is different from zero. \( r = \sum_{i=1}^{m} r_i \) indicates the relative degree of the nonlinear system. When the relative degree \( r < n \), the internal dynamics of the system have to be stable.

4. SIMULATION RESULTS

We demonstrate the performance of the proposed adaptive control system using a dynamical model of a planar, two-link, articulated robotic manipulator [4], [14]. The dynamics of this robotic system are nonlinear with strong coupling between the two degrees of freedom. The equations of motion in terms of the generalized coordinates \( q_1 \) and \( q_2 \), representing the angular positions of joints 1 and 2 and applied torques \( \tau_1 \) and \( \tau_2 \) at these joints is given by:

\[
\begin{bmatrix}
H_{11} & H_{12} \\
H_{21} & H_{22}
\end{bmatrix}
\begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2
\end{bmatrix}
+
\begin{bmatrix}
-hq_2 \\
-h(\dot{q}_1 + \dot{q}_2)
\end{bmatrix}
\begin{bmatrix}
\ddot{q}_1 \\
\ddot{q}_2
\end{bmatrix}
=
\begin{bmatrix}
\tau_1 \\
\tau_2
\end{bmatrix}
\]  

(59)

where,

\[
H_{11} = a_1 + 2a_3 \cos(q_2) + 2a_4 \sin(q_2)
\]

\[
H_{12} = H_{21} = a_2 + a_3 \cos(q_2) + a_4 \sin(q_2)
\]

\[
H_{22} = a_2
\]

\[
h = a_3 \sin(q_2) - a_4 \cos(q_2)
\]

with

\[
a_1 = I_1 + m_1 l_1^2 + I_e + m_e l_e^2
\]

\[
a_2 = I_e + m_e l_e^2
\]

\[
a_3 = m_e l_e \cos(\delta_e)
\]
\[ a_4 = m_x l_x l_{ee} \sin(\delta_x) \]

The numerical values used for simulation purposes are:
\[ m_1 = 1, \quad l_1 = 1, \quad m_x = 2, \quad \delta_x = 30^\circ, \quad I_i = 0.12, \quad l_{ai} = 0.5, \quad l_{ex} = 0.25, \quad l_{ee} = 0.6 \]

The robot initially is at rest at the position \( q_1 = 0, q_2 = 0 \). It is desired to determine control inputs \( \tau_1(t) \) and \( \tau_2(t) \) such that \( q_1 \) and \( q_2 \) follow a desired trajectory defined by:
\[ q_{d_1}(t) = 30^\circ \cos(2\pi t) \] (60a)
\[ q_{d_2}(t) = 45^\circ \cos(2\pi t) \] (60b)
with the tracking performance defined by:
\[ |e_1| = |q_1 - q_{d_1}| \leq 1.0^\circ; \quad |e_2| = |q_2 - q_{d_2}| \leq 1.0^\circ \] (61)

Since the inertia matrix \( H(q) \) is positive definite, (59) can be written as:
\[
\begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2
\end{bmatrix} = -\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}^{-1} \begin{bmatrix} -h \dot{q}_2 - h(q_1 + \dot{q}_2) \\ h \ddot{q}_1 - 0 \end{bmatrix} + \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}^{-1} \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}
\] (62)

The above equation is in the general nonlinear form of (1) with \( B(x) = H^{-1} \). Because the workspace is a closed set, it is easy to show that both \( H \) and \( H^{-1} \) are uniformly positive definite for all positions \( q \) in the robot’s workspace. In this simulation, the lower bound of the minimum singular value of \( H^{-1} \) is \( \bar{b} = 0.152 \). Assuming a completely unknown nonlinear plant, one layer radial basis neural network was used to approximate the unknown functions \( f \) and \( \mu \). The upper bounds for the approximation errors are estimated using offline training as \( |f_i - f_i^a| \leq 1.0^\circ, \quad i = 1,2 \) and \( |\mu - \mu^a| \leq 0.01, \) i.e., \( \varepsilon_f = 1.0 \) and \( \varepsilon_\mu = 0.01 \). The value of \( \delta_\mu \) is chosen to be 0.002. Using (40a-c), we calculate \( \delta_1 = 0.12, \quad \delta_2 = 0.141, \quad \delta_3 = 0.141 \). Based on (54b-c), we choose \( \sigma_f = 0.15, \quad \sigma_x = 0.15, \quad \sigma_\mu = 0.15 \). Selecting \( k_y = 20 \) and \( \Phi = 0.3 \), such that (54a) is satisfied. Taking \( \lambda_1 = 20, \quad \lambda_2 = 20 \), the performance requirements for tracking errors are satisfied, i.e., \( |e_1| \leq \Phi/\lambda_1 = 0.9^\circ < 1^\circ \) and \( |e_2| \leq \Phi/\lambda_2 = 0.9^\circ < 1^\circ \). In this simulation, the system is assumed initially at rest and the initial conditions for the parameter estimates are taken to be zero, reflecting the fact that the
system is completely unknown. Figures 1, 2 show the simulation results for the tracking errors. Figure 3 shows the action of the switching function $\rho$. 

Fig. 1. Tracking error response of link 1 during the first 2 seconds
The dotted lines indicate the required error bound of ±1 degree

Fig. 2. Tracking error response of link 2 during the first 2 seconds
The dotted lines indicate the required error bound of ±1 degree
The simulation results in Fig. 1,2 demonstrate the theory and calculated steady state error bounds. Let us now change the performance requirements to have instead of (61) the following desired error bounds

\[ e_1 = \varepsilon_1 - q_d \leq 0.5^\circ \; ; \; e_2 = \varepsilon_2 - q_d \leq 0.5^\circ \] (63)

In this case we use the design inequalities (54a-c) to choose \( \lambda_1 = 30 \), \( \lambda_2 = 30 \), \( k_2 = 20 \) and \( \Phi = 0.26 \), leading to \( |e_1| \leq \Phi / \lambda_1 = 0.5^\circ \) and \( |e_2| \leq \Phi / \lambda_2 = 0.5^\circ \). As demonstrated in Fig. 4,5 the algorithm meets the new error bound requirements at steady state. We should also note that as shown Fig. 3,6 the switching function \( \rho \) reaches a steady state in a short period of time after which no more switching takes place.
Fig. 4. Tracking error response of link 1 during the first 2 seconds. The dotted lines indicate the required error bounds ±0.5 degree.

Fig. 5. Tracking error response of link 2 during the first 2 seconds. The dotted lines indicate the required error bounds ±0.5 degree.
5. CONCLUSIONS

In this paper, we consider the control problem of a class of nonlinear MIMO with unknown nonlinearities. The nonlinearities are assumed to be smooth functions and as such can be approximated and estimated on-line using a single layer neural network. A nonlinear robust adaptive control algorithm is designed and analyzed. The controller guarantees closed loop semi-global stability and convergence of the tracking error to a small residual set. The size of the residual set for the tracking error depends solely on design parameters, which can be chosen to meet desired upper bounds for the tracking error. Consequently the proposed methodology provides a design procedure to meet apriori specified performance guarantees for the steady state tracking error.

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APPENDIX

A. Proof of Lemma 1

Since any matrix can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix, $B(x)$ can be written as:

$$B(x) = \frac{B(x) + B^T(x)}{2} + \frac{B(x) - B^T(x)}{2} \quad (A1)$$

Then

$$S^T(x, y_d)B(x)S(x, y_d) = S^T(x, y_d)\left(\frac{B(x) + B^T(x)}{2}\right)S(x, y_d) \quad (A2)$$

because the quadratic form associated with a skew-symmetric matrix is always zero.

If $\|S(x, y_d)\| \neq 0$, let $S_e(x, y_d) = S(x, y_d)/\|S(x, y_d)\|$ be the unit vector. Then

$$S^T(x, y_d)\left(\frac{B(x) + B^T(x)}{2}\right)S(x, y_d) = S_e^T(x, y_d)\left(\frac{B(x) + B^T(x)}{2}\right)S_e(x, y_d)\|S(x, y_d)\|^2 \quad (A3)$$

Since $\frac{1}{2}(B(x) + B^T(x))$ is uniformly symmetric definite, all eigenvectors are orthogonal and all eigenvalues are real. Let $e_1(x), \cdots, e_m(x)$ indicate a set of unit eigenvectors of the span of $\frac{1}{2}(B(x) + B^T(x))$. $e_1(x), \cdots, e_m(x)$ are orthonormal and form a set of basis in $\mathbb{R}^m$. Thus the unit vector $S_e(x, y_d)$ can be expressed as a linear combination of $e_1(x), \cdots, e_m(x)$.

$$S_e(x, y_d) = c_1(x, y_d)e_1(x) + c_2(x, y_d)e_2(x) + \cdots + c_m(x, y_d)e_m(x) \quad (A4)$$

where $c_1(x, y_d), \cdots, c_m(x, y_d)$ are corresponding coefficients and $c_1^2(x, y_d) + c_2^2(x, y_d) + \cdots + c_m^2(x, y_d) = 1$.

Let $\gamma_1(x), \gamma_2(x), \cdots, \gamma_m(x)$ be the eigenvalues of $\frac{1}{2}(B(x) + B^T(x))$. One has

$$S^T(x, y_d)\left(\frac{B(x) + B^T(x)}{2}\right)S(x, y_d) = (c_1^2\gamma_1(x) + c_2^2\gamma_2(x) + \cdots + c_m^2\gamma_m(x))\|S(x, y_d)\|^2$$

$$= c_1^2(x, y_d)\gamma_1(x) + c_2^2(x, y_d)\gamma_2(x) + \cdots + c_m^2(x, y_d)\gamma_m(x)\|S(x, y_d)\|^2 \quad (A5)$$

Where,
\[ \mu(x, y_d) = c_1^2 \gamma_1 + c_2^2 \gamma_2 + \cdots + c_m^2 \gamma_m \]  

(A6)

and,

\[ |\mu(x, y_d)| \geq \min(|\gamma_1|, |\gamma_2|, \ldots, |\gamma_m|) \geq \bar{b} \]  

(A7)

\( \mu(x, y_d) \) is a linear combination of all eigenvalues of the matrix \( \frac{1}{2}(B(x) + B^T(x)) \) and is real for all \( x \).

Let \( x^0, y^0_d \) be these values such that \( \|S(x^0, y^0_d)\| = 0 \), (A5) always holds. However, the value of \( \mu(x^0, y^0_d) \) is arbitrary. Define:

\[
\mu(x^0, y^0_d) \overset{\text{def.}}{=} \lim_{x \to (u^0, y^0_d) \to (0, 0)} \mu(x, y_d) 
\]

(A8)

Note that both \( S(x, y_d), B(x) \) are smooth functions of \( x, y_d \). From (A5), It is easy to verify that \( \mu(x, y_d) \) is a smooth function of \( x, y_d \). Here we omit the proof.

\section*{B. Proof of the inequality (41).}

The last term in (39) can be expressed as:

\[
d^\prime_{\mu}(x, y_d)u_0 - \delta_\mu u' = \rho [d^\prime_{\mu}(x, y_d)u_0 - \delta_\mu u'] + (1 - \rho) [d^\prime_{\mu}(x, y_d)u_0 - \delta_\mu u'] 
\]

(B1)

From the expression of (27), (36), the scalar control function \( u' \) can be written as:

\[
u' = \frac{1}{(\hat{\mu}^a(x, y_d, t))^2 + \delta_\mu} \bar{u}
\]

(B2)

From (B2), we obtain

\[
\{(\hat{\mu}^a(x, y_d, t))^2\}u' + \delta_\mu u' = \bar{u}
\]

(B3)

Since \( \hat{\mu}^a(x, y_d, t)u' = u_0 \), (B3) can be written as

\[
\hat{\mu}^a(x, y_d, t)u_0 = \bar{u} - \delta_\mu u' 
\]

(B4)

Substituting \( \hat{\mu}^a(x, y_d, t) = \mu^a(x, y_d) + \mu^a(x, y_d, t) \) into (B4), we have

\[
\mu^a(x, y_d)u_0 = \bar{u} - \delta_\mu u' - \mu^a(x, y_d, t)u_0 
\]

(B5)

Using (B2), it follows that
\[
\mu^a(x, y_d) u_0 = \left\{ \frac{1 - \delta^a}{(\mu^a(x, y_d), t)^2 + \delta^a} \right\} \bar{\mu} - \bar{\mu}^a(x, y_d, t) u_0 \\
= \frac{\left(\mu^a(x, y_d, t)^2\right)^2}{1 - \mu^a(x, y_d, t)^2} \tilde{u} - \tilde{\mu}^a(x, y_d, t) u_0
\]

Therefore, the scalar control law \(u_0\) can be expressed as:

\[
u_0 = \left\{ \frac{\left(\mu^a(x, y_d, t)^2\right)^2}{1 - \mu^a(x, y_d, t)^2} \tilde{u} - \tilde{\mu}^a(x, y_d, t) u_0 \right\}
\]

(B6)

Using the fact \(\mu^a(x, y_d) = |\mu(x, y_d) - d_\mu(x, y_d)| \geq b - c_\mu\), one has

\[
|\tilde{u}| \leq \frac{1}{\mu^a(x, y_d)} |\tilde{\mu}^a(x, y_d, t)| u_0 \\
\leq \frac{1}{b - c_\mu} |\tilde{\mu}^a(x, y_d, t)| u_0
\]

(B8)

Since (B3) can also be written as

\[
\{[\mu^a(x, y_d) + \tilde{\mu}^a(x, y_d, t)]^2 + \delta^a u' = \bar{\mu}
\]

(B9)

We have

\[
\{[\mu^a(x, y_d)]^2 + (\tilde{\mu}^a(x, y_d, t)]^2 + \delta^a u' = \bar{\mu} - 2\mu^a(x, y_d) \tilde{\mu}^a(x, y_d, t) u'
\]

(B10)

From (B10), \(u'\) can be expressed as

\[
u' = \frac{1}{\mu^a(x, y_d)^2 + (\tilde{\mu}^a(x, y_d, t)]^2 + \delta^a} \bar{\mu} - \frac{2\mu^a(x, y_d)}{[\mu^a(x, y_d)]^2 + (\tilde{\mu}^a(x, y_d, t)]^2 + \delta^a} \tilde{\mu}^a(x, y_d, t) u'
\]

(B11)

Therefore, we have

\[
|\tilde{u}| \leq \frac{1}{[\mu^a(x, y_d)]^2 + (\tilde{\mu}^a(x, y_d, t)]^2 + \delta^a} |\tilde{\mu}^a(x, y_d, t)| u' \]

\[
\leq \frac{1}{(\mu^a(x, y_d)]^2 + \delta^a} |\tilde{\mu}^a(x, y_d, t)| u' \\
\leq \frac{1}{(\mu^a(x, y_d)]^2 + \delta^a} |\tilde{\mu}^a(x, y_d, t)| u'
\]

(B12)

The absolute value of the first term in (B1) can be written in the following form:
\[ |d_\mu (x, y_d)u_0 - \delta_\mu u'| \leq \rho |d_\mu (x, y_d)|u_0| + \rho |\delta_\mu u'| \]  \hspace{1cm} (B13)

Noting that (B8), (B12), and \( |d_\mu (x, y_d)| \leq \epsilon_\mu \), (B13) can be rewritten as

\[
|d_\mu (x, y_d)u_0 - \delta_\mu u'| \leq \rho \left( \frac{\epsilon_\mu}{b - \epsilon_\mu} + \frac{\delta_\mu}{(b - \epsilon_\mu)^2 + \delta_\mu} \right) |u| + \rho \frac{2\delta_\mu}{b - \epsilon_\mu} ||\overset{\mu}{\mu}(x, y_d, t)||u| + \rho \frac{\epsilon_\mu}{b - \epsilon_\mu} ||\overset{\mu}{\mu}(x, y_d, t)||u_0| \leq \rho \left( \frac{\epsilon_\mu}{b - \epsilon_\mu - \Delta} + \frac{\delta_\mu}{(b - \epsilon_\mu - \Delta)^2 + \delta_\mu} \right) |u| + \rho \frac{2\delta_\mu}{b - \epsilon_\mu} ||\overset{\mu}{\mu}(x, y_d, t)||u| + \rho \frac{\epsilon_\mu}{b - \epsilon_\mu} ||\overset{\mu}{\mu}(x, y_d, t)||u_0| \]  \hspace{1cm} (B14)

Also, for the second term in (B1), we obtain:

\[
(1 - \rho)|d_\mu (x, y_d)u_0 - \delta_\mu u'| = (1 - \rho) \frac{d_\mu (x, y_d)|\hat{\mu}^\mu (x, y_d, t)|}{(\hat{\mu}^\mu (x, y_d, t))^2 + \delta_\mu} - (1 - \rho) \frac{\delta_\mu}{(\hat{\mu}^\mu (x, y_d, t))^2 + \delta_\mu} \] \hspace{1cm} (B15)

Using the fact \((1 - \rho) \neq 0\) only if \(|\hat{\mu}^\mu (x, y_d, t)| > b - \epsilon_\mu - \Delta\), we have

\[
|(1 - \rho)|d_\mu (x, y_d)u_0 - \delta_\mu u'| \leq (1 - \rho) \left( \frac{|d_\mu (x, y_d)|}{|\hat{\mu}^\mu (x, y_d, t)|} |u| + (1 - \rho) \frac{\delta_\mu}{(\hat{\mu}^\mu (x, y_d, t))^2 + \delta_\mu} |u| \right) \leq (1 - \rho) \left( \frac{\epsilon_\mu}{b - \epsilon_\mu} + \frac{\delta_\mu}{(b - \epsilon_\mu - \Delta)^2 + \delta_\mu} \right) |u| \] \hspace{1cm} (B16)

Combining the results of (B14) and (B16), it follows that:

\[
|d_\mu (x, y_d)u_0 - \delta_\mu u'| \leq \left( \frac{\epsilon_\mu}{b - \epsilon_\mu - \Delta} + \frac{\delta_\mu}{(b - \epsilon_\mu - \Delta)^2 + \delta_\mu} \right) |u| + \rho \frac{2\delta_\mu}{b - \epsilon_\mu} ||\overset{\mu}{\mu}(x, y_d, t)||u| + \rho \frac{\epsilon_\mu}{b - \epsilon_\mu} ||\overset{\mu}{\mu}(x, y_d, t)||u_0| \] \hspace{1cm} (B17)

where \(\delta_1, \delta_2, \delta_3\) are defined in (40a-c).

**REFERENCES**


