Learning Laws with Exponential Error Convergence for Recurrent Neural Networks

ELIAS B. KOSMATOPoulos1, MANOLIS A. CHRISTODOULOU1, PETROS A. IOANNOU2

1 Dept. of Electronic & Computer Engineering  
Technical University of Crete  
72100 Chania, Crete, GREECE

2 Dept. of Electrical Engineering-Systems  
University of Southern California  
Los Angeles, CA 90089-2562, U.S.A.

Abstract

In this paper, we propose new learning laws for adjusting the weights of recurrent high order neural networks (RHN) when they are used to system identification problems. The main advantages of these learning laws over the classical robust adaptive ones, is that the identification error converges to zero exponentially fast, and that such a convergence is independent of the number of high order connections of the RHN.

I Introduction

Recently, there has been a concentrated effort towards the design and analysis of learning algorithms for neural networks that are based on the Lyapunov stability theory [1, 2]. The main drawback of the above methods is that they require very large number of neurons or high order connections in order to guarantee error convergence to zero; if the number of neurons or the high order connections is not sufficiently large, then only the error convergence to a ball whose radius is proportional to the modeling error can be guaranteed. This is due to the fact that the above approaches make use of adaptive algorithms borrowed from classical adaptive identification and control. Such algorithms like the gradient or the least squares parameter estimation ones, are capable to ensure error convergence to zero, only in the case where there is no modeling error, while, in the case where the modeling error is not zero, only the error convergence to a ball whose radius is proportional to the modeling error can be guaranteed. Therefore, in the case of identifying unknown systems using neural networks, the classical or robust adaptive methods that may be applied, are able to ensure error boundedness but they are unable to guarantee error convergence to zero, due to the unavoidable emergence of the modeling error term.

In this paper, we propose new adaptive algorithms for identifying nonlinear systems using RHNs. Among the properties of the proposed adaptive laws, the most important are:

- Despite the conventional parameter estimation and adaptive algorithms, the proposed architectures guarantee exponential error convergence, which is the best convergence that can be achieved.
- Contrary to the existing methods, the proposed algorithms guarantee that the error converges to zero for any selection of high order connections. More precisely, the proposed learning laws ensure that, even in the case where there is modeling error (i.e., the neural network’s number of high order connections is not sufficiently large), the error converges to zero exponentially fast, provided that either the regressor terms are persistently exciting or the neural network’s adjustable weights are kept bounded.
- The convergence properties of the parameter error - which is defined as the difference between the neural network weights and the weight values that minimize the worst case difference between the RHN and the unknown system vector fields - are similar with those of classical adaptive schemes. Moreover, we show that robust modifications of the originally proposed algorithm possess the same robustness capabilities with those of, let’s say, switching or modification scheme.

The proofs of the main results of this paper can be found in [3].

II The Neural Model

The neural network model used in this paper is the so-called Recurrent High Order Neural Network (RHN) consisting of n neurons and m inputs. The state of each neuron is governed by a differential equation of the form

\[ \dot{x}_i = -a_i x_i + \sum_{k=1}^{n} w_{ik} x_k + \sum_{k=1}^{m} u_{ik} z_k \]

where \( \{x_1, \ldots, x_n\} \) is a collection of L not ordered subsets of \( \{1, \ldots, m + n\} \), \( a_i \) are real coefficients, \( w_{ik} \) are the (adjustable) synaptic weights of the neural network, \( d_i(k) \) are non-negative integers, and \( x_k = \prod_{i=1}^{n} d_i(k) \). The state of the \( k \)-th neuron is again represented by \( x_k \), and \( y \) is the vector consisting of inputs to each neuron, defined by \( y = [x_1, \ldots, x_n, S(x_1), \ldots, S(x_n)] \), where \( S(x) \) is the external input vector to the network.

The function \( S(\cdot) \) is a positive, monotone increasing, differentiable sigmoidal function.

In [1] we have proved that the RHN model (2.1) is capable of approximating - with arbitrary accuracy, any dynamical system of the form (3.1) (see below).

III The Learning Law

In this section we consider the learning problem of adaptively adjusting the weights such that the RHN model identifies general dynamical systems of the form

\[ \dot{x} = f(x; u) \]

A classical approach to the identification of a dynamical system is by writing the unknown dynamical system (3.1) as follows

\[ \dot{x} = Ax + \dot{w} \dot{z} (x, u) + \nu(t) \]

I.e., by considering that the dynamical system vector fields can be decomposed into RHN type vector fields \( Ax + \dot{w} \dot{z} \) plus a "disturbance", or modeling error term \( \nu(t) \). Using this approach, we can use standard parameter estimation robust adaptive laws in order to estimate the "unknown" parameters \( \dot{w} \). Polycarpou & Ioannou [2], and Kosmatopoulos et al. [1] have used this approach in order to develop efficient learning laws for adjusting \( \dot{w} \). Unfortunately, the proposed approaches, although they prove that the identification error is bounded, and moreover, its square integral is proportional to the modeling error square integral, they cannot ensure convergence of the identification error \( x - x \) to zero, but for few special cases (i.e. the case where the modeling error term \( \nu(t) \) is either zero, or square integrable). In this paper, we present new learning laws that ensure that the identification error converges asymptotically to zero, and, moreover, that this convergence is exponential, which is the best kind of convergence that can be achieved.

As in common in all identification procedures, we will assume that the state \( x(t) \) and the time-derivative \( \dot{x}(t) \) are bounded for all admissible bounded inputs \( u(t) \), i.e. we will assume that

\[ (A1) \quad x, \dot{x} \in L_\infty. \]

Using the classical in adaptive control filtered regressor technique (see e.g. [1, 2]), we can rewrite the RHN model (2.1) as

\[ x_i = u^T \dot{z}_i \]

\[ z_i = \eta(z) \theta_i (x; u) \]

\[ \theta_i (x; u) > 0 \quad \forall x, u \in L_\infty \]

where \( \dot{z}_i \) is a filtered version of the vector \( z \) given by

\[ \dot{z}_i = -a_i \dot{z}_i + z_i \]

\[ \eta(z) \theta_i (x; u) > 0 \quad \forall x, u \in L_\infty \quad \text{and } c_i \text{ is an exponentially decaying term.} \]

In order to simplify analysis, the exponentially decaying term \( c_i(t) \) will be omitted in (3.3) since it does not affect the convergence properties of the scheme.

Before we proceed to the presentation of the proposed learning laws, let us demonstrate the main idea that led to the development of such learning laws. At first consider the identification error \( c_i \) defined as the difference between the RHN and the unknown system states, i.e.,

\[ c_i = x_i - x_i \]
Using (3.3), we can rewrite (3.5) as
\[ e_i = w_i^T \zeta_i - y_i \]
and by differentiating (3.6) with respect to time, we readily obtain that
\[ \dot{e}_i = w_i^T \dot{\zeta}_i + w_i^T \dot{\zeta}_i - \dot{y}_i \]
(3.7)
Treating the \( w_i \) as a control variable, we can easily see that if we choose
\[ w_{i,k} = -\frac{e_i n_k}{\zeta_i} + \frac{x_{i,n_k}}{\zeta_i} - \frac{\dot{e}_i n_k}{\zeta_i} \]
(3.8)
where \( n_k \) are design constants which satisfy \( \sum_{k=1}^{J} n_k = 1 \), \( \forall i \) and \( \gamma \) is a positive constant denoting the adaptive gain (learning rate), the error differential equation becomes \( \dot{e}_i = -\gamma e_i \), i.e., the identification error converges to zero exponentially fast. Note that the implementation of the adaptive law (3.3) is not possible because it requires the time derivative \( \dot{\zeta}_i \) to be measured; however, the time-differentiation of a signal is not desirable in most engineering applications. This problem can be overcome if we rewrite (3.8) as follows
\[ w_{i,k}(t) = \dot{w}_{i,k}(t) + \frac{\dot{e}_i}{\zeta_i} \]
(3.9)
\[ \dot{w}_{i,k} = -\gamma \frac{e_i n_k}{\zeta_i} \]
(3.10)
\[ \phi_{i,k} = \frac{x_{i,n_k}}{\zeta_i} - \sum_{l=1}^{L} \frac{w_{l,k}}{\zeta_l} \]
(3.11)
Note that the last equation can be rewritten as
\[ \phi_{i,k}(t) = \int_{0}^{t} \zeta(t) \frac{d}{dt} \frac{w_{i,k}}{\zeta_i} \]
(3.12)
and using the integration-by-parts rule, we finally obtain that
\[ \phi_{i,k}(t) = \frac{x(t) n_k}{\zeta_i} - \int_{0}^{t} \zeta \frac{d}{dt} \left( \frac{w_{i,k}}{\zeta_i} \right) \frac{d}{dt} \int_{0}^{t} \zeta \frac{d}{dt} \left( \frac{w_{i,k}}{\zeta_i} \right) \]
(3.13)
Observe now that the above equation can be rewritten as
\[ \phi_{i,k}(t) = \frac{x(t) n_k}{\zeta_i} = \frac{w_{i,k}}{\zeta_i} \]
(3.14)
where
\[ \dot{w}_{i,k} = -\frac{d}{dt} \left( \frac{n_k}{\zeta_i} \right) - \sum_{l=1}^{L} \frac{w_{l,k}}{\zeta_l} \]
(3.15)
(note that \( \dot{\zeta}_i \) can be computed by (3.4) where \( \dot{\zeta}_i = -a_i \zeta_i + \gamma \). Therefore

Proposition III.1 Consider the unknown system (3.1) and assume that assumption (A1) holds. Moreover consider the RHON model (3.3), (3.4) whose weights are adjusted according to (3.9), (3.10), (3.14), (3.15). Then, if \( w_i \in \mathcal{L}_\infty \), the error \( e_i \) satisfies
\[ e_i(t) = e^{-\gamma t} e_i(0) \]
and the identification error converges to zero exponentially fast.

IV Parameter Convergence and Robust Modifications of the Learning Law
Proposition III.1 ensures that the identification error will converge to zero exponentially fast based on the assumption that the weights \( \dot{w}_i \) are bounded. However, there is no a priori guarantee that the learning law (3.9), (3.10), (3.14), (3.15) ensures that the synaptic weights are kept bounded during the whole identification procedure. Another aspect that we have also to study is the weight convergence. Proposition III.1 does not make any statement about the convergence of the synaptic weights to the optimal ones. In order to examine such aspects we proceed as follows. Obviously by optimal weights we mean a vector \( w^*_i \) such that
\[ w^*_i = \arg \min_{w_i} \left\{ \sup_{x,\dot{x}} |F_i(x,u) + a_i x - w_i^T \zeta(x,u)| \right\} \]
(4.1)
Let us first, examine the performance of the learning law (3.9), (3.10), (3.14), (3.15) in the case where the modeling error term \( w_i(t) \) is zero. The next theorem establishes the learning law (3.9), (3.10), (3.14), (3.15) properties in this case.

Theorem IV.1 Consider the system (3.1) and assume that assumption (A1) holds. Also assume that \( w_i(t) = 0 \) for all \( t \). Moreover consider the RHON model (3.3), (3.4) whose weights are adjusted according to (3.9), (3.10), (3.14), (3.15). Also assume that
(A2) The regressor vector \( \zeta_i \) is persistently exciting.
(A3) There exist positive constants \( k_0, \delta \) such that for all \( t \geq 0 \)
\[ \int_{0}^{t} \|e_i(t)\|^2 dt \leq k_0 \]
(4.2)
where \( Q_i(t) \) is a \( L \times L \) matrix whose \( k \)-th entry is defined as follows
\[ Q_{ik} = \frac{n_k}{\zeta_i^T} \frac{n_k}{\zeta_i} \]
Then

(a) The identification error converges to zero exponentially fast.
(b) The parameter error \( \dot{w}_i = \dot{w}_i - w^*_i \) converges to zero exponentially fast.

Theorem IV.1 shows that the proposed learning law, ensures that the neural network weights will converge to their optimal values, under the assumption that there is no modelling error term. However - due to insufficient number of high-order terms in the RHON model - such an assumption seems to be unrealistic. It is not difficult for someone to see, that in the case where the modelling error is not zero, the solutions of the learning equations may become unbounded, even in the case where the modelling error is bounded; therefore, in the case where the modelling error is not zero, the RHON synaptic weights may drift to infinity, even if the identification error converges to zero. Therefore, the learning law (3.9), (3.10), (3.14), (3.15) has to be modified in order to avoid the parameter drift problem. For this, we propose the following modification of the learning law (3.9), (3.10), (3.14), (3.15)
\[ w_{i,k}(t) = \dot{w}_{i,k}(t) + \phi_{i,k}(t) \]
(4.3)
\[ \dot{w}_{i,k} = -\frac{e_i n_k}{\zeta_i} - \delta w_{i,k} + \xi \phi_{i,k} \]
(4.4)
\[ \phi_{i,k}(t) = \frac{x(t) n_k}{\zeta_i} + n_i(t) \]
(4.5)
\[ \theta_{i,k} = -\frac{d}{dt} \left( \frac{n_k}{\zeta_i} \right) - \sum_{l=1}^{L} \frac{w_{l,k}}{\zeta_l} \]
(4.6)
where \( \delta \) is a scalar time-varying term, which is defined as in the classical switching \( \sigma \)-modification adaptive laws see [3] for more details). The next theorem establishes the properties of the learning law (4.3)-(4.5):

Theorem IV.2 Consider the unknown system (3.1) and the RHON model (3.3), (3.4) whose weights are adjusted according to (4.3)-(4.5). Assume that assumption (A1) holds. Then
(a) The identification error \( e_i \) converges to zero exponentially fast.
(b) The parameter error \( \dot{w}_i, \dot{w}_i \in \mathcal{L}_\infty \).
(c) Moreover, if assumptions (A2), (A3) hold, then the parameter error \( \dot{w}_i \) converges exponentially to the residual set \( \mathcal{E}_i = \{|w_i|/\zeta_i| \leq \delta \} \) for some \( \delta \in \mathbb{R}_+ \), where \( \dot{w}_i = \sup_{x,\dot{x}} |F_i(x,u) + a_i x - w_i^T \zeta(x,u)| \) is the \( L \)-dimensional signal where \( k \)-th entry is defined as \( w_{i,k} = \theta_{i,k} + \sigma \phi_{i,k} \).

References

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