Comparison of a Nonlinear Adaptive Controller with Certainty-Equivalence Type Adaptive Controllers

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Abstract

Recently a new class of adaptive controllers for linear time invariant minimum phase plants has been proposed and analyzed. The design of these controllers is based on nonlinear tools and appears to be different from that of the more traditional adaptive controllers developed using the certainty-equivalence approach.

The purpose of this paper is to compare the two classes of adaptive controllers by analyzing their design and stability properties. It is shown that for plants with relative degree 1, the new class of adaptive controllers is a special case of a class of indirect model reference adaptive controllers developed using the certainty-equivalence approach. For relative degree greater than 1, the two classes of controllers are different.

1 Introduction

A new class of adaptive control schemes was recently proposed for minimum phase linear time invariant plants by Kretic et. al. [6, 5]. The design of these schemes is based on nonlinear design tools such as integrator backstepping, nonlinear damping[2, 3], tuning[4] and Lyapunov-like functions. The new design approach is different from the certainty-equivalence (CE) approach used to design most of the more traditional adaptive controllers and leads to more complex adaptive controllers with high order nonlinear terms. The stability and parameter convergence properties of the new designs are, however, very similar to those of the CE based adaptive controllers [1, 7].

The purpose of this paper is to compare the two designs and use the simplicity of the CE based controllers to understand the complexity and behavior of the new controllers where possible. We have shown that for plants with relative degree 1 the new adaptive controller is a special case of a CE based indirect model reference adaptive controller. For relative degree greater than 1 the two classes of controllers differ considerably.

2 Certainty-Equivalence Based Adaptive Controllers

Consider the following Single-Input Single-Output (SISO) plant described by

\[ y = R(x)u = \frac{b_0 + b_1 + \ldots + b_m}{a_0 + a_1 + \ldots + a_m} u \]

(1)

with unknown parameters \(a_0, \ldots, a_m, b_0, \ldots, b_m\), and a known reference model

\[ y_r = W_m(s)y_r = \frac{Z_m(s)}{R_m(s)} \]

(2)

with the following assumptions:

* Plant: \(Z(s)\) is Hurwitz and of degree \(m\); the plant degree \(n\), the plant relative degree \(\rho = n - m\) and the sign of the high frequency gain \(b_m\) are known.

* Reference model: \(Z_m(s)\) and \(R_m(s)\) are monic and Hurwitz; the degree of \(R_m\) is \(\leq n\), and the relative degree of \(W_m(s)\) is equal to \(\rho\). (In particular, one can choose \(Z_m(s) = 1\) and \(R_m(s)\) a Hurwitz polynomial of degree \(\rho\).

The control objective is to choose \(u\) as a function of signals that can be measured to that all signals in the closed loop are bounded and \(y, y_r, \dot{y}, \dot{y}_r \) as close as possible for any bounded piecewise continuous reference input signal \(r(t)\).

If the plant parameters are known, the following control law

\[ u = \frac{V_{h_{n-1}}(s) u + V_{h_{n-1}}(s) y + \dot{y} + \rho \dot{y}}{R_m(s)} \]

(3)

will meet the control objective, where

\[ h_i(s) = (s^{i-1})^T \]

(4)

\[ \phi_1, \phi_2 \in \mathbb{R}^n, \phi_i \in \mathbb{R}^1; K_n(s) = K_m(z_n(s)) = s^n + b_1 s^{n-1} + \ldots + b_m \]

and \(K_m(s)\) is an arbitrary Hurwitz polynomial of degree \(n = \deg(z_n(s))\). The parameters \(\phi_i\) satisfy the Diophantine equation

\[ [K_m(s) - \phi_{n-1}^T h_{n-1}(s) R(s) - \phi_{n-2}^T h_{n-2}(s) + \phi_{n-1} Y(s) Z(s) = \frac{\phi_1}{K_m} Z(s) K_1(s) R_m(s)] \]

(5)

Equation (5) is solved for \(\phi_1\) as follows:

\[ \phi_1 = K_m p \]

(6)

\[ \phi_1^T h_{n-1}(s) = K_n(s) - \rho Z(s) Q(s) \]

(7)

\[ \phi_2^T h_{n-1}(s) + \phi_1 K_n(s) = p Q(s) R(s) - K_n(s) R_m(s) \]

(8)

where \(p = \frac{1}{K_m}\) and \(Q(s)\) is the quotient of \(K_m(s) R_m(s)\).

For the case of \(\rho = 1\), we have \(Q(s) = s + b_1\), and it is clear from (7), (8) that for \(\rho = 1\), the relationship between the plant parameters \(a, b, h, \dot{h}\) and controller parameters \(\phi_1, \phi_2\) is linear; for \(\rho > 1\), however, it can be shown that this mapping becomes highly nonlinear.

When the plant parameters are unknown, the CE approach may be used to form an adaptive controller by replacing the unknown controller parameters \(\phi_i\) in the control law (3), say, with their estimate \(\hat{\phi}_i\), i.e.,

\[ u = \frac{\hat{V}_{h_{n-1}}(s) u + \hat{V}_{h_{n-1}}(s) y + \dot{y} + \rho \dot{y}}{R_m(s)} \]

(9)

The \(\hat{\phi}_i\)'s are generated directly using an adaptive law leading to what is called direct model reference adaptive control (MRAC) or can be calculated using the equations

\[ \phi_1 = K_m p \]

(10)

\[ \phi_1^T h_{n-1}(s) = K_n(s) - \rho Z(s) \hat{Q}(s) \]

(11)

\[ \phi_2^T h_{n-1}(s) + \phi_1 K_n(s) = \hat{Q}(s) \hat{R}(s) - K_n(s) R_m(s) \]

(12)

where \(\hat{R}(s) = s^n + \hat{a}^T h_{n-1}(s), \hat{Z}(s, \hat{\theta}) = \hat{b}^T h_{n}(s)\), and \(\hat{a}, \hat{b}, \hat{\theta}\) the estimates of \(a, b, \theta\), are generated using an adaptive law leading to what is called indirect MRAC. In [1], it has been shown that both direct and indirect MRAC have the same stability properties.

3 The New Adaptive Controller for Relative Degree \(\rho = 1\)

For the new adaptive controller, the problem formulation is slightly different. Instead of requiring a reference model \(W_m(s)\), the reference signal \(y_r\), and its derivatives up to the order \(\rho\) are assumed to be measurable. Of course such an \(y_r\) can be generated using the reference model (2).

The new adaptive controller also uses a series of filter signals generated as

\[ \zeta_i = A_0 \zeta_i + b_0 y, \quad 0 \leq i \leq n - 1 \]

(13)

\[ \xi_i = A_0 \xi_i + c_i + y_r, \quad 0 \leq i \leq m \]

(14)

\[ \dot{\xi}_i = A_0 \dot{\xi}_i + c_i + y_r, \quad 0 \leq i \leq m \]

(15)
For relative degree 1, the new design scheme gives the following control and adaptive laws

\[ u = -\beta_t [\xi_{\omega} + \hat{\theta} \omega - \hat{\gamma} + (c_1 + d_1) z_1] \]  
\[ \hat{\omega} = \gamma \Omega \omega \]  
\[ \hat{\theta} = \gamma \Omega \hat{\theta} \]  
\[ \omega = (\xi_{\omega,2} + \cdots \xi_{\omega,2} \cdots v_{\omega,2} v_{\omega,1} - \xi_{\omega,2} \cdots v_{\omega,1} - \xi_{\omega,1}) \]  
\[ \theta = (-a_{\omega,1} - \cdots \cdots a_0 b_0 \cdots b_{n-1}) \]  
\[ z_1 = y - \hat{y} \]  
where \( \hat{\gamma} \) is the tracking error, and \( c_1, d_1 \) are positive design constants.

Let us first analyze the control law (16) assuming known plant parameters, i.e., \( \hat{\theta} = \theta, \hat{\gamma} = \gamma \).

**Lemma 3.1** The non-adaptive control law (16) with \( \hat{\theta} = \theta, \hat{\gamma} = \gamma \) is equivalent to the control law (9) with a reference model

\[ y_r = W_m(s)r = -\frac{1}{s + c_1 + d_1}r \]  
where the parameters \( \phi \)'s are solved using the algebraic equations (6), (7), (8), and \( r \) is a signal defined as \( r = y - (c_1 + d_1)u \).

**Proof:** The proof follows by substituting the following equalities

\[ \xi_{\omega,3} = \frac{sK_\omega(s) - (s + h_k)s^2y}{K_\omega(s)} \]  
\[ \xi_{\omega,1} = \frac{(s + h_k)s^2}{K_\omega(s)}y - y \]  
\[ \xi_{\omega,2} = \frac{(s + h_k)s}{K_\omega(s)}y, \quad i = 0, \cdots, n - 2 \]  
\[ v_{\omega,3} = \frac{(s + h_k)s^2 - K_\omega(s)}{K_\omega(s)}u \]  
\[ v_{\omega,1} = \frac{(s + h_k)s}{K_\omega(s)}u, \quad i = 0, \cdots, n - 1 \]

into (21) and substituting the \( \phi \)'s in (3) using equations (7), (8).

Let us now consider the case where the plant parameters are unknown. If we use the control law (21) and the CE approach to replace the unknown parameters \( \theta, \hat{\gamma} \) with their estimates \( \hat{\theta}, \hat{\gamma} \), we will obtain an adaptive controller. The following lemma establishes that the CE adaptive controller developed using (21) is exactly the same as the adaptive controller (16)–(18) developed using the nonlinear design approach.

**Lemma 3.2** The adaptive control law (16)–(18) can be developed by applying the CE approach based on the control law (21).

**Proof:** Neglecting the effect of possible non-zero initial conditions in of \( \eta(0) = y_r(0) \), the error system written in the SPR transfer function form is given by

\[ z_1 = \frac{[\beta_\omega]}{s + c_1 + d_1} \]  
where \( z_1 = y - \hat{y} \). Using the Lyaupnov-like function

\[ V = \frac{1}{2} \xi^2 + \frac{1}{2} \eta^2 \]  

the adaptive law (17), (18) follows directly by using the familiar Lyapunov design approach [1].

In view of Lemma 3.2, the adaptive control law (16)–(18) can be classified as an indirect MRAC scheme based on the CE approach with a specific model reference. Therefore the adaptive controller (16)–(18) is a special case of the CE based indirect MRAC for plants with \( \rho \geq 1 \).

**4 Plants with \( \rho \geq 1 \)**

If the plant relative degree is 2 or higher, then the new class of controllers deviates from the class of the CE based adaptive controllers. We demonstrate the difference using an example of a second order plant as follows:

We consider the plant

\[ y = \frac{1}{s^2 + c_2 s + d_2}u \]  

where the parameter \( c_2 \) is a constant scalar. When \( a \) is known, the control law based on the new class of controllers is given by

\[ u = -\frac{(c_2 + d_2 s^2 + \gamma)}{K(s)} \frac{s^2}{K(s)} \]  

The above controller guarantees that \( y \) follows a given signal \( y_r \) that is bounded and has bounded first and second derivatives. The above control law can be expressed in the form of (3) with the \( \phi \)'s defined appropriately, and \( r = R_m(s) - y \).

We can verify that the \( \phi \)'s can be obtained by solving the Diophantine equation

\[ [K(s) - \phi^T \psi(s)](\phi^T \psi(s)) - \phi^T h(s) = R_m(s)K(s) \]  

In other words, the control law (31) is a model reference controller with reference model

\[ y_r = \frac{1}{R_m(s)}r \]  

In this case, \( R_m(s) \) is dependent on the plant parameter \( a \) thought \( \sigma \). When the parameter \( a \) is unknown, the adaptive controller is

\[ u = \hat{\omega} + \hat{\theta} \omega \]  

\[ \dot{\hat{\omega}} = s(y - \hat{y}) - \hat{\gamma} + (c_1 + d_1)z_1 \]  

into (21) and substituting the \( \phi \)'s in (3) using equations (7), (8).

5 Conclusion

A nonlinear class of adaptive controllers designed for minimum phase LTI plants is compared with the traditional class of the CE based MRAC. We showed that for the plant relative degree \( \rho \geq 1 \) the nonlinear class is a special case of the CE based indirect MRAC for plants with \( \rho \geq 1 \). As we can see from above, even for this simple example the new controller differs from the CE one. First of all, the new controller is no longer certainty equivalence since the adaptive control law can not be simply obtained by replacing unknown parameters with their estimates in the non-adaptive control law, as the CE type adaptive controllers do. Second, the reference model is no longer LTI but depends on the parameter estimate, and is not known a priori.

**References**


