Adaptive Control of Linear Time Varying Systems

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Abstract
A new class of adaptive controllers for linear time varying systems is designed and analyzed using nonlinear design techniques and the certainty-equivalence approach. It is established that this new class of controllers guarantees robustness with respect to unknown but slow parameter variations in the global sense. In addition, asymptotic tracking is achieved when the parameter variations, not necessarily the plant parameters themselves, are fully known. Simulation results are presented to demonstrate the robustness and performance of the new class of controllers.

1 Introduction
A major motivation for using adaptive control is that in many applications the plant parameters vary with time. Existing adaptive controllers for linear time varying (LTV) systems are based on certainty-equivalence in combination with a fixed controller structure whose parameters are computed by solving a Diophantine equation [5, 15, 14]. These controllers are based on the notion that slow time variations of the plant parameters act as a perturbation to the system in the same manner as unmodeled dynamics. Therefore robust adaptive control schemes for linear time invariant (LTI) systems can be used to guarantee signal boundedness and small tracking error in the order of the time variations of the plant parameters. Adaptive controllers that take into account the time varying (TV) nature of the plant and some a priori information about the parameter variations, that allowed the system to be fast TV, are presented in [16, 18]. These traditional adaptive control schemes for LTV systems and the associated design and analysis tools are summarized in [17]. These controllers, like those designed for LTI systems, can not guarantee good transient behavior [19, 11], and generally cannot be extended to nonlinear systems.

Recently, there has been several successful adaptive control designs for a class of nonlinear systems [12, 3, 13, 9]. These controllers use nonlinear design tools, notably integrator backstepping, nonlinear damping [4], tuning [6], and control terms [2]. Based on the same design tools, a new class of adaptive controllers was proposed for LTI systems [8, 7]. In the absence of modeling uncertainties, these controllers achieve global boundedness, asymptotic tracking, and most importantly, arbitrarily improved transient perfor-

formance [10]. These controllers have later on been modified so that they can tolerate a class of modeling uncertainties, especially high frequency unmodeled dynamics, in a global sense [20, 21]. Due to their superior performance improvement over the traditional ones, this new class of adaptive controllers deserves further investigation.

In this paper, we design a new class of adaptive controllers using integrator backstepping and nonlinear damping together with the certainty-equivalence approach. As in [20], we abandon the tuning design, which introduces higher order nonlinearities, and is unfavorable to robustness [23]. We show that the proposed controllers are robust with respect to the unknown but slow parameter variations. The advantage of the proposed new class of adaptive controllers as applied to LTV plants are the following: First, only the unknown plant parameters are estimated and are required to be slowly TV. The known plant parameters are allowed to have any finite speed of variations. Second, performance bounds can be developed and used to choose certain design parameters for improved performance.

Notations The following notation is used throughout the paper, unless otherwise stated.

\[ f_i : \text{ the } i\text{th element of vector } f \]
\[ c_i : \text{ the } i\text{th coordinate column vector in } \mathbb{R}^n \]
\[ \|f(\cdot)\|_{\rho, t} : \text{ the exponentially weighted } \mathcal{L}_{\rho, t} \text{ norm} \]
\[ \|f(\cdot)\|_{\rho, t} = \left( \int_0^t e^{-\rho(t-s)} |f(s)|^\rho ds \right)^{1/\rho}, \rho \geq 1 \]
\[ I_n : \text{ the } n \times n \text{ identity matrix (or } F \text{ for short)} \]
\[ \gamma : \text{ any exponentially decaying scalar/} \]

2 Problem Statement
Consider the LTV plant

\[ R_l(s,t)y = Z_l(s,t)u \]

(1)

where \( Z_l, R_l \) are the polynomial differential operators (PDO) [15, 17] defined as

\[ R_l(s,t) = s^n + s^{n-1} a_{n-1}(t) + \cdots + a_1(t) + a_0(t) \]
\[ Z_l(s,t) = s^m b_m(t) + \cdots + b_1(t) + b_0(t) \]

The plant can also be represented in the following observable canonical form [15, 17]

\[
\begin{cases}
\dot{x}(t) = A x(t) + a(t) x_1(t) + b(t) u(t) \\
y(t) = x_1(t)
\end{cases}
\]

(2)

where \( x(t) = (x_1(t) \ldots x_n(t))^T \) represents the state, \( a(t) = (a_{n-1}(t) \ldots a_0(t))^T \) and \( b(t) = (0 \ldots b_m(t) \ldots b_0(t))^T \) are the parameter vectors, and \( A = \begin{bmatrix} 0 & I_{n-1} \\ 0 & 0 \end{bmatrix} \).
We want to design an output feedback control law so that the closed-loop system is uniformly stable, and the plant output $y$ tracks a reference signal $y_r$ as close as possible.

We make the following standard assumptions about the plant and the reference signal:

(A1) The plant parameters $a_i(t), b_i(t)$ are unknown smooth time functions which are bounded and have bounded derivatives.

(A2) The PDO's $Z_i(t, s), R_i(t, s)$ are strongly coprime for all $t > 0$, with known orders $m$ and $n > m$.

(A3) The sign of the high frequency gain $b_m$ is known and constant, and a lower bound of $\bar{b} \leq |b_m|$ is known.

(A4) The operator $Z_i(s, t)^{-1}$ is exponentially stable, i.e., the transition matrix $\Phi(t, \tau)$ of the system $Z_i(t, s) = u$ satisfies $\|\Phi(t, \tau)\| \leq c_0 e^{-\gamma_0(t-\tau)}$ for some constants $c_0, \gamma_0 > 0$.

(A5) The reference signal $y_r$ and its first $\rho$ derivatives are known and bounded.

3 Structured Parameter Variations and State Estimation

In this section, we consider the structured parameter variations [17] and show how we can use the knowledge of the fast TV components of the parameters to construct a state estimator.

Assume the plant parameters $a_i(t), b_i(t)$ have the following known structure:

$$\begin{bmatrix}
-a(t) \\
 b(t)
\end{bmatrix} = \Psi(t) \dot{\theta} + \theta_0(t) = \begin{bmatrix}
\Psi_a(t) \\
\Psi_b(t)
\end{bmatrix} \dot{\theta} + \begin{bmatrix}
\theta_{a_0} \\
\theta_{b_0}
\end{bmatrix}$$

(3)

where $\Psi_a \in \mathbb{R}^{n \times N}, \Psi_b \in \mathbb{R}^{n \times N}$ form the decomposition of $\Psi \in \mathbb{R}^{n \times 1}$ which is a matrix of known time functions, $\theta(t) \in \mathbb{R}^{N \times 1}$ is a new parameter vector that is not known, $\theta_0(t) \in \mathbb{R}^{n \times 1}$ is a known parameter vector which can be decomposed to $\theta_{a_0}(t), \theta_{b_0}(t) \in \mathbb{R}^{n \times 1}$. Notice that the leading $n - m + 1$ rows of $\Psi_b, \theta_{b_0}$ are zeros. From (3), we have

$$a_i(t) = -\Psi_a(t) \theta_i(t) - \theta_{a_0}, \quad b_i(t) = \Psi_b(t) \theta_i(t) + \theta_{b_0}$$

(4)

We construct the following filters

$$\begin{align*}
\dot{\xi} &= A_0 \xi + \Psi_a y \\
\dot{V} &= A_0 V + \Psi_b u \\
\dot{\xi} &= A_0 \xi + (k + \theta_{b_0}) y + \theta_{b_0} u
\end{align*}$$

(5)

where $\xi, V \in \mathbb{R}^{n \times N}, k = (k_1, k_2, \ldots, k_n)^T$ is such that $A_0 = A - k e_1^T$ is exponentially stable with stability margin $\gamma_0$, i.e., $A_0 - \gamma_0 I$ is non-singular. Consider the following virtual state observer

$$\ddot{\xi} = \xi + \Xi + V \dot{\theta}$$

(6)

We can verify that the observation error $\ddot{\xi} = x - \dot{x}$ satisfies

$$\ddot{x} = A_0 \dot{x} - (\Xi + V \dot{\theta})$$

(7)

When $\dot{\theta} = 0, \ddot{x} \rightarrow 0$ as $t \rightarrow \infty$. Hence (6) is a true state observer for (2) when the parameter vector $\theta$ is known and constant. If $\dot{\theta}$ is not constant, then the observation error $\ddot{x}$ is non-vanishing and is represented in the following transfer function form

$$\ddot{x} = -(sI - A_0)^{-1}(\Xi + V \dot{\theta}) + c_1$$

(8)

Using (8), we obtain the following plant parameterization

$$\ddot{y} = w_0 + \omega_0 T \ddot{x} + \ddot{x}_2$$

(9)

where

$$\omega = \Psi_{a_1} y + \Xi_0 + V_2, \quad w_0 = \theta_{a_0} y + \xi_2$$

$$\ddot{x}_2 = \Psi_{a_2} \theta(t) + \xi_2, \quad V_2(s) = \theta_{b_0} (sI - A_0)^{-1}$$

(10)

The parameterization (9) appears to be the same form as in the LTI case [2] except for the $\theta$ term in $\ddot{x}_2$, and is suitable for applying the adaptive backstepping design.

Remark Note that (3) covers the general case including the fully structured, unstructured and known parameter cases. If the parameters are unstructured, then we simply have $\Psi(t) = I_{2n}$. If the parameters are fully structured, then $\theta$ is constant but unknown. The case where $\theta \equiv 0$ corresponds to the known parameter case.

4 Adaptive Backstepping Controller

We assume that the unstructured parameters are slowly TV such that the following holds

$$\dot{\theta} = c_0 \theta$$

(11)

where $c_0 = c_{0n} + c_{0m}$. The parameter vector $\theta$ is differentiable with respect to time and satisfies $\sqrt{\|\dot{\theta}\|} \leq \mathcal{L}(\mu) \cap \mathcal{L}_\infty$, i.e., for some $c > 0$

$$\int_0^t |\dot{\theta}(\tau)| d\tau \leq c + \mu T, \quad \forall t, T \geq 0$$

4.1 Certainty-equivalence control law

Let us assume that $\Xi = \begin{bmatrix} \Xi_1 \\
\Xi_2 \\
\Xi_3 \\
\vdots \\
\Xi_m \end{bmatrix}, \quad V = \begin{bmatrix} V_1 \\
V_2 \\
V_3 \\
\vdots \\
V_m \end{bmatrix}, \quad \Psi_a = \begin{bmatrix} \Psi_{a_1} \\
\Psi_{a_2} \\
\vdots \\
\Psi_{a_m} \end{bmatrix}, \quad \Psi_b = \begin{bmatrix} \Psi_{b_1} \\
\Psi_{b_2} \\
\vdots \\
\Psi_{b_m} \end{bmatrix}$

(12)

The controller design follows the same procedure as in the known parameter case. The idea is to recursively treat $V_i, \theta$ as virtual control, and apply the backstepping procedure using certainty-equivalence, i.e., replacing the unknown parameter vector $\theta$ with its on-line estimate $\hat{\theta}$. The design steps are as follows:

$$z_1 = y - y_r$$

$$\alpha_1 = -c_1 z_1 - d_1 z_1 - w_0 + (\Psi_{a_1} y + \Xi_2) \dot{\theta} + \dot{y}_r$$

$$z_i = V_i^T \theta - \alpha_{i-1}$$

$$\alpha_{i-1} = z_{i-1} - c_i z_i - d_i z_i - w_0 + (\Psi_{a_i} y + \Xi_2) \dot{\theta} + \frac{\alpha_{i-1}}{\theta_{b_0}} (w_0 + \omega_0 \ddot{\theta})$$

$$+ \sum_{j=1}^{i-1} \frac{\alpha_{j-1}}{\theta_{b_0}} (-k_j \xi_j + \xi_j + (k + \theta_{b_0}) y)$$

$$+ \sum_{j=1}^{i-1} \frac{\alpha_{j-1}}{\theta_{b_0}^2} (-k_j \Xi_1 + \Xi_2 + \Psi_{a_j} y) + \sum_{j=0}^{i-1} \frac{\alpha_{j-1}}{\theta_{b_0}^2} (y_j + 1) + \sum_{j=0}^{i-1} \frac{\alpha_{j-1}}{\theta_{b_0}^2} (y_j + 1) + \sum_{j=0}^{i-1} \frac{\alpha_{j-1}}{\theta_{b_0}^2} (y_j + 1)$$

$$\dot{s}_i = \left(\frac{\alpha_{i-1}}{\theta_{b_0}^2}\right)^2 + \|\Psi_{a_i} y + \Xi_2 + \|\Psi_{a_i} y + \Xi_2\|^2 + \|\Psi_{a_i} y + \Xi_2\|^2 + \|\Psi_{a_i} y + \Xi_2\|^2 + \|\Psi_{a_i} y + \Xi_2\|^2$$

(13)
where $\| \cdot \|_2^2$ denotes the sum of squares of all the elements of a matrix. In step $\rho$, the control $u$ appears in the form of $\Psi_{\theta, \bar{\theta}, \bar{\theta}}^T b_m u = b_m u$, therefore the control law can be chosen as

$$
u = \begin{cases} 
\frac{a_{\nu}}{b_m} \frac{\partial \nu}{\partial \theta} & \text{if } m = 0 \\
\frac{a_{\nu}}{b_m} \frac{\partial \nu}{\partial \theta} & \text{if } m \geq 1
\end{cases}$$

(14)

Note that for the control law (14) to exist, the adaptive law must assure that $|\Psi_{\theta, \bar{\theta}, \bar{\theta}}^T b_m| = |b_m(t)| \geq b, \forall t \geq 0$.

With control law (14), the underlying error system is

$$
\dot{z} = A_z(z, \theta, t)z + b_z(z, \theta, t)(\bar{\theta}^T \omega + \hat{\theta}) + D_z(z, \hat{\theta}, t)\hat{\theta}
$$

(15)

where $z = (z_1, z_2, \ldots, z_p)^T$ and

$$
A_z = \begin{bmatrix}
-c_1 - d_1 & 1 & \cdots & 0 \\
-1 - c_2 - d_2 s_2^2 & -1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & -1 - c_p - d_p s_p^2
\end{bmatrix}
$$

$$
b_z = \begin{bmatrix}
k_1 & k_2 & k_3 & \cdots & k_p
\end{bmatrix}^T, \quad k_i = -\frac{b_{\nu,i}}{b_{\nu}} - \frac{\partial \nu}{\partial \theta},
$$

$$
D_z = \begin{bmatrix}
l_1 & l_2 & l_3 & \cdots & l_p
\end{bmatrix}^T, \quad l_i = -\frac{b_{\nu,i}}{b_{\nu}} + \nu
$$

(16)

4.2 Adaptive law with auxiliary filter

The adaptive law is based on the idea of introducing an auxiliary filter to counteract the effect of $\bar{\theta}$ term in the $z$ equation, therefore ending up with a new error system that is suitable for synthesizing adaptive law based on a Lyapunov function.

We define the following auxiliary filter:

$$
\dot{\phi} = A_z \phi - D_z \hat{\theta}
$$

(17)

and auxiliary error signal

$$
\epsilon_1 = z + \phi
$$

(18)

Then the error signal $\epsilon_1$ satisfies the following equation

$$
\dot{\epsilon}_1 = A_z \epsilon_1 + b_z (\bar{\theta}^T \omega + \hat{\theta})
$$

(19)

Since $\hat{\theta}$ is not guaranteed to be bounded, we introduce the following normalizing signal

$$
m_s = \sqrt{1 + \delta_1 (\|y(t)\|_{\dot{m}_s}^2 + \|y(t)\|_{\dot{m}_s}^2)}
$$

(20)

where $0 < \delta_1 < 2 \min \{\gamma_2, c_1\}$ and $\delta_1 > 0$ are design constants. Some important properties of $m_s$ have been established in [20, 22] and are listed below:

Lemma 4.1

$$
\frac{n_s}{m_s}, \frac{d_s}{m_s}, \frac{\dot{m}_s}{m_s}, \frac{\dot{m}_s}{m_s} \in \mathbb{L}_\infty
$$

(21)

$$
\frac{n_s}{m_s} \leq c_0 |\hat{\theta}|^2 + |\dot{\theta}|, \quad \forall \theta \in \{0, \delta_0\}
$$

(22)

$$
\frac{m_s}{m_s} \geq -\frac{\delta_0}{2}
$$

(23)

Now define the normalized estimation error

$$
\epsilon = \frac{\epsilon_1}{m_s}
$$

(24)

which can be shown to satisfy

$$
\dot{\epsilon} = (A_z - \frac{\dot{m}_s}{m_s}) \epsilon + b_z (\bar{\theta}^T \omega + \hat{\theta})
$$

(25)

By considering (25), (23) and the following Lyapunov function

$$
\nu_{\nu} = \frac{1}{2} \dot{\epsilon}^T \epsilon + \hat{\theta}^T \Gamma^{-1} \hat{\theta}
$$

(26)

the following robust adaptive law with a switching $\sigma$-modification and projection for $b_m$ can be chosen:

$$
\dot{\hat{\theta}} = \text{Pr} \left[ \Gamma_{\nu}^T \epsilon + \frac{\nu}{m_s} - \Gamma \dot{\nu} \right]
$$

(27)

where $\text{Pr}$ is the projection operator to project $\hat{\theta}$ along the boundary $|b_m| = \text{sign}(b_m) \Psi_{\theta, \bar{\theta}, \bar{\theta}}^T b_m \geq b$, which is defined as

$$
\text{Pr}[\nu] = \begin{cases}
\nu - \Gamma_{\nu}^T \frac{\nu}{m_s} - \Gamma \nu, & \text{if } |b_m| = b \& \text{sign}(b_m) \Psi_{\theta, \bar{\theta}, \bar{\theta}}^T b_m \leq 0 \\
\nu, & \text{otherwise}
\end{cases}
$$

and $\sigma$ is the leakage coefficient defined as

$$
\sigma = \begin{cases}
\sigma_0 & \text{if } |\theta| > 2M \\
\sigma_0 \left( \frac{|\theta|^2}{M^2} - 1 \right) & \text{if } M < |\theta| < 2M \\
0 & \text{if } |\theta| < M
\end{cases}
$$

$M$ is a known upper bound for $|\theta|$, and $\sigma_0 > 0$ is a small constant.

The stability properties of the adaptive law are described by the following lemma:

Lemma 4.2 Assume that $\sqrt{|\theta|^2} \in S(\mu) \cap L_\infty$, then the adaptive law (27) guarantees that

(i) $|b_m| \geq b$, $\sigma \bar{\theta} \hat{\theta} \leq 0$, $|\dot{\theta}| \leq \frac{\sigma \bar{\theta} \hat{\theta} + \bar{\theta} \hat{\theta}}{M - |\theta|}$, $\forall t \geq 0$.

(ii) $\hat{\theta}, \bar{\theta}, c_1, \hat{\theta} \in \mathbb{L}_\infty$

(iii) $c_1, b_z, \sqrt{\sigma \bar{\theta} \hat{\theta}}, \sigma \hat{\theta}, \bar{\theta}, \frac{\nu}{m_s} \in S(\mu)$. In particular, if $\hat{\theta} \in L_1 \cap L_2 \cap L_\infty$, then $c_1, \frac{\nu}{m_s} \in L_2$ and $\rightarrow 0$ as $t \rightarrow \infty$.

Proof: Using (25), (27) and the properties of the projection operator [1], we compute the derivative of $\nu_\nu$ as

$$
\dot{\nu}_\nu \leq -c_0 |\epsilon|^2 - \frac{d_0}{4} \left( \frac{b_z \epsilon^2}{c_1 \epsilon} \right)^2 + \sigma \bar{\theta}^T \epsilon + \bar{\theta}^T \Gamma^{-1} \hat{\theta} + \frac{\delta_0^2}{d_0 m_s^2}
$$

(28)

where

$$
c_0 = \min_{1 \leq i \leq p} c_i, \quad d_0 = \min_{1 \leq i \leq p} d_i
$$

(29)

In view of (22), we have $\frac{\nu}{m_s} \in S(\mu) \cap L_\infty$, the proof then follows from the same procedure as in [1, 17] by using (26), (28), Lemma 4.1, and the properties of the switching $\sigma$.

4.3 Stability analysis

The closed-loop stability properties of the adaptive control scheme introduced above are described by the following theorem:

Theorem 4.3 The adaptive controller described by the equations (27), (14), and applied to the LTV plant (2), guarantees that there exists a constant $\mu^* > 0$ such that $\forall \mu \in [0, \mu^*]$, all the closed-loop signals are uniformly bounded, and the tracking error is of order $\mu$ in the mean square sense, i.e., for some $c > 0$,

$$
\int_t^{t+T} (y(t) - y_s(t))^2 dt \leq c\mu T + c, \quad \forall t \geq 0
$$

(30)
Proof: The proof is similar to those in [20, 22] and is presented below for the sake of completeness. It has been shown [20] that the control law (14) can be represented as

\[ u = \sum_{i=1}^{p} L_i z_i + \sum_{i=0}^{m} F_i \frac{s}{K(s)} u + \sum_{i=0}^{n} G_i \frac{s}{K(s)} y + \sum_{i=0}^{n} H_i s^i y_i \]

for some \( C \) functions \( L_i, F_i, G_i, H_i \) of \( \hat{\theta} \) and \( t \) only. Substituting in \( u = Z(t,s)^{-1} R(t,s)y \), we obtain

\[ u = \sum_{i=1}^{p} L_i z_i + \left( \sum_{i=0}^{m} F_i \frac{s}{Z_i(s)K(s)} R_i + \sum_{i=0}^{n} G_i \frac{s}{K(s)} \right) y + H_r r \]

where \( r = W_m(s)y_m \), \( W_m(s) \) is some Hurwitz polynomial of order \( \rho \), \( H_r(s) = \sum_{i=1}^{m} H_i W_m(s) \). Let us define a fictitious normalizing signal

\[ m_f = \sqrt{1 + \|u_2\|^2_{\infty} + \|y_2\|^2_{\infty}} \]

where \( \delta \in (0, \delta_0) \) is a constant such that \( Z(t, s - \frac{\delta}{2})^{-1} \) is an exponentially stable operator. Then using \( y = y_2 + z_1 = z_1 + W_m(s)^{-1}r \) and the properties of the \( \|\cdot\|_{\infty} \) norm, we can derive

\[ m_f^2 \leq 1 + c_m \|r\|^2_{\infty} + c_m \|z_1\|^2_{\infty} \]

where \( c_m > 0 \) is a constant dependent on the plant parameters, parameter estimates, and design parameters. Since \( |z| = \frac{\|z_1\|}{m_s} m_s \leq \sqrt{1 + \delta_1} \frac{1}{m_s} \) \( m_f \) and \( \frac{1}{m_s} = \frac{c - \frac{\delta}{2}}{m_s} \in \mathcal{L}_\infty \cap \mathcal{S}(\mu), \) we have

\[ m_f^2 \leq 1 + c_m \|r\|^2_{\infty} + c_m \|z_1\|^2_{\infty} \]

where \( \mu = \sqrt{1 + \delta_1} \frac{1}{m_s} \in \mathcal{S}(\mu) \), i.e., \( \exists \mu, c \geq 0 \) such that

\[ \int_{t}^{t+T} g(T)^2 dT \leq c \mu T + c_c, \quad \forall t, T \geq 0 \]

Applying the Bellman-Gronwall lemma, we obtain

\[ m_f^2 \leq \left( 1 + c_m \|r\|^2_{\infty} \right) \left( 1 + c_m \int_{0}^{t} e^{-(\delta - c_m \mu)(t-t') + c_c} g(T)^2 dT \right) \]

Let \( \mu = \frac{1}{c_m c_c} \), then \( \forall \mu \in [0, \mu^*] \), we have \( m_f \in \mathcal{L}_\infty \). Since \( m_f \) bounds \( m_s^* \), which bounds all the closed-loop signals, it follows that all signals are uniformly bounded. In addition, the tracking error \( z_1 = \frac{z_1}{m_s} \in \mathcal{S}(\mu) \).

Remark As we can see from the above design and analysis, \( m_f \) is the only signal that uses the knowledge of the stability margin of the inverse plant \( Z_i^{-1} \). However, since it is a fictitious one and is not implemented, the actual controller requires no such knowledge. The knowledge of the stability margin of \( Z_i^{-1} \) has been required for the standard certainty-equivalence type adaptive controllers [15] and was later on relaxed using a new controller structure [16].

4.4 Fully structured parameter variations

The case of fully structured parameter variations corresponds to \( \theta \) being constant. We generalize it to the situation that we allow \( \theta \) to be TV with vanishing rate, that is, \( \theta \in \mathcal{L}_1 \cap \mathcal{L}_2 \cap \mathcal{L}_\infty \). For this special class of LTV plant, the proposed adaptive controller (27), (14) has the following properties, as a direct consequence of Theorem 4.3:

Corollary 4.4 If the speed of parameter variations is asymptotically vanishing in the sense that \( \dot{\theta} \in \mathcal{L}_1 \cap \mathcal{L}_2 \cap \mathcal{L}_\infty \), then the adaptive controller (27), (14) guarantees that all the closed-loop signals are uniformly bounded, and the tracking error converges to zero asymptotically.

Due to the transformation (3), the parameter vector \( \theta \) may not reflect the plant parameters themselves, and can contain more or less than \( n+m+1 \) elements, which corresponds to the overparameterized and the underparameterized case, respectively. Corollary 4.4 indicates that when full knowledge of the parameter variations is available, then regardless of the speed of the parameter variations of the plant, global stability is guaranteed, and asymptotic tracking is achieved.

In addition, in the case of fully structured parameter variations, \( z_2 \) is exponentially vanishing. Therefore in this case, we can apply the tuning design of [8, 7] instead of the certainty-equivalence approach by using parameterization (9). The advantage is a guaranteed performance improvement, as in the T1 case [10].

5 Performance of the Adaptive Controller

In this section, we evaluate the mean square error (MSE) performance and \( \mathcal{L}_\infty \) performance of the adaptive controller given in Section 4, in a qualitative manner. The incentive is to provide guidelines for performance improvement for practical purposes. We assume that closed-loop stability condition given by Theorem 4.3 is satisfied so that \( m_s \) is bounded from above by a constant \( \bar{m} = \sup_{t \geq 0} (m_s(t)) \).

Lemma 5.1 (performance of the estimation error)

The adaptive controller (14), (27) guarantees the following performance bounds:

\[ \text{MSE}(e) = \lim_{t \to \infty} \sup_{t_0 \geq 0} \left[ \frac{1}{t_0} \int_{t_0}^{t} |e(t)|^2 dt \right] \leq \frac{c^2}{c_0} \left( \frac{1}{d_0^2} + \frac{1}{\|\hat{\theta}\|^2_{\infty}} \right) \]

\[ \text{MSE}(\dot{e}) = \frac{c}{c_0} \left( \frac{1}{d_0^2} + \frac{\|\hat{\theta}\|^2_{\infty}}{\|\hat{\theta}\|_{\infty}} \right) \]

\[ |e(t)| \leq \frac{c}{c_0} \left( \frac{\|\hat{\theta}\|_{\infty}}{\|\hat{\theta}\|_{\infty}} + \sqrt{\frac{\|\hat{\theta}\|_{\infty}^2}{\|\hat{\theta}\|_{\infty}^2}} \right) + |e(0)| e^{-c_0 t} \]

where \( c \) denotes a generic constant that is non-increasing with respect to \( c_0, d_0 \) and \( \Gamma \).

Proof: Using the fact that \( \sqrt{\|\hat{\theta}\|_{\infty}^2} \in \mathcal{S}(\mu) \cap \mathcal{L}_\infty \), we integrate (28) and get

\[ \int_0^t |e|^2 dt \leq \frac{1}{c_0} (e^{-c_0 t} + c_0^2 e^{-c_0 t} + \max_{t}) \]

\[ \int_0^t |\dot{e}|^2 dt \leq \frac{1}{d_0} (e^{-d_0 t} + c_0^2 e^{-d_0 t} + \max_{t}) \]

where \( \max_{t} \) is the maximum value of \( V \) which is a non-increasing function of \( c_0, d_0, \Gamma \) and \( c \) is independent of \( c_0, d_0, \Gamma \), from which (36), (37) follows. In addition, we have

\[ \frac{1}{2} \int_0^t |e|^2 dt \leq -c_0 |e|^2 + c_0^2 \|\hat{\theta}\|_{\infty}^2 + c \|\hat{\theta}\|_{\infty}^2 \]

from which (38) follows.
Lemma 5.2 (performance of the auxiliary signal)
The auxiliary filter signal $\phi$ defined in (17) satisfies the following performance bounds:

$$MSE(\phi) \leq c(||\Gamma||\sigma) \frac{\mu}{c_0 d_0} \bar{m}, \quad |\phi(t)| \leq c(||\Gamma||\sigma) \frac{\delta}{\sqrt{c_0 d_0}} \bar{m}$$

where $c(||\Gamma||\sigma)$ is some constant that is non-increasing with respect to $c_0, d_0$.

Proof: Using (17), (23), and the definition of $s^2_i$, we obtain:

$$\frac{d |\phi|^2}{dt} \leq -c_0 \frac{|\phi|^2}{m^2_0} + \frac{(\Xi^2 m^2 + \nu^2 m^2 + \xi^2 m^2 + \nu^2 m^2)}{m^2_0} \frac{\delta}{d_0}$$

Since $\Xi, \nu, \xi$ depend on $u, y, k, \Psi$ only, hence $\frac{(\Xi^2 m^2 + \nu^2 m^2 + \xi^2 m^2 + \nu^2 m^2)}{m^2_0} \leq c(||\Gamma||\sigma)$ which is non-increasing with respect to $c_0$ and $d_0$. From (41), (40) follows immediately using (27) and (37).

Note that $c(||\Gamma||\sigma)$ in (40) may be an increasing function of $||\Gamma||$, due to the $\sigma$ term in (27). Otherwise, it is independent of $||\Gamma||$.

Theorem 5.3 (performance of the error signals)
For $\mu \in [0, \mu^*)$, the error signals $z_i$ satisfy the following performance bounds:

$$MSE(z_i) \leq \frac{c}{c_0} \left( \frac{\delta + c(||\Gamma||\sigma)}{m_0} \right) \bar{m}$$

$$|z_i(t)| \leq \frac{c}{\sqrt{c_0}} \left( \frac{\sqrt{\delta} + c(||\Gamma||\sigma)}{\sqrt{m_0}} \right) \bar{m}$$

where $c$ is non-increasing with respect to $c_0, d_0$, and $c(||\Gamma||\sigma)$ is non-increasing with respect to $c_0, d_0$. If $\mu \in [0, \frac{\delta}{2}]$, then

$$MSE(z_i) \leq \frac{c}{c_0} \bar{m}$$

where $c$ is a constant independent of $c_0$.

Proof: The inequalities in (42) follow immediately from Lemma 5.1, 5.2. To prove (43), we consider (35). Using $c_m c_m \mu \leq \frac{\delta}{2}$, we obtain

$$\frac{c^2}{m_0^2} \leq \left( 1 + c_m \sqrt{\delta} \right) \left( 1 + c_m \int_0^t e^{-\frac{\delta}{2}}(t - \tau) + c\gamma(\tau) d\tau \right)$$

Therefore

$$\mu^2 m_0^2 \leq \left( \mu + \frac{\delta}{2c_c} \sqrt{\delta} \right) \left( \mu + \frac{\delta}{2c_c} \right)$$

which is independent of $c_0$. Therefore (43) follows from (42).

The above theorem shows that the $MSE$ or $L_2$ performance can be improved by increasing $c_0, d_0$, and possibly $\Gamma$ for $\mu$ sufficiently small, provided the stability condition is not violated. Unlike [10], arbitrarily good performance improvement is not assured, due to the dependence of the performance bounds on $\bar{m}$. We can reduce $\delta$, however, and obtain a weaker normalization and therefore a smaller $\bar{m}$.

The above performance bounds provide qualitative guidelines for performance improvement. We demonstrate this fact via simulations in the subsequent section.

6 Simulation Results

Let us consider a simple second order system in the following state space form:

$$\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
\alpha + b \sin(10t) & 1 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix} u$$

$$y = x_1$$

where $\alpha = 2 + \sin(0.1t), b = 1$ are assumed to be unknown.

We design a controller so that the plant output $y$ tracks the reference signal $y_r = \sin(t)$ as close as possible.

Figure 1 shows the tracking error $x_1$, parameter estimates $\hat{a}, \hat{b}$, and the control signal $u$ for a period of 100 seconds, when the design parameters are chosen as $c_1 = c_2 = c_3 = 1$, $\Gamma = 1, \delta_0 = 2, \delta_1 = 1$. As we see, the system is stabilized, and the tracking error stays in the neighborhood of 0 for all time.

![Figure 1: $c_1 = c_2 = c_3 = 1, \Gamma = 1, \delta_0 = 2, \delta_1 = 1$](a)

![Figure 2: $c_1 = c_2 = 5, c_3 = 1, \Gamma = 1, \delta_0 = 2, \delta_1 = 1$](a)

Next we amplify the design parameters $c_1 = c_2 = 5$ while the others remain unchanged. We see from Figure 2 that the tracking performance is improved (note difference in scale).

![Figure 2: $c_1 = c_2 = 5, c_3 = 1, \Gamma = 1, \delta_0 = 2, \delta_1 = 1$](b)

By increasing the adaptive gain to $\Gamma = 5$, the tracking performance is further improved as shown in Figure 3.

Finally, we reduce $\delta_1 = 0.1$, and Figure 4 shows a further improvement of performance.

In the above simulations, we demonstrate that the parameter estimates adapt to the parameter changes. Since the only unknown TV parameter $a$ is slowly time varying, stability is guaranteed.
7 Discussions and Conclusions

In this paper, we introduced a new class of adaptive controllers for LTV systems, based on the nonlinear design techniques motivated from [8, 7]. We showed that for the LTV system with partially structured parameter variations, the proposed adaptive controller guarantees global uniform signal boundedness and small tracking error in the order of the speed of the unstructured parameter variations, which is required to be small. When the parameter variations are fully structured, then asymptotic tracking is achieved. In addition, we developed performance bounds for the tracking error that could be used to choose design parameters and improve performance.

References


