ABSTRACT

This paper considers the robust redesign of a direct adaptive controller applied to a linear time-invariant plant with unmodeled stable dynamics. In order to achieve robustness the adaptive law is modified by retarding the updated parameters whenever they drift to large values and switching off retardation when their norm is less than an arbitrary upper bound. The modified adaptive law guarantees boundedness for any bounded initial conditions and convergence of the tracking error to a bounded residual set whose size depends on the bound for the reference input signal and the characteristics of the unmodeled dynamics. In the regulation case or in the absence of unmodeled dynamics the modified adaptive law guarantees zero tracking error as $t \to \infty$ in addition to boundedness.

1. INTRODUCTION

Recently several attempts have been made to formulate and analyze the adaptive control of plants with unmodeled dynamics and bounded disturbances. In [1-7] it is shown that unmodeled dynamics or even small bounded disturbances can cause most of the present adaptive control schemes to go unstable. These instability phenomena are basically due to two factors: First the strict positive realness property of a certain error transfer function $W(s)$ which is vital for the stability proof, can be destroyed by the unmodeled dynamics. [3,4,6,9]. Second the pure integral action of the adaptation laws used to update the controller parameters, can cause the controller parameters to drift to infinity [1-7] when disturbances or unmodeled dynamics are present and the measurement vector is not persistently exciting.

To avoid instability due to disturbances modifications have been introduced which eliminate the integral action of the adaptive laws. Peterson and Narendra [2] introduced a dead zone in the adaptive law which consists of switching off the adaptation when the tracking error is within a selected zone. The drawback of this method is that several terms which depend on the unknown plant parameters and disturbances are needed to calculate and implement the dead zone. Kreisselmeier and Narendra [10] use the knowledge of an upper bound for the norm of the desired controller vector to retard adaptation whenever the norm of the controller parameter vector exceeds this bound. Ioannou and Kokotovic [4] use a forgetting factor which retards adaptation and eliminates parameter drift caused by disturbances. Egardt [1], Sanson [11] use similar type of modifications to guarantee boundedness in the case of discrete-time adaptive controllers.

When unmodeled fast dynamics are present, global stability cannot be guaranteed by simply eliminating the integral action of the adaptive laws as it was the case with the disturbances. Unmodeled dynamics can destroy the positive realness property of $W(s)$ and may lead to instability. However, even if $W(s)$ is not positive real, the modification introduced by Ioannou and Kokotovic [4,7] guarantees the existence of a large region of attraction from which all signals converge to a bounded residual set. In the discrete-time case, Praly [12] introduced a different modification which consists of slowing down the adaptation by a proper normalization of the signals used in the adaptive laws and by projecting the controller parameters to a bounded set. This modification guarantees global boundedness for small "size" unmodeled dynamics and high signal to noise ratio. Other approaches [13-17] retain the integral action of the adaptation law and use persistent of excitation conditions to guarantee boundedness in the presence of disturbances and local stability in the presence of unmodeled dynamics.

In this paper we analyze the stability properties of a modified adaptive controller applied to a plant with unmodeled dynamics which are assumed to be stable and have a certain structure. We first show that if the size of the unmodeled dynamics is small then the strict positive realness property of $W(s)$ is preserved. The modified adaptive law is then used to show boundedness for any bounded initial conditions and convergence of the tracking error to a small residual set despite the presence of unmodeled dynamics. In the case of regulation or when no unmodeled dynamics are present, the modified adaptive controller guarantee zero residual errors.
2. **PROBLEM STATEMENT**

Consider the linear time invariant plant

\[
\begin{align*}
\dot{x} &= A_0 x + b_0 u \\
\dot{z} &= A_z x + b_z u \\
y &= T_0 x + c_0^T e
\end{align*}
\]  

(2.1) (2.2) (2.3)

where \(x \in \mathbb{R}^n, \dot{z} \in \mathbb{R}^m, u, y \in \mathbb{R}^p\) and \(e\) is a small scalar. System (2.2) represents the unmodeled part of the plant which gives rise to the unmodeled term \(c_0^T e\) appearing in the plant output. When \(c = 0\) the unmodeled system (2.2) is completely isolated and does not interfere with the modeled part of the plant (2.1)-(2.3). The nominal transfer function of (2.1)-(2.3) with \(c = 0\) is defined as

\[
G_0(s) = \frac{c_0^T(s - \lambda_1) \cdots (s - \lambda_n) k_p}{b_0^T s + h_p^T h_p}
\]

(2.4)

is assumed to satisfy the following conditions:

(i) \(c_0^T\) is a monic Hurwitz polynomial of degree \(n \leq n - 1\).

(ii) \(b_0^T\) is a monic Hurwitz polynomial of degree \(n\).

(iii) The sign of \(k_p\) and the values of \(m, n\) are known (without loss of generality we shall assume that \(k_p > 0\)).

The location of the zeros and poles of \(G_0(s)\) as well as the value of \(k_p\) are unknown. For the transfer function

\[
L(s) = \frac{c_0^T(s - \lambda_1) \cdots (s - \lambda_n) b_0^T}{p(s)}
\]

(2.5)

associated with the unmodeled subsystem (2.2) we assume that \(p(s)\) is a Hurwitz polynomial. No assumptions are made about the degrees of \(c_0^T\) and \(p(s)\) and \(L(s)\) is allowed to have both "slow" and "fast" modes.

The adaptive control problem can be stated as follows:

Given the reference model

\[
\begin{align*}
\dot{x}_r(s) &= A_r x_r(s) + b_r u_r(s) \\
\dot{z}_r(s) &= T_r x_r(s) + c_0^T e
\end{align*}
\]

(2.6)

where \(x_r(s), z_r(s)\) are monic Hurwitz polynomials of degree \(m, n\) respectively, \(k_p > 0\) is a constant and \(e\) is a uniformly bounded reference input signal, design an adaptive controller so that the output \(y\) of the plant (2.1)-(2.3) tracks the output \(y_r\) of the reference model (2.6) as well as possible. In the absence of unmodeled dynamics, i.e., when \(c = 0\) the adaptive controller by Narendra and Valavani [10] guarantees that \(y(t) \rightarrow y_r(t)\) as \(t \rightarrow \infty\) and all the signals in the feedback system remain bounded.

When \(c \neq 0\), however, such a result cannot be ascertained and the boundedness of all the signals in the closed loop is questionable. In fact, it was shown in [3,4,5,7] that unmodeled dynamics and even small disturbances can make an adaptive controller, such as that of Narendra and Valavani, unstable. Our objective is to devise an adaptive controller which guarantees the stability of the resulting closed-loop plant despite the presence of the small plant uncertainties rated by the scalar \(e\). By stability we mean that for any bounded initial conditions and any arbitrary, piecewise continuous, uniformly bounded external reference inputs \(r(t)\) all the signals in the closed loop remain bounded and the tracking error converges to a small residual set. The design and stability analysis of such an adaptive controller is given in the following section.

3. **ROBUST ADAPTIVE CONTROLLER**

We consider the case of plants whose dominant parts have \(m = n - 1\) and \(M_i(s)\) is chosen to be strictly positive real. The basic structure of the adaptive controller is summarized below. The input \(u\) and output \(y\) are used to generate the states \(v_1(s), v_2(s)\) as follows:

\[
\begin{align*}
v_1(s) &= Q(s) u(s) \\
v_2(s) &= Q(s) y(s)
\end{align*}
\]

(3.1) (3.2)

where \(Q(s) = [1, s, s^2, \ldots, s^{2n-2}]^T\).

The measurement vector \(w(t)\) is chosen as \(w(t) = [r(t), y_1(t), y_2(t)]^T\) and the control input \(u\) is given by

\[
u = \theta^T_w
\]

(3.3)

where \(\theta(t)\) is the controller parameter vector with \(2n\) adjustable parameters. If \(c = 0\) and the parameters of the dominant plant are known then it can be shown as in [18] that a constant vector \(\theta^*\) can be found such that for \(\theta = \theta^*\) the transfer function \(L(s)\) of the plant together with the controller (3.1) to (3.3) is equal to \(M_i(s)\), the transfer function of the reference model. Since \(\theta^*\) is not known, \(\theta(t)\) is adjusted according to the adaptive law

\[
\dot{\theta} = -\theta_0 - \Gamma \theta^*_0 \\
\Gamma = \tau \Gamma^T > 0
\]

(3.4)

where

\[
\begin{align*}
\phi = \phi_0 \quad \text{if} \quad |\theta| > \phi_0 \\
\phi = 0 \quad \text{if} \quad |\theta| \leq \phi_0
\end{align*}
\]

(3.5)

and \(\phi > 0, \phi_0 \geq \|\theta^*_0\|\) are positive design parameters and \(\Theta_0^* \theta^*_0 - \theta^*_0 \) is the tracking error.

Let us now apply the adaptive controller (3.1)-(3.5) to the plant (2.1)-(2.3) with unmodeled
dynamics. Using * and following [18] it can be shown that the output error $e_1(s)$ satisfies the equation

$$e_1 = W_0(c,s)\omega + c\alpha(c,s)\eta$$  \hspace{1cm} (3.6)

where $\eta = \gamma_0 \ast e^*$

$$W_0(c,s) = \frac{k_0}{\mu} W_0(s) + c \frac{k_0}{\mu} \alpha(c,s)$$

$$\alpha(c,s) = \frac{k_0}{\mu} \eta(s) + c \gamma_0\eta(s)$$  \hspace{1cm} (3.7)

$e_1(s)$ is an Hurwitz polynomial and the degree of $e_1(s)$ is greater than the degree of $e_0(s)$ and $f_0(s)$. It is clear from (3.6) that the effect of the plant unmodeled dynamics is to perturb the strictly positive real (SPR) transfer function $k_0 W_0(s)$ by $c \frac{k_0}{\mu} \alpha(c,s)$ and introduce the disturbance term $c\alpha(c,s)\eta$. In general, the SPR property of $W_0(c,s)$ does not imply that for small $c$, $W_0(c,s)$ is SPR and therefore $\alpha(c,s)$ has stable poles. The SPR property of $W_0(c,s)$ together with the boundedness of the disturbance term $c\alpha(c,s)\eta$ enables us to establish the stability properties of the closed-loop adaptive control system described in (3.4)-(3.6).

**Theorem 1.** If $k_0 W_0(s)$ is SPR, then there exists an $c^* > 0$ such that for each $c \in [0, c^*)$ the transfer function $W_0(c,s)$ given by (3.7) is SPR.

**Proof.** A necessary and sufficient condition for $W_0(c,s)$ to be SPR is that (i) all the poles of $W_0(c,s)$ are in the half-plane $\text{Re}[s] < 0$ and (ii) $\text{Re}[W_0(c,s)] \geq 0$ for all real $u \in [-\infty, \infty]$ [19]. Choose

$$c^* = \min_{u \in [-\infty, \infty]} \left| \frac{e_1(s)}{e_2(s)} \right|$$  \hspace{1cm} (3.8)

Since $e_2(s)$ has no zeros on the imaginary axis and the degree of $e_1(s)$ is higher than the degree of $e_2(s)$ then $c^* > 0$. Hence for each $c \in [0,c^*)$ the zeros of $e_1(s) + c e_2(s)$ are in the half-plane $\text{Re}[s] < 0$ and therefore all the poles of $W_0(c,s)$ are in the half-plane $\text{Re}[s] < 0$ for $c < c^*$. Let $k_0 W_0(s) = \frac{k_0}{\mu} W_0(s)$ and $f_0(s) = k_0 f_0(s)$. Then

$$\text{Re} \left[ \frac{1}{P_0(s)^2} \frac{1}{\left[ f_0(s) + c e_2(s) \right]^2} \right] \left[ \frac{1}{2} \frac{e_1(s)}{f_0(s)} \right] + c \text{Re} \left[ \frac{e_1(s)}{f_0(s)} \right] + c e_2(s) f_0(s)$$

Let

$$g_1(u) = \frac{1}{g_0(u) \gamma_0(u) - \gamma_0(u) g_0(u)} \left[ \frac{e_1(s)}{f_0(s)} \right]$$

$$g_2(u) = \frac{1}{g_0(u) \gamma_0(u) - \gamma_0(u) g_0(u)} \left[ \frac{e_1(s)}{f_0(s)} \right]$$

$$g_3(u) = \frac{1}{g_0(u) \gamma_0(u) - \gamma_0(u) g_0(u)} \left[ \frac{e_1(s)}{f_0(s)} \right]$$

The SPR property of $W_0(s)$ implies that $\text{Re} \left[ g_0(u) \gamma_0(u) - \gamma_0(u) g_0(u) \right] > 0$ for $u \in [-\infty, \infty]$. Hence the functions $g_i(u)$, $i = 1, 2, 3$ are positive for all $u \in [-\infty, \infty]$. In fact

$$g_1(u) > 0 \quad \text{for} \quad u \in [-\infty, \infty]$$

and $i = 1, 2, 3$. Condition (3.13) follows from the fact that $f_0(s)$ has no zeros on the imaginary axis and the degree of the numerator of $g_i(u)$ is greater or equal to its denominator. Set

$$e_1^* = \min_{u \in [-\infty, \infty]} g_1(u)$$

and choose

$$c^* = \min \left[ c \gamma_1, c^2 \gamma_2 \right]$$

Then for each $c \in [0, c^*)$, $\alpha(c,s)$ has stable poles and $W_0(c,s)$ is SPR.

**Theorem 2.** For each $c \in [0, c^*)$ where $c^* > 0$ is defined by (3.15), the plant (2.1) to (2.3) with the controller (3.1)- (3.5) has the following stability properties:

1. All the signals in the closed-loop plant are bounded.
2. There exists a finite constant $c_0 > 0$ such that as $t \to \infty$ the tracking error $e_1(t)$ converges to the residual set

$$\mathcal{R}_0 = \left\{ e_1^* : e_1^* \leq \frac{e_1^*}{c_0} \right\}$$

where $r_0 = \sup_{t \in [0, \infty]} \left| r(t) \right|$.

4. If $c = 0$, in addition to (1) we have $\lim_{t \to \infty} e_1(t) = 0$.  

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Proof. As in [18] a nominal state representation of (3.6) (with no hidden unstable modes) can be obtained, i.e.,
\[ e = A_v e + b_0 u + c_d c \]  
(3.17)
\[ e_t = [1 0 \ldots 0] e + b_0 u \]  
(3.18)
where \( e \) is the state error between the states of the plant together with the controller and the states of a nonminimal representation of the reference model,
\[ \eta_t = [1 - A_v]^{-1} b_0 w + a_0 r \]  
(3.19)
and \( a_0 \) is a disturbance term introduced by \( a(c,s) \) \( r \). Since \( |r(t)| < a_0 \), for some finite positive constant \( a_0 \), then there exists a finite positive constant \( a_0 \) such that
\[ |e_t(t)| < a_0 v_0 \]  
(3.20)
For each \( c \in [0,\infty) \), \( \bar{w}(c,s) \) is SPR hence there exists a \( P(c) = P_1(c) > 0 \) such that
\[ P_1(c) > A_v c \]  
(3.21)
\[ P_1(c) > b_0 \]  
(3.22)
for some vector \( a \), matrix \( b_0 = b_0^T \), \( c > 0 \) and scalar \( r > 0 \). Equation (3.17),(3.18) together with (3.4), (3.5) describe the stability properties of the closed-loop adaptive control system. Choose
\[ V = e^T P e + a_0^2 e^T b_0 \]  
(3.23)
The time derivative of \( V \) along the solution of (3.4), (3.5), (3.17), (3.18) is
\[ \dot{V} = -2e^T P e + 2ce^T P_1 c - 2v_0^2 \]  
(3.24)
Using (3.20) and defining \( \lambda_e = \min(|v(c)|) \) we can write (3.24) as
\[ \dot{V} \leq -\lambda_e^2 e^T b_0 |e| (|e| - \theta_0) - 2v_0^2 (|e| - \theta_0) + e^T P e \]  
(3.25)
where
\[ v_0 = 2a_0 \sup_{c \in [0,\infty]} \frac{P_1(c)}{P_1(c)} \]  
(3.26)
is finite since \( \lambda_e(c) > 0 \) for all \( c \geq 0 \). Since \( \theta_0 > \theta_0 \), \( |e| \) and \( |e| e - e_0 - \theta_0 \) \( v_0 \) whenever \( \lambda_e^2 |e|^2 > v_0^2 e_0^2 \). Hence \( e(t) \) is bounded and there exists a finite positive constant \( e_0 \) such that as \( t \to \infty \), \( e_0(t) \) converges to the residual set \( D_0 \). Furthermore, for \( |e| > e_0 \), \( e = e_0 \) and \( V < 0 \) when \( 0 < |e| \). The error transfer function \( e_0(t) \) converges to the residual set \( D_0 \). Hence there exists a finite positive constant \( V_0 \) such that for \( V > V_0 \), \( V < 0 \) and therefore \( e(t) \) is also bounded. Since \( e(t) \) is bounded and \( |e| \), \( |e| \) is bounded for some finite constant \( k_2 > 0 \), it follows that \( |e| \) and therefore \( u \) are bounded. Hence all the signals in the closed loop are bounded and the proof for part (1) and (3) is complete.
To prove part (3), we set \( r = 0 \) or \( c = 0 \) then
\[ V = -\frac{3}{2} |e|^2 \leq 0 \]  
(3.27)
and part (3) still holds. Set \( W = \frac{3}{2} |e|^2 \). Since \( \bar{W} \) is bounded, \( W \) is uniformly continuous. From (3.27) we have
\[ \lim_{t \to \infty} W = \infty \]  
(3.28)
for any bounded initial conditions which implies that \( \lim_{t \to \infty} |e(t)| = 0 \).

Remark 1. For implementing the adaptive law (3.4), (3.5) an upper bound \( a_0 \) for the norm of the desired parameter vector \( |a| \) is needed. Since \( |a| \) is not known, \( a_0 \) can be overestimated. If, however, \( 0 < a_0 < |a| \), it can be shown that boundedness can still be preserved but the residual error for \( e_1 \) can be larger and non-zero even when \( c = 0 \) or \( r = 0 \).

Remark 2. The unmodeled dynamics considered in (2.17)-(2.18) appear in parallel with the modeled part of the plant. The results of theorem 1 and 2 can be shown to apply to plants with unmodeled dynamics which are in series with the modeled part of the plant, i.e., of the form \( G(c,s) = G_2^{(1+c)} + c(1+c) \), where \( G(s) \), a proper transfer function with stable poles, is the unmodeled part of the plant.

4. Conclusions. In this paper we consider the robust redesign of an adaptive controller in the presence of unmodeled dynamics. We first showed that a wide class of unmodeled dynamics exists for which the SPR property of the error transfer function \( \bar{W} \) can be preserved. We then used a modified adaptive law to counteract the disturbance terms introduced by the unmodeled dynamics and guarantee boundedness for any bounded initial conditions and convergence of the tracking error to a small residual set. The size of this set depends on the characteristics of the unmodeled dynamics and the magnitude of the reference input signal. In the regulation case or in the absence of unmodeled dynamics the modified adaptive law guarantees boundedness and zero residual tracking errors.
5. REFERENCES


