THE THEORY AND DESIGN OF
ROBUST ADAPTIVE CONTROLLERS

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Abstract
A general approach for designing and analyzing robust adaptive control schemes for continuous-time plants is presented. The design approach involves the development of a general robust adaptive law and the use of the certainty equivalence principle to combine it with model reference, pole placement and linear quadratic control structures. The global stability properties and robustness of the adaptive control schemes that are established. The developed theory and design approach can be used to analyze and compare the robustness properties and performance of a wide class of robust adaptive controllers.

Notation
- $a \in \mathbb{R}^n$: $a$ is a finite nonnegative constant.
- $\|x\|$ or $\|x(t)\|$ denotes the Euclidean norm of the vector $x$ at time $t$.
- $f \in L_T$ : $\|f\|$ is bounded over the interval $[t_0, t_0 + T]$ where $f$ is defined.
- $f \in L_{\infty}$, $f$ is uniformly bounded over the interval $[t_0, \infty)$, $\forall t_0 \geq 0$.
- $A(s, \theta_i)$ denotes a polynomial in $s$ whose coefficients form the vector $\theta_i$, i.e., $A(s, \theta_i) = \theta_0 s^{m-1} + \ldots + \theta_i s^{m-i} + \ldots + \theta_m$ and $\theta_i = [\theta_0, \ldots, \theta_m]^T$.
- $\delta_T$ denotes the order of the polynomial $X(s)$.

1. INTRODUCTION
Recently several robust adaptive control schemes have been presented in literature which guarantee global stability and robustness in the presence of unmodeled dynamics and bounded disturbance[1]-[7], parameter variations [8],[9] etc. These schemes are developed using the certainty equivalence principle[11] to combine the model reference control (MRC) or pole placement control (PPC) structures, which could be used if the pa-

rameters of the plant are known, with appropriate robust adaptive laws. More recently[10], we have shown that a similar procedure can be used to design a robust adaptive control scheme based on the linear quadratic control (LQC) structure.

In this paper, we show that robust adaptive control schemes based on MRC, PPC or LQC structure can be developed and analyzed in a unified manner. The procedure for developing such schemes consists of two parts: In the first part we show that the desired parameter vector $\theta$ of the MRC, PPC and LQC structure, which can be calculated when the parameters of the dominant part of the plant are known, satisfies a general time varying equation. We use this equation to define the measured estimation error $e$, which reflects the error in using the estimated parameter $\hat{\theta}(t)$ instead of $\theta$. Then, a general robust adaptive law (GRAL) is developed for updating $\theta(t)$ by considering the minimization of certain cost function which penalizes $e$. The properties of the GRAL are independent of the control law. In the second part we show that the GRAL can be combined with the MRC, PPC and LQC controller structures to develop robust adaptive control schemes. The GRAL consists a design of time-varying control $f(t)$ which for stability and robustness has to satisfy certain sufficient conditions. A wide class of robust adaptive laws can be generated by choosing different functions $f(t)$. We have shown that recently proposed modifications such as the deadzone[2],[3],[5], fixed $\sigma$ and switching $\sigma(t)[7]$ are particular choices for $f(t)$ and therefore can be analyzed in a unified manner.

2. PLANT AND CONTROL SCHEMES
Consider the single-input single-output plant:

$$ y = G(s)x $$

where

$$ G(s) = G_0(s)(1 + \mu_1 \Delta_1(s)) + \mu_0(s) $$

is a strictly proper transfer function; $G_0(s)$ represents the modeled part of the plant; $\mu_1 \Delta_1(s), \mu_0(s)$ is a multiplicative and an additive plant perturbation respec-
tively rated by the positive scalar $\mu$. For the plant perturbation $\Delta_u(s)$, $\Delta_d(s)$, we make the following assumptions:

- (A1) $\Delta_u(s)$ is strictly proper.
- (A2) A lower bound $p_0 > 0$ on the stability margin $\rho > 0$ for which the poles of $\Delta_u(s - p)$, $\Delta_d(s - p)$ are stable is known.

The assumptions about $G_d(s)$, the modeled part of the plant transfer function differ for different control objectives. These assumptions are given in the following subsections together with the control objectives.

2.1. Model Reference Controller

The output $y(t)$ is required to track the output $y_m(t)$ of a reference model given by:

$$y_m(t) = W_m(s)r(t) = \frac{k_m}{D_m(s)}r(t) \quad (2.3)$$

where $r(t)$ is a bounded piecewise continuous input signal, $k_m > 0$ and $D_m(s)$ is a monic Hurwitz polynomial. In order to achieve this objective, the following assumptions are made for the modeled plant transfer function $G_d(s) = \frac{k_d}{D_d(s)}$:

- (M1) $D_d(s)$ is a monic Hurwitz polynomial and $\delta_d \leq n - 1$.
- (M2) $R_d(s)$ is a monic polynomial and $\delta_R = n$.
- (M3) $k_d$ is a constant with known sign.
- (M4) The relative degree $n^*$ $= \delta_R - \delta_d$ is known.
- (M5) An upper bound $\bar{n}$ for $\delta_R$ is known.

With assumptions (M1)-(M5), we can choose $\delta_R = n^*$ and use the following controller structure:

$$u = \frac{\lambda(s)}{\Lambda(s)} (+ g(s,s)\bar{p} + g(s,s))^T \cdot \frac{1}{\Lambda(s)}\Lambda(s) \quad (2.4)$$

where $\lambda(s)$ is an arbitrary monic Hurwitz polynomial and $\delta_{\lambda} = \bar{n} - 1$. $g(s,s)$ is a polynomial in $s$ of order $n - 2$ and $\bar{p} = [\bar{p}_1, \bar{p}_2, \bar{p}_3]^T$ contains the coefficients of $g(s,s)$, respectively and $r^T \bar{p}$ satisfy the following matching condition:

$$\bar{p} L(s) = \Lambda(s)\Lambda(s) = W_m(s) \quad (2.5)$$

If the coefficients of the polynomials $R_d(s), Z_d(s)$ and $k_d$ are known, $k^* = [k_1^*, k_2^*, k_3^*, k_4^*, k_5^*, k_6^*, k_7^*$] can be obtained from (2.5) uniquely [12]. In this case the stability and robustness properties of (2.1) with controller (2.4) are given by the following lemma.

**Lemma 2.1.** There exists a $\mu^* > 0$ such that for $\mu \in [0, \mu^*]$, all the signals in the closed loop system (2.1), (2.4) are bounded for any bounded initial condition. Furthermore, the tracking error $e_i = y - y_m$ converges exponentially to the residual set:

$$D_\delta^* = \{e_i \mid |e_i| \leq \gamma_\mu \} \quad (2.6)$$

where $\gamma_\mu \in R^+$ is proportional to the upper bound of $|r(t)|$.

2.2. Pole Placement Controller (PPC)

The plant output $y(t)$ is required to track a reference signal $y_m(t)$ which satisfies:

$$Q(s)y_m = 0 \quad (2.7)$$

where $Q(s)$ is a polynomial of order $q_0$ with non-repeated roots on the $j\omega$-axis. We make the following assumptions about the modeled part of the plant transfer function:

- (P1) $R_d(s) = s^{q_0} + R(s, s^*)$, $R(s, s^*)$ is a polynomial and $\delta_R = n - 1$.
- (P2) $k_d Z_d(s) = \bar{Z}(s, s^*)$ and $\delta_d \leq n - 1$.
- (P3) $Q(s)R_d(s), k_d Z_d(s)$ are relatively prime.

The control law is given as:

$$u = \frac{\lambda(s)}{\Lambda(s)} P(s)Q(s)\left(\frac{1}{\Lambda(s)}\Lambda(s) - 1\right) L(s) \quad (2.8)$$

where $\Lambda(s)$ is a Hurwitz polynomial with $\delta_{\lambda} = n + q_0 - 1$ and $L(s), P(s)$ are polynomials of degree $n = q_0 - 1$ and $n - 1$, respectively, obtained by solving the Bezout equation:

$$P(s)Q(s)R(s) + k_d L(s) Z(s) = A^*(s) \quad (2.9)$$

where $A^*(s)$ is the desired closed-loop Hurwitz polynomial with $\delta_{A^*} = 2n + q_0 - 1$. Assumption (P3) guarantees that the Bezout equation (2.9) has a unique solution $L(s), P(s)[11]$ and therefore the control law (2.8) can be implemented when the parameters of $G_d(s)$, i.e., $\sigma = [s^*, s^*, \bar{p}]^T$ are known. The robustness properties of PPC (2.3), (2.9) when applied to the full order plant (2.1) are given as follows:

**Lemma 2.2.** There exists a $\mu^* > 0$ such that for $\mu \in [0, \mu^*]$, the closed loop plant (2.1), (2.3) and (2.9) is exponentially stable and the tracking error $e_i = y - y_m$ converges exponentially to zero, i.e.,

$$\lim_{t \to \infty} e_i(t) = 0 \quad (2.10)$$

**Remark 2.1.** It is shown that PPC guarantees the robustness of the closed loop system as well as the convergence of the tracking error to zero. In contrast to the MRC scheme, the PPC scheme does not require the dominant part of the plant to be inversely stable. The trade-off between the MRC and PPC, however, is that
the class of signals which can be tracked in the PPC case is much smaller than those in the MRC case. Furthermore, these signals have to be known a priori in the PPC case.

2.3. Linear Quadratic Controller (LQC)

The linear quadratic controller is originally designed for the regulation problem. However, for the tracking problem, i.e., the output $y$ of the plant is required to track a class of signals $y_m$, which satisfies:

$$ Q(s) y_m = 0 $$

(2.11)

where $Q(s)$ is the same as that in the PPC case, we can convert the tracking problem to a regulation problem since the tracking error $e_t$ satisfies:

$$ R(s) Q(s) e_t = h_y Z(s) Q(s) e_t + \mu (b_y Q(s) Z(s) \Delta y(s) + Q(s) R(s) \Delta y(s)) w. $$

(2.12)

In view of (2.12), the LQC can be designed by minimizing the criterion:

$$ J(T) = \int_{0}^{T} (\delta^T(t) + \lambda \delta^2(t)) dt + e^T(l_0 + T) $$

(2.13)

where $\lambda \in R^+$ under the following assumptions about $G(s)$:

- (L1) $\theta_a = n$ and $\theta_b \leq n - 1$.
- (L2) The numerator of $G(s)$, $G_0(s)$, does not have common unstable factors with $Q(s)$.

Let $(A, b, C)$ be the minimal state space realization of $G(s)$, i.e.,

$$ \frac{\delta}{\delta}(t) = \frac{A}{B} \delta(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} $$

(2.14)

and $a = [a_1, a_2, \ldots, a_{n-1}]^T$, $b = [b_1, b_2, \ldots, b_{n-1}]^T$ are the coefficient vectors of $(R(s) Q(s) - \sigma^2)$, $b_y Q(s) Z(s)$ respectively. If the parameters of the plant are known, i.e., $A, b$ are known, then a state observer can be constructed as:

$$ \dot{\hat{e}} = A \hat{e} + b \hat{y} + K (C \hat{e} - e) $$

(2.15)

where $K$ is the gain matrix of the observer and is chosen such that $A + KC$ is stable. The control law is then given by:

$$ u = \frac{Q_1(s)}{Q(s)} \hat{e} = -L(t) \hat{e} = \lambda B \hat{P}_0(t) $$

(2.16)

where $P_0(t)$ is the solution of the following Riccati equation:

$$ \dot{P}_0 = -A^T P_0(t) - P_0(t) A + P_0(t) \lambda^{-1} L^T P_0(t) - C^T C $$

(2.17)

with the terminal condition $P_0(T) = C^T C$.

Since $(A, b)$ is stabilizable which follows from assumption (L2), we can solve the differential Riccati equation (2.17) off-line and then apply the control law (2.16) to the full order plant (2.1). The robustness properties of the LQC are given by the following lemma:

**Lemma 2.3.** There exists a $\mu^* > 0$ such that for $\mu < 0$, the closed loop plant (2.1), (2.16) with $T = \infty$ is exponentially stable and the tracking error $e_t$ converges exponentially to zero. Furthermore

$$ |J(\infty)| < \gamma \mu; \ \gamma \in R^+ $$

(2.18)

where $J(\infty)$ is the cost given by (2.13) in the case of $\mu = 0$.

The proofs of Lemma 2.1-2.3 are given in [7] and [10].

2.1-2.3 established the robustness properties of the fixed control schemes. The significance of this section is that it shows what can be achieved by MRC, PPC and LQC schemes in an ideal situation where the coefficients of $G(s)$ are exactly known. Our goal in the following sections is to combine the MRC, PPC and LQC with a suitable adaptive law and establish similar robustness and performance results for the case where the coefficients of $G(s)$ are constant but unknown.

3. THE GENERAL ADAPTIVE LAW

The robustness properties for the MRC, PPC and LQC are established in section 2 by assuming that the coefficients of $G(s)$, the modelled part of the plant, are known so that the desired parameter vector $\theta$ can be calculated. If the coefficients of $G(s)$ are unknown, a suitable estimation scheme or adaptive law has to be developed for estimating $\theta$ in such a way that the robustness properties of MRC, PPC and LQC are preserved. A wide class of adaptive laws can be developed by examining the properties of a measured estimation error $e_t$. To obtain a general expression for $e_t$, we use the following lemma:

**Lemma 3.1.** The desired parameter vector $\theta$ of the closed loop MRC, PPC and LQC schemes satisfies the following equation:

$$ \rho_0 \theta = \rho_0 W_1(s) u + \rho_0 W_2(s) y + \rho_0 (y - y_m) + \mu_\eta $$

(3.1)

$$ \zeta = H(s) (u, y, y_m)^T; \ \eta = \Delta(s) \eta $$

(3.2)

where $W_1(s), W_2(s)$ are known proper transfer functions with stable poles; $H(s)$ is a known proper transfer function matrix which is designed to be stable. $\Delta(s)$ is a strictly proper transfer function with the same poles as $\Delta_m(s), \Delta_u(s)$ and $H(s)$; $\rho_0$ is 1 or 0 depending on the scheme and $\mu_\eta \neq 0$ is a constant whose sign is known.

The proof of Lemma 3.1 is given in [7].
Let $\hat{\theta}(t), \hat{\psi}(t)$ be the estimate of $\theta, \psi$ respectively. Then from Lemma 3.1, the measured estimation error $e_1$ can be defined as:

$$e_1 = \psi(t)(\theta^T(t)\zeta - W_1(s)u - W_2(s)\psi) + \rho(t)(\nu - \mu).$$

(3.3)

Since all the signals in the right hand side of (3.3) are available for measurement, $e_1$ is also measurable. Defining the parameter error as: $\phi(t) = \hat{\theta}(t) - \theta \ast, \phi(t) = \hat{\psi}(t) - \psi$, then $e_1$ can be expressed as:

$$e_1 = \psi \xi + \phi \beta \xi + \mu$$

(3.4)

where $\xi = \phi \beta \xi + \mu$ is available for measurement. From estimation theory, the adaptive law for estimating $\phi, \psi$ can be obtained by minimizing the following performance criterion:

$$J_1 = \frac{\beta^T \beta}{1 + \beta^T \zeta + \zeta \beta}$$

(3.5)

with respect to $\phi$ and $\psi$. However, when $\phi \neq 0$, $e_1$ is not bounded by $(1 + \beta^T \zeta + \zeta \beta)$ and therefore $J_1$ cannot be guaranteed to be bounded even when $\phi, \psi$ are bounded. Since the performance criterion has to be bounded, when $\phi, \psi$ are bounded, independent of $\phi, \psi$ of other signals in the system, instead of (3.5), we consider the following cost function:

$$J = \frac{\beta^T \beta}{m}$$

(3.6)

where $m$ is a normalizing signal [1] which has to be designed such that $m/n$ is bounded.

If $m(t)$ is given by:

$$m(t) = -d_0m(t) + \delta(t)(\nu(t) + |y(t)| + 1); \quad m(0) > 0$$

(3.7)

and $H(s)$ is designed so that $H(s - p_0)$ has stable poles, then it can be shown [4, 17] that for $d_0 + \delta(t) > (0, p_0)$, $\zeta/m, m \in L_\infty$ and therefore $J \rightarrow \infty \Rightarrow \phi, \psi \rightarrow \infty$.

Taking the derivative of $J$ with respect to $\phi, \psi$, we have:

$$\frac{\partial J}{\partial \phi} = \frac{n \beta}{m^2}, \quad \frac{\partial J}{\partial \psi} = \frac{n \beta \xi}{m^2}$$

(3.8)

therefore, the adaptive law can be obtained using the gradient technique:

$$\phi = \frac{-\text{sign}(\beta \xi) \xi}{m^2}, \quad \psi = \frac{-\rho \xi}{m^2}$$

(3.9)

where $\Gamma > 0$ and $\gamma \in R^+$. The adaptive law (3.9) guarantees, due to the normalizing signal $m(t)$, that the input to the adaptive law is bounded and the measured estimation error $e_1$ will converge to zero in the absence of unmodeled dynamics. However, the normalizing signal cannot guarantee the boundedness of $\theta, \psi$ in the presence of $\nu$. In fact, it has been shown in several papers [13-15] that even a small bounded disturbance and/or unmodeled dynamics can drive adaptive laws with pure integral action such as the one given by (3.9) unstable. In order to counteract instability, (3.9) is modified as:

$$\dot{\theta} = -\gamma \text{sign}(\beta \xi) \xi + \Gamma \xi, \quad \psi = \frac{-\rho \xi}{m^2}$$

(3.10)

where $f_1, f_2$ are some nonlinear functions to be chosen for stability and robustness.

**Definition 3.1** If $f_1, f_2$ are chosen such that:

- (i) $\theta, \psi \in L_\infty$ for $\mu \in [0, \mu_0]$ with $\mu_0 > 0$.
- (ii) If $\theta, \psi \in L_\infty$, then, $f_1, f_2$ are globally Lipschitz functions of time and $\|f_1\| + \|f_2\| \leq \frac{\mu_0}{\gamma} + \|g(t)\| h_2 \leq \alpha_1(\mu + 1)$.

(3.11)

then, (3.11) is called a robust adaptive law.

Several choices for $f_1, f_2$ are given by the following lemmas:

**Lemma 3.2** If $f_1, f_2$ are chosen as:

$$f_1 = \sigma_1 \theta, \quad f_2 = \sigma_2 \psi$$

(3.12)

where $\sigma_1 = \sigma_2 = 0$ if $\beta(t) \leq \sigma_1 \mu_0$, $\sigma_0$ if $\beta(t) \leq \sigma_0 \mu_0 - 1$, $\sigma_1 \mu_0$ if $\beta(t) \geq \sigma_0 \mu_0$, $\alpha_1(\mu + 1)$.

**Remark 3.1.** Instead of the gradient method, we can use the Newton's method to obtain the least square algorithm:

$$\theta = -\text{sign}(\beta \xi) \xi + \Gamma \xi, \quad \Gamma(t) = \frac{-\Gamma(t) \xi \beta \xi}{m^2}$$

(3.10)

where $\theta = [\theta^T, \psi^T]^T, \Gamma = [\Gamma^T, \xi]^T$ and $\Gamma(t) = \Gamma(t)(t) > 0$.

**Lemma 3.3** If $f_1, f_2$ are chosen as:

$$f_1 = \frac{\tau \theta}{\mu \theta \mu}, \quad f_2 = \tau \psi$$

(3.13)

where $\psi = \theta > 0$ is a design parameter. Then, (3.11) is a robust adaptive law with $\mu = \tau \mu_0$ for some $\gamma \in R^+$. 

**Lemma 3.4** If $f_1, f_2$ are chosen as:

$$f_1 = \sigma \theta, \quad f_2 = \sigma \psi$$

(3.14)
where \( \sigma > 0 \) is a design parameter. Then, (3.11) is a robust adaptive law with \( \bar{\beta} = \gamma \sigma \) for some \( \gamma \in R^+ \).

**Lemma 3.8** If \( f_1, f_2 \) are chosen as:

\[
\begin{align*}
 f_1 &= \lambda_1 \frac{1}{m} y & f_2 &= \lambda_2 \frac{1}{m} y \\
&\enspace \text{where } \lambda_1 > 0 \text{ is a design parameter, Then, (3.11) is a} &\enspace \text{robust adaptive law with } \bar{\beta} = \gamma \lambda_2 \text{ for some } \gamma \in R^+.
\end{align*}
\]

The proofs of Lemma 3.2-3.5 can be found in [7]

### 4. ROBUST ADAPTIVE SCHEMES

In this section, we use the certainty equivalence principle to combine the MBC, PPC, and LQC schemes developed in Section 2 with the general robust adaptive law (3.11) and establish their stability and robustness properties.

#### 4.1. Robust Model Reference Adaptive Controller (RMRAC)

In RMRAC, the control law has the same form as the one that we would use if the parameters of \( G_2(s) \) were known, i.e.,

\[
\begin{align*}
 u = g(s, \theta_1) \frac{1}{\Lambda(s)} u + p(s, \theta_2) \frac{1}{\Lambda(s)} y + \beta \Delta + \omega. \quad (4.1)
\end{align*}
\]

and the adaptive law for updating \( \theta = [\theta_1^T, \theta_2^T, \theta_3, \theta_4]^T \) is given as follows:

\[
\begin{align*}
 \dot{\theta} = \Gamma f_1 - \gamma \frac{\Lambda(s)}{m^2} \Gamma f_2. \quad (4.2)
\end{align*}
\]

\[
\begin{align*}
 \eta = \phi \epsilon \Delta s + \gamma \Delta \theta + \gamma \eta \Delta t. \quad (4.3)
\end{align*}
\]

\[
\begin{align*}
 \zeta = \frac{r^T(s)}{\Lambda(s)} W_0(s) y, \quad \frac{r^T(s)}{\Lambda(s)} W_0(s) y, \quad (4.4)
\end{align*}
\]

\[
\begin{align*}
 m = -m + \delta_1(|u| + |y| + 1), \quad m(0) > 0. \quad (4.5)
\end{align*}
\]

where \( r(s) = [s^{n-1}, \ldots, s, 1]^T \), \( g(s, \theta_1) = \theta_1^T r(s) \) and \( \Lambda(s) \) are designed so that \( \Lambda(s) = \Lambda(s) \Delta(s) \) for some \( \Lambda(s) \Delta(s) \) are Hurwitz polynomials.

The robustness properties of the closed-loop model reference adaptive control system are described by the following theorem:

**Theorem 4.1.** Let \( f_1, f_2 \) be chosen so that (4.2) is a robust adaptive law, then, there exists a \( \mu^* > 0 \) such that for \( \mu \in [0, \mu^*] \), the existence of \( \mu^* > 0 \) is guaranteed such that the model reference adaptive control system is globally stable for \( \mu \in [0, \mu^*] \) and any bounded initial condition. Furthermore, the tracking error \( \epsilon = y - y_m \) belongs to the residual set:

\[
\begin{align*}
 D_e = \left\{ \epsilon_1 | \lim_{t \to \infty} \sup_{u \in u} \int_0^T \epsilon_1^2 dt \leq \beta \epsilon_1 + \epsilon_m \right\} \quad (4.6)
\end{align*}
\]

where \( \beta \epsilon_1 \in R^+ \) and \( \epsilon_m > 0 \) is an arbitrary small number.

**Corollary 4.1** If \( \tilde{\beta} \) can be expressed as \( \tilde{\beta} = h(\mu) \) where \( h(\mu) > 0 \) for \( \mu > 0 \) and \( h(0) = 0 \), then, in the absence of the unmodeled dynamics, i.e., \( \mu = 0 \), the tracking error \( \epsilon_1 \) reduces to zero asymptotically with time.

#### 4.2. Robust Pole Placement Adaptive Control (RPPAC)

In RPPAC, the control law has the same form as the one would be used if the parameters of \( G_2(s) \) were known, i.e.,

\[
\begin{align*}
 (s^2 + \beta(s, \theta_1)) P(s, p, \theta_2) Q(s) + L(s, \theta_3) Z(s, \theta_4) = A^T(s) \quad (4.7)
\end{align*}
\]

\[
\begin{align*}
 P(s, p, \theta_2) Q(s) \frac{1}{\Lambda_1(s)} u = L(s, \theta_3) \frac{1}{\Lambda_1(s)} (y_m - y) \quad (4.8)
\end{align*}
\]

and the adaptive law for updating \( \theta = [\theta_1^T, \theta_2^T, \theta_3, \theta_4]^T \) in (4.7) is given as follows:

\[
\begin{align*}
 \dot{\theta} = -\Gamma f_1 - \gamma \frac{\Lambda(s)}{m^2} \Gamma f_2 \quad (4.9)
\end{align*}
\]

\[
\begin{align*}
 \epsilon_1 = \theta_1^T \zeta - \frac{s^2}{\Lambda_1(s)} y, \quad \zeta = \left[ \begin{array}{c} r(s) \frac{s^2}{\Lambda_1(s)} y \\ r(s) \frac{s^2}{\Lambda_1(s)} y \\ \varepsilon \end{array} \right] \quad (4.10)
\end{align*}
\]

\[
\begin{align*}
 m = -m + \delta_1(|u| + |y| + 1), \quad m(0) > 0. \quad (4.11)
\end{align*}
\]

where \( r(s) = [s^{n-1}, \ldots, s, 1]^T, \Lambda_1(s) \) is chosen so that \( \Lambda_1(s) = \Lambda_1(s) \Delta(s) \) for some \( \Lambda_1(s) \Delta(s) \) is Hurwitz. The stability and robustness properties of the pole placement adaptive controller are given by the following theorem:

**Theorem 4.2**. Let \( f_1, f_2 \) be chosen so that (4.9) be a robust adaptive law, if \( (s^2 + \beta(s, \theta_1), Z(s, \theta_4) \) are relatively prime at each time \( t \), then, there exists a \( \mu^* > 0 \) such that for \( \mu \in [0, \mu^*] \), the existence of \( \mu^* > 0 \) is guaranteed and the closed-loop pole placement adaptive control scheme is globally stable for \( \mu \in [0, \mu^*] \). Furthermore, the tracking error \( \epsilon_1 \) \( D_e \) where

\[
\begin{align*}
 D_e = \left\{ \epsilon_1 | \lim_{t \to \infty} \sup_{u \in u} \int_0^T |\epsilon_1(t)| dt \leq \beta \epsilon_1 + \epsilon_m \right\} \quad (4.12)
\end{align*}
\]

for some \( \beta \epsilon_1 \in R^+ \) and an arbitrary small number \( \epsilon_m \).

**Corollary 4.2** If \( \tilde{\beta} \) can be expressed as \( \tilde{\beta} = h(\mu) \) where \( h(\mu) > 0 \) for \( \mu > 0 \) and \( h(0) = 0 \), then, in the absence of unmodeled dynamics, i.e., \( \mu = 0 \), the tracking error \( \epsilon_1(t) \) reduces to zero asymptotically with time.

#### 4.3. Robust Linear Quadratic Controller (RLQAC)

In the RLQAC, the control law is given as:

\[
\begin{align*}
 \dot{x} = A^T + \tilde{B} u + K(t) C \epsilon_1 \quad (4.13)
\end{align*}
\]
\[ u = \frac{Q(x)}{Q(x)} \text{,} \quad \dot{u} = -L(t)\dot{x} \quad L(t) = -\lambda^{-1} \tilde{B}^T(t) P(t) \]

where \( A, \tilde{B} \) are obtained from \( A, \tilde{B} \) by replacing \( a, b \) by \( \tilde{a}, \tilde{b} \) respectively. \( K(t) \) is chosen such that \( \dot{\lambda}(t) + K(t)C = A^T \) for some constant stable matrix \( A^T \), and \( P(t) \) is solved from the following Riccati equation:

\[ \dot{P} = \tilde{A}^T(t)P(t) + P(t)\tilde{A}(t) - P(t)\tilde{B}^T(t) \lambda^{-1} \tilde{B} T(t)P(t) + C^T C \]

with the initial condition \( P(0) = C^T C \). The adaptive law for updating \( \theta = [\theta^T, \theta^T]^T \) is given as follows:

\[ \dot{\theta} = -\gamma_{\theta} \sum \left[ \lambda_x^T(s) \right] \]

\[ c_i = \theta^T \left[ \sum \left[ \lambda_x^T(s) \right] - \sum \left[ \lambda_x^T(s) \right] \right] \]

\[ m = -\sum \left[ \lambda_x^T(s) \right] \]

where \( \lambda_x^T(s) = [s^{x-m+1} + \sum_{x=1}^{m} s^{x-1}] \). The stability and robustness properties of the LQAC are given by the following theorem:

**Theorem 4.3** Let \( f \) be chosen such that (4.16) be a robust adaptive law. If \( \mathbb{B} \) is asymptotically stable, then there exists a \( \beta^* > 0 \) such that for \( \beta \in [0, \beta^*] \), the existence of \( \mu > 0 \) is guaranteed and the closed loop linear quadratic adaptive controller is globally stable for \( \mu \in [0, \mu^*] \) and any bounded initial condition. Furthermore, the tracking error \( e(t) = y - \mu \) belongs to the residual set:

\[ D_\beta = \left\{ e(t) \mid m \leq \sum \left[ \lambda_x^T(s) \right] \right\} \]

for some \( \beta_i \in \mathbb{R}^n \) and an arbitrary small number \( \epsilon \).

**Corollary 4.4** If \( \beta \) can be expressed as \( \beta = \mu(\mu) \) where \( \lambda(\mu) > 0 \) for \( \mu > 0 \) and \( \lambda(0) = 0 \), then, in the absence of unmodeled dynamics, i.e., \( \mu = 0 \), the tracking error \( e(t) \) reduces to zero asymptotically with time.

The proofs of Theorem 4.1-4.3 and their corollaries are given in [7] and [10].

5. CONCLUSION

In this paper, we develop a general robust adaptive law whose properties are independent of the structure of the controller. We show that a wide class of robust adaptive control schemes can be developed by using the certainty equivalence principle to combine the robust adaptive law with model reference, pole placement and linear quadratic control structures. These robust adaptive control schemes are shown to guarantee global stability in the presence of unmodeled dynamics and residual tracking errors which are small in the mean.

References


