ON THE MODEL REFERENCE ADAPTIVE CONTROL OF TIME-VARYING PLANTS

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Abstract

In this paper we study the problem of developing an adaptive control law which can force the output of a linear time-varying plant to track the output of a stable linear time-invariant reference model and at the same time guarantee the internal stability of the closed-loop plant. We use the new model reference controller structure, proposed in [1], that guarantees stability and zero tracking error for a general class of linear time-varying plants with known parameters and design an adaptive law to estimate the controller parameters. We show that the assumption of slow parameter variations in the adaptive case can be relaxed if some information about the frequency or the form of the fast varying parameters is available a priori. Such information can be incorporated in the adaptive law so that stability and tracking performance is considerably improved for a wide class of linear time-varying plants which includes plants with fast varying parameters.

1. Introduction

In the study of the adaptive control problem of linear time-varying (LTV) plants it is usually assumed that the plant parameter variations are unstructured, that is they either vary slowly with time but in an unknown fashion ([1,2,3,4,5,6]) or they are small in the norm perturbations of some nominal fixed parameters ([7,8]), or the time variations are structured but restricted to a particular class (exponential or 1/t decay, finite number of jumps/discontinuities) ([9,10,11]). In the first two classes of studies, the dominant idea is that the effect of the plant parameter variations can be treated as a small disturbance, because of the smallness in the norm of the parameter variations or their derivatives (slowly time varying (TV) parameters). In the third class of studies, a rather restrictive structure of parameter variations is considered to show the asymptotic stability of the closed-loop plant.

In the recent, more general results of [2,4,5,6] it was shown that one can solve the adaptive control problem (model reference or pole placement) of slowly TV plants, without any persistence of excitation requirements, using a robust adaptive law to update the estimates of the controller or the plant parameters. The design of the adaptive law is done as if the parameters to be estimated were frozen at each time instant.

More specifically, for the model reference control (MRC) case, it was shown in [4,12] that the standard MRC structure of [13] offers only an approximate solution to the MRC problem, with known slowly TV plant parameters. The standard MRC was also combined with an appropriate adaptive law to estimate the controller parameters in the case of unknown plant parameters. It was then shown that the resulting model reference adaptive controller (MRAC) can guarantee the closed loop stability and good performance, provided that the plant parameters are slowly TV and the minimum stability margin of the inverse plant is known a priori.

In [1,12], a new MRC structure was proposed that can force a rather general LTV plant to behave, from an input-output (I/O) point of view, exactly as a linear time-invariant (LTI) reference model, provided that the plant parameters are known for all time but without requiring their slow time variation ([13]). The new MRC law was also combined with an appropriate adaptive law to estimate the controller parameters when the plant parameters are unknown. The resulting new MRAC was then shown to guarantee closed loop stability and good tracking performance in the case of slowly TV plant parameters and, in contrast to the standard MRAC, without requiring the a priori knowledge of the stability margin of the inverse plant. The need to assume that the plant parameters are slowly TV can be intuitively interpreted as follows: Since the speed of adaptation is finite and the knowledge about the unknown plant parameters has to be learned by the adaptive law on-line from I/O measurements only, then for stability the speed of variation of the plant parameters has to be small relative to the speed of variation of the signals and the speed of adaptation. Furthermore, due to the finite speed of adaptation, the effect of the parameter variations, even though slow, cannot be completely accommodated by the adaptive controller and therefore, the tracking error should not be expected to go to zero as \( t \to \infty \).

The ability of the new MRC law to solve the MRC problem exactly for not necessarily slowly TV plants, gives rise to the following question: Is the complete knowledge of the TV plant parameters necessary to guarantee exact tracking between the plant and the reference model as well as the stability of the closed-loop plant? The purpose of this paper is to give an answer to this basic question. We start by presenting, in Section 2, the appropriate mathematical preliminaries that can be used to describe and analyze the I/O properties of continuous-time LTV systems in a rather similar way as in the case of LTI systems. We assume that the plant is described by a general linear ordinary differential equation with TV coefficients and express it in a compact form by using the notions of polynomial differential operators (PDOs) [14,15] and polynomial integral operators (PIOs) [13]. The properties of the PDOs and PIOs enable us to describe the I/O properties of a LTV system in a compact way which is very similar to that of a transfer function in the LTI case. The mathematical preliminaries of Section 2 are used in Section 3 to describe the plant and the control objective. In Section 3 we also present the new MRC law and give the conditions which guarantee the exact...
solution of the MRC problem for LTV plants with arbitrary, but finite, speed of plant parameter variations. These conditions are the generalizations of the ones required in the LTI case (13), to the LTV case. The properties of the new MRC structure are significant when compared with the standard one which, for stability, requires the plant parameters to vary sufficiently slowly with time and cannot guarantee zero tracking error even for arbitrarily slow TV parameters. In Section 4 we first use intuition to claim that if some information about the form or structure of the parameter variations is available a priori and if that information is appropriately incorporated in the adaptive law then the requirement of slow parameter variations can be relaxed. We then show that such a claim is true by developing an appropriate adaptive law which can use the information about the form of the parameter variations in such a way that global stability and zero tracking error can be guaranteed for a wide class of LTV plants which are not necessarily slowly TV. This result is achieved at the expense of updating additional parameters in the adaptive law and, of course, requiring some a priori information about the form of the parameter time variations.

2. Mathematical Preliminaries

2.1 I/O Representation of LTV Systems

In this section we present some definitions which will be used to describe the I/O properties of LTV systems in a similar way as in the case of LTI systems.

Definition 2.1: A LTV PDO of degree \( n \) is defined by
\[
P(x, t) = a_0(t)x^n + a_1(t)x^{n-1} + \cdots + a_n(t)
\]
where \( x \in \mathbb{R}^d \); \( a_i(t), i = 0, 1, \ldots, n \) are bounded piecewise continuous functions of time, \( a_0(t) \neq 0 \) \( \forall t \geq 0 \). When \( a_0(t) \equiv 1 \) \( \forall t \geq 0 \), \( P(x, t) \) is referred to as a monic PDO.

The properties of the PDO follow directly from the rules of differentiation and are given in [16], pp. 47.

Using Definition 2.1, we can express the \( n \)-order linear ordinary differential equation
\[
P(t, y) = u
\]
in state-space as
\[
\dot{x} = A(t)x + b(t) ; \quad y = c^T x
\]
with \( x = [x_1, \ldots, x_d]^T \), \( b = [0, \ldots, 0, 1]^T \), \( y = [y_1, \ldots, y_{d-1}]^T \) and \( A(t) \) containing the TV coefficients of \( P(t, x) \). The solution of (2.3) exists and is unique [16, 17] and is given by
\[
y(t) = \int_{0}^{t} \Phi(t, \tau)u(\tau) d\tau + \int_{0}^{t} \Phi(t, \tau)b(\tau) d\tau
\]
where \( \Phi(t, \tau) \) is the state transition matrix (STM) of (2.3).

The first term in (2.4) is the zero state response (ZSR) of (2.3) and the second term is the zero input response (ZIR) of (2.3) [17].

Following the same procedures as in the LTI case, the ZSR can be expressed in a compact form by introducing the notion of the PDO \( P^{-1}(x, t) \).

Definition 2.2: A LTV PIO of order \( n \) is defined as the operator that maps the input \( u \) to the ZSR of the differential equation
\[
P(x, t)u = y
\]
where \( P^{-1}(x, t) \) is a monic PDO of degree \( n \). We will denote the PIO by \( P^{-1}(x, t) \) and write
\[
P^{-1}(x, t)u = \int_{0}^{t} \Phi(t, \tau)u(\tau) d\tau
\]
(2.5)

Using the notion of PDOs and PIOs we can express the ZSR of more general LTV differential equations than (2.2) in a compact form. The operator that maps the input \( u \) to the ZSR of a general linear differential equation
\[
\dot{x} = A(t)x + b(t)u ; \quad x(0) = x_0
\]
\[
y = c^T x + d(t)u
\]
(2.6)

where \( A(t), b(t), c(t), d(t) \) are bounded, piecewise continuous functions of time, will be referred to as the LTV I/O operator of (2.6). In the following example we show that for a general representation of LTV systems the I/O operator may be expressed as a combination of PDOs and PIOs.

Example 2.1: Consider the system described by the differential equation
\[
D(x, t)u = y ; \quad x(0) = x_0
\]
\[
y = K(t)N(x, t)x
\]
(2.7)

where \( x \) is an internal state, \( x(0) = [x(0), \ldots, x(n-1)]^T \) is the vector with the initial conditions, \( u, y \in \mathbb{R}^d \) is the input and the output of the system respectively, \( K(t) \) is a scalar function of time and \( D(x, t), N(x, t) \) are monic PDOs of degree \( n, m \) respectively. As shown in [18] the LTV system (2.6) can be transformed to the form (2.7) if it is uniformly controllable. Using Definition 2.2 the LTV I/O operator of (2.7) \( G(t, x) \) is given by
\[
G(t, x) = K(t)N(t)D(t)\Phi(t, 0)
\]
(2.8)

Remark 2.1: If (2.6) represents a LTI system \((A(t), b(t), c(t), d(t) \) are constant) with transfer function \( H(s) = \Phi(s)/P(s) \) and \( \hat{s} \) is the Laplace complex variable, its I/O operator is given by \( G(s) = Q(s)\Phi(s) + d = P^{-1}(s)Q(s) + d \).

It should be noted that due to the time dependence of the coefficients of the PDOs, the multiplication (cascade combination) of two LTV I/O operators is not commutative. Despite the non-commutativity of I/O operators their properties are quite similar to those of transfer functions of LTI systems.

2.2 Stability of LTV I/O Operators

The stability properties of the PIOs and, in general, the LTV I/O operators are defined through the stability properties of the respective space-time representation [16, 17].

Definition 2.3: A LTV PDO, \( P^{-1}(x, t) \), is exponentially stable (or uniformly asymptotically stable) with rate \( -\alpha_t, \alpha_t > 0 \), if the state transition matrix \( \Phi(t, \tau) \), associated with the linear differential equation (2.2) satisfies
\[
\|\Phi(t, \tau)\| \leq k \exp[-\alpha_t(t - \tau)] \quad \forall t \geq \tau \geq 0
\]
(2.9)

for some positive constants \( k, \alpha_t \).

The same definition can be used for defining the stability of LTV I/O operators by replacing the STM of (2.2) with the STM of the space-time representation of the LTV I/O op-
erat. It can then be shown ([12]) that

**Theorem 2.1**: A proper LTV 1/0 system is exponentially stable if all the PIOs it contains are exponentially stable.

### 2.3 Coprimeness of TV PDOs

The important in control systems design notion of coprimeness of two polynomials can be extended to the case of PDOs by using the following general definition:

**Definition 2.4**: Two monic PDOs \(D(s,t), N(s,t)\), of degree \(n, m\) respectively with smooth coefficients, are right coprime for all \(t \geq 0\) if

\[
Q(s,t)D(s,t) + P(s,t)N(s,t) \neq 0, \quad \forall t \geq 0 \tag{2.10}
\]

is satisfied for all PDOs \(Q(s,t), P(s,t)\) of degree \(i \leq m - 1, j \leq n - 1\) respectively when \(Q(s,t), P(s,t)\) are not both equal to zero for any \(t \geq 0\).

The coprimeness of two TV PDOs can be checked in a similar way as in the LTI case by first extending the definition of the Sylvester matrix to the TV case and then examining its properties. The following definition of the TV Sylvester matrix is similar to the one used for polynomials and is motivated by writing (2.10) as a system of linear algebraic equations w.r.t. the coefficients of \(Q(s,t), P(s,t)\).

**Definition 2.5**: The TV Sylvester matrix \(S_{i}\) of the PDOs \(D(s,t), N(s,t)\), of degree \(n, m\) respectively is defined as

\[
S_i = [C_i, B_i, A_i] \tag{2.11}
\]

where

\[
C_i = [0, \ldots, 0, C_{ij}]^T; \quad i \in \mathbb{N}; \quad B_i = [0, \ldots, 0, B_{ij}]^T; \quad j \in \mathbb{N} \tag{2.12}
\]

and \(C_{ij}, B_{ij}\) are vectors containing the coefficients of the PDOs \(s^{n-i}D(s,t), s^{m-j}N(s,t)\) respectively.

### 3. LTV Plant and the New MRC Structure

Consider a single-input single-output (SISO) LTV plant that can be described by one of the following forms of differential equations.

\[
P - 1: \quad D_{1}(s,\rho)\dot{y}_{1}(t) = u_{1}(t) + X_{1}(0) \tag{3.1}
\]

\[
D_{2}(s,\rho)\dot{y}_{2}(t) = \dot{y}_{2}(t) + X_{2}(0) \tag{3.2}
\]

where \(x_{p} \in \mathbb{R}^{p}\) is an internal state; \(u_{1}, y_{1} \in \mathbb{R}^{m}\) are the input and the output of the plant respectively; \(D_{1}(s,\rho), N_{1}(s,\rho)\) are monic LTV PDOs i.e.,

\[
D_{1}(s,\rho) \triangleq \rho^{n} + \sum_{i=1}^{n} \alpha_{i}(\rho)\rho^{n-i} \tag{3.3}
\]

\[
D_{2}(s,\rho) \triangleq \rho^{m} + \sum_{j=1}^{m} \alpha_{j}(\rho)\rho^{m-j} \tag{3.4}
\]

and \(\alpha_{i}(\rho), \beta_{j}(\rho) \neq 0\) \(\forall \rho \geq 0\). Let the initial conditions \(X_{1}(0) = [x_{0}(0), \ldots, x_{n-1}(0)]^{T}, Y_{1}(0) = [y_{0}(0), \ldots, y_{m-1}(0)]^{T}\) be the initial conditions for (3.1),(3.2) respectively.

We will assume that the plant parameters \(\alpha_{i}(\rho), \beta_{j}(\rho)\) are uniformly bounded, smooth functions of \(\rho\), where \(\rho = \mu t\) is the parameter time scale and \(\mu \geq 0\), is the ratio of the time scale of the plant parameters versus the time scale of the system of the plant. When \(\mu < 1\) the plant parameters vary slowly with time and in the limit as \(\mu \to 0\) the plant parameters become constant. For finite \(\mu \geq 1\), (3.1),(3.2) represent a LTV plant whose parameter bounds are smooth, smooth, but other than that, arbitrary functions of time.

The control objective is to design a control law \(u_{1}(t)\) such that the output \(y_{1}(t)\) of the plant tracks the output \(y_{1}(t)\) of the LTI reference model

\[
y_{m} = W_{m}(s,\rho) e^{s^m} \tag{3.4}
\]

where \(D_{m}(s)\) is a monic Hurwitz polynomial; \(\lambda_{m} > 0\) and \(r(t)\) is a uniformly bounded reference input signal. In order to achieve such an objective we make the following assumptions:

1. \(n, m\) are constant and known
2. \(D_{1}(s,\rho), N_{1}(s,\rho)\) are strongly right coprime PDOs, according to Definition 2.6.
3. The sign of \(\alpha_{i}(\rho)\) (or \(\beta_{j}(\rho)\)) is constant and known. Without loss of generality we will assume that \(\alpha_{i}(\rho) > 0\) (or \(\beta_{j}(\rho) > 0\)) \(\forall \rho \geq 0\). Furthermore, the range of \(\alpha_{i}(\rho)\) (or \(\beta_{j}(\rho)\)) is a subset of a closed interval on the real axis that does not contain 0.
4. \(N_{1}^{\star}(s,\rho)\) (or \(N_{1}^{\star}(s,\rho)\)) is an exponentially stable PIO with rate bounded from above by \(-\alpha\), for some \(\alpha > 0\).
(8.5) \( D_r(s, \rho) \) is designed so that \( \text{deg}(D_r(s)) = n - m \).

**Remark 3.1:** For the plant in the form F-2 the PDOS \( D_r(s), N_r(s, \rho) \) are not required to be coprime. As it is shown in [12], due to the form F-2 and the new MRC structure, that allows the direct cancellation of the plant FDOs, \( N_r(s, \rho) \), the controller design equation for plants in the form F-2 has always a unique solution. For plants in the form F-1 however, a full pole-placement of the nonminimal closed-loop plant is required and Assumption 3.2 is needed to guarantee the existence of a unique, bounded solution for the desired controller parameters.

**Assumptions (S.1)-(S.5)** are the extensions of those in the MRC of LTI plants to the TV case. When \( \mu = 0 \) i.e., the plant parameters are constant, the standard MRC structure can be used to meet the control objectives exactly, both in the case of known and unknown plant parameters [19,20]. When \( \mu \neq 0 \) i.e., the plant parameters are TV, the standard MRC law cannot, in general, meet the control objective exactly, for either one of the representations F-1 or F-2 ([4,12]). Furthermore, as shown in [4], when the plant parameters are unknown, the standard MRC structure can be combined with a suitable adaptive law to update the controller parameters and guarantee global stability for slowly TV plant parameters. In this case however, the implementation of the adaptive law requires the a priori knowledge of no upper bound for the exponential rate \( -\alpha \) of \( N_r(s, \rho) \).

Next we present the new MRC structure ([11,12]) which is applicable to LTV plants in either one of the forms F-1 or F-2 and has the following advantages when compared with the standard MRC structure:

1. It guarantees the exact solution to the MRC problem (known plant parameters) of LTV plants for any finite \( \mu \geq 0 \).

2. In the MRAC case the adaptive law for updating the parameters of the new controller requires no a priori knowledge about the stability margin of the inverse plant.

3. As it will be shown in Section 4, it guarantees stability and exact tracking in the MRAC case for a wide class of TV plants whose parameters may not necessarily vary slowly with time.

The new controller is described as follows. The plant output \( y_p \) and input \( u_p \) are used to generate a 2n - 1 dimensional auxiliary vector \( \omega \) as

\[
\omega = \begin{bmatrix} \omega_1 & \omega_2 & \cdots & \omega_n \end{bmatrix}^T \quad \omega = \begin{bmatrix} \omega_1 & \omega_2 & \cdots & \omega_n \end{bmatrix}^T
\]

\[
\omega = \begin{bmatrix} \omega_1 & \omega_2 & \cdots & \omega_n \end{bmatrix}^T \quad \theta = \begin{bmatrix} \theta_1 & \theta_2 & \cdots & \theta_n \end{bmatrix}^T
\]

\[
\theta = \begin{bmatrix} \theta_1 & \theta_2 & \cdots & \theta_n \end{bmatrix}^T
\]

\[
\omega = \begin{bmatrix} \omega_1 & \omega_2 & \cdots & \omega_n \end{bmatrix}^T
\]

\[
\omega = \begin{bmatrix} \omega_1 & \omega_2 & \cdots & \omega_n \end{bmatrix}^T
\]

The input to the plant is taken as

\[
y_p = \begin{bmatrix} y_p & y_r \end{bmatrix}^T
\]

where \( \begin{bmatrix} y_p & y_r \end{bmatrix}^T \) is a constant vector such that \( \begin{bmatrix} y_p & F \end{bmatrix} \) is an observable pair and \( \begin{bmatrix} y_p & y_r \end{bmatrix} \) is a scalar parameter.

The difference between the new MRC structure and that of [4,13,21] is that in the new controller the auxiliary signal \( \omega \) is obtained by filtering the parameter vectors \( \theta_1, \theta_2 \) scaled by \( y_r \) and \( y_p \) respectively, whereas in the standard MRC \( \omega \) is formed by filtering \( y_r \) and \( y_p \) alone.

The stability and tracking properties of the new MRC law, when applied to the LTV plant represented by either F-1 or F-2 form, is given by the following theorem:

**Theorem 3.1:** ([12]) Consider a plant represented by either one of the forms given by (3.1), (3.2) whose parameters are known, bounded, smooth functions of \( \rho \) that satisfy Assumptions S.1 - S.4. Then, for any finite \( \mu \geq 0 \) a vector \( R^+ \rightarrow R^{(n-1)} \) and a scalar \( q_r(\rho) : R^+ \rightarrow R \) exist so that, for \( \theta = \theta_r(\rho) \) and \( \theta_0 = \theta_0(\rho) \) in (3.5), (3.6), the closed-loop TV plant (3.1) (or (3.2)), (3.5),(3.6) is internally stable and its I/O operator \( G_r(s, \rho) : s \mapsto y_r \) is equal to that of the Ti reference model (3.4). Furthermore, if the plant parameters possess \( I + q \) bounded derivatives with respect to \( \mu \) then \( \theta_r(\rho) \), \( q_r(\rho) \) are bounded functions of \( \rho \), possessing at least \( I \) bounded derivatives with respect to \( \rho \), where \( I \) is an arbitrary positive integer and \( q \geq 0 \) depends on \( n, m \) and the form of the plant.

The new MRC law guarantees exact I/O matching between the closed-loop plant and the LTI reference model for any finite \( \mu \geq 0 \) i.e., the plant does not have to be slowly TV. In contrast, the standard MRC law not only requires the plant to be slowly TV (i.e., \( \mu \in [0,\alpha] \) for some \( \alpha > 0 \)) but also fails to guarantee exact I/O matching between the closed-loop plant and the reference model in general, even for \( \mu \) sufficiently small ([11,12]).

4. MRAC for TV Plants

One of the distinct features of the new MRC structure is to guarantee the convergence of the tracking error to zero in the case of known plant parameters. As we mentioned earlier, it is unreasonable to expect stability and zero tracking error in the adaptive case where the information about the TV plant parameters has to be learned on-line from I/O data only. However, if some information about the form of the parameter time variations is available a priori and that information is incorporated beforehand in the adaptive law then intuitively the performance of the closed-loop plant should be improved and stability should be maintained for a wider class of parameter variations which may include fast time variations as well.

In this section we establish that the exact knowledge of the TV plant parameters is not necessary for the implementation of the new MRC law so that exact tracking as well as closed-loop stability are guaranteed. We propose an adaptive law for estimating \( \theta_r(\rho), q_r(\rho) \) that can utilize any available a priori knowledge of the structure of the parameter time variations and guarantee stability and improved tracking performance for a wide class of linear, not necessarily slowly TV plants.

Let us assume that some of the plant parameters, whose values are unknown, vary with certain frequencies or constitute certain known functions of time so that the desired controller parameter vector may be expressed in the form

\[
\theta_r(\rho) = \theta_0^\rho + F_1(\rho)\theta_1^\rho + \ldots + F_\ell(\rho)\theta_\ell^\rho
\]

\[
q_r(\rho) = q_0^\rho
\]

where \( \theta_i^\rho \) i = 0, 1, \ldots, \ell are unknown but constant vectors; \( F_i(\rho) : R^+ \rightarrow R^{(n-1)} \), i = 1, 2, \ldots, \ell are known TV matrices whose entries are smooth uniformly bounded functions of time \( \rho \); \( f_\ell(\rho) \) is a known smooth function of time \( \rho \) which arises when the high frequency gain \( k_\ell \) is of the form

\[
k_\ell(\rho) = k_\ell f_\ell(\rho)
\]

with \( k_\ell \) constant and \( f_\ell(\rho) \geq 0 \) for \( \rho \geq 0 \) such that Assumption 3.3 is satisfied. We will refer to this class of parameter variations as structured parameter variations.

Now, instead of estimating \( \theta_r(\rho), q_r(\rho) \) directly, we first es-
estimate $R_1$ and $Q_2$ and then use (4.1) to form the estimate of $\phi(t)$ and $\xi_2(t)$ of $Q_2$. That is, the estimates $\hat{\phi}(t)$ of $\phi(t)$ and $\hat{\xi}_2(t)$ of $Q_2$ are given by

$$\hat{\phi}(t) = \hat{\phi}_0(t) + F(t)\hat{\phi}(t) + \ldots + F(t)\hat{\phi}_0(t) \quad \hat{\xi}_2(t) = \hat{\xi}_2(t) + F(t)\hat{\xi}_2(t)$$

(4.2)

where $\hat{\phi}_0(t)$ is the estimate of $R_1$ and $\hat{\xi}_2(t)$ the estimate of $Q_2$ at time $t$. We use the following adaptive law for updating the augmented parameter vector $\sigma = [\hat{\phi}_0; \hat{\xi}_2; \hat{\xi}_3; \hat{\xi}_4; \hat{\xi}_5; \hat{\xi}_6]$ and $\hat{\xi}_7$, the estimate of $Q_2$ is 1/$\hat{Q}_2$.

$$\dot{\sigma} = -\frac{1}{\hat{\xi}_7} - \Gamma \sigma \dot{\xi}_7 \quad \hat{\xi}_7 = -\frac{1}{\hat{\xi}_7} \Gamma \sigma \Xi \hat{\xi}_7$$

(4.3)

where $\Gamma = \Gamma^T > 0$, $\sigma > 0$.

$$\mu = -\hat{\xi}_7 + \delta_4 |\hat{\xi}_7| + |\hat{\xi}_7| + 1 \quad m(\hat{\xi}_7) \geq \delta_5 \mu$$

(4.4)

$$\delta_5 \leq 1$$

(4.5)

and $\sigma$ is chosen to be the same as $\sigma$. The $M_3$, $M_4$, replaced by $\hat{\sigma}_0$, $M_5$, respectively; the signals $\xi$, $\hat{\xi}$ and $\sigma$ are given by

$$\xi = \xi_0 + \hat{\xi}_7 + \frac{1}{\hat{\xi}_7}$$

(4.6)

$$\hat{\xi}_7 = \hat{\xi}_7 + W_1(x)\hat{f}_1(x)u_1$$

(4.7)

where $\eta_1 = 0, \ldots, 2(\eta - m)$ are constant vectors and $u_0$ is any finite constant. For plants in the form $P(t)$ (3.1), Theorem 3.1 can be used to express $\theta(t)$ in the form (4.1). However, in this case, the nonlinear dependence of $\phi(t)$ on the plant parameters may cause $\theta(t)$ in (4.1) to be TV. In a similar way, TV $\theta(t)$ may result in (4.1) if the functions $F(t)$ are approximately known. Corollary 3.1 is then applicable provided, of course, that the time variations of the $\xi(t)$ are slow.

$$\xi(t) = \xi_0 + \xi_1 sin[\eta_1(t)] + \xi_2 cos[\eta_2(t)] + \ldots + \xi_{2(\eta - m)} sin[(\eta - m)\eta_1(t)] + \xi_{2(\eta - m)} cos[(\eta - m)\eta_2(t)]$$

(4.8)

where $\eta_1 = 0, \ldots, 2(\eta - m)$ are constant vectors and $u_0$ is any finite constant. For plants in the form $P(t)$ (3.1), Theorem 3.1 can be used to express $\theta(t)$ in the form (4.1). However, in this case, the nonlinear dependence of $\phi(t)$ on the plant parameters may cause $\theta(t)$ in (4.1) to be TV. In a similar way, TV $\theta(t)$ may result in (4.1) if the functions $F(t)$ are approximately known. Corollary 3.1 is then applicable provided, of course, that the time variations of the $\xi(t)$ are slow.

$$\xi(t) = \xi_0 + \xi_1 sin[\eta_1(t)] + \xi_2 cos[\eta_2(t)] + \ldots + \xi_{2(\eta - m)} sin[(\eta - m)\eta_1(t)] + \xi_{2(\eta - m)} cos[(\eta - m)\eta_2(t)]$$

(4.9)

The parameter $\delta_4$ is designed so that the normalization signal $m(t)$ given in (4.4), decays slower than all the signals appearing in the adaptive law. This is achieved by selecting $\delta_4 > 0$ to satisfy

$$\delta_4 + \delta_5 \leq \delta_5$$

(4.10)

for some $\delta_4 > 0$ and by designing $W_1(x)$ and the matrix $F$ of the filters (3.5) so that the poles of $W_1(x)$ $\gamma_0$ and the eigenvalues of $F + \gamma_0 I$ are stable (121). Furthermore, $G(x)$ in (4.8) should have components that are first order filters that is $F$ should be a diagonal matrix. This requirement however, can be easily satisfied since $F$ is at the disposal of the designer.

The parameter vector $\theta$ in the control law (3.5)-(3.6) is now updated by using (4.3) and the expressions (4.2) to form $\theta(t)$. The stability properties of the control law (3.5)-(3.6) whose parameter vector $\theta(t)$ is formed by (4.2) and updated by the adaptive law (4.3)-(4.7) are given by the following theorem.

**Theorem 4.1.** Structured parameter variations 

The closed-loop plant (3.5) of (3.2) with the control law (3.5)-(3.6), (4.2) and adaptive law (4.3)-(4.7) is globally stable for any finite $\mu > 0$. Furthermore, the tracking error $e_1 = y_0 - \hat{y}_0$ reduces to zero asymptotically with time.

**Corollary 4.1:** (122) If the $\hat{\xi}_7 = 0, 1, \ldots, 2$ and $m(t)$ in (4.1) are smooth functions of $\mu = \mu$ and $\mu \in \mathbb{R}^2$ is fixed then for all $\nu \in [0, \nu^*]$ and some $\nu^* > 0$ the closed-loop plant (3.5) or (3.2), (3.5)-(3.6) with the adaptive law (4.3)-(4.7) is globally stable and the tracking error $e_1 = y_0 - \hat{y}_0$ belongs to the residual set

$$D = \left\{ e_1 : \lim_{T \to \infty} \frac{1}{T} \int_0^T |e_1^T| \leq \nu + \nu^* \right\}$$

(4.11)

where $\nu^* > 0$ and $\nu$ is a small number.

**Remark 4.1:** In Corollary 4.1 the time scale $\hat{\xi}_7 = \mu$ of $R_1$, $Q_2$ is used. This is done in order to distinguish the time scale $\mu$ of the TV matrices $F(t)$, which may depend on fast TV plant parameters, from the time scale $\hat{\xi}_7$ of $R_1$, $Q_2$ which for stability are required to be constant or slow TV.

The result of Theorem 4.1 is based on the assumption that, given a certain a priori information about the way the plant parameters vary with time, the desired parameter vector $\theta(t)$ for exact TV operator matching between the plant and the reference model can be expressed in the form (4.1) with $F(t)$, $f_1(t)$ known and $\theta(t)$ constant. This is always true for plants in the form $P(t)$ (3.2). For example, consider an $n$th order TV plant with degree $[N_{i}(s, \nu)] = m$, $\nu_1(t) = 1$ and $\nu = \nu^*$, whose parameters vary as sinusoids with a known frequency $\omega_0$ and unknown amplitude. Using Theorem 3.1, $\theta(t)$ may then be expressed as

$$\theta(t) = \theta_0 + \theta_1 \sin(\omega_0 t) + \theta_2 \cos(\omega_0 t) + \ldots + \theta_{2n+m-1} \sin((\omega_0 - m) t) + \theta_{2n+m} \cos((\omega_0 - m) t)$$

where $\theta_1 = 0, \ldots, 2(\eta - m)$ are constant vectors and $u_0$ is any finite constant. For plants in the form $P(t)$ (3.1), Theorem 3.1 can be used to express $\theta(t)$ in the form (4.1). However, in this case, the nonlinear dependence of $\phi(t)$ on the plant parameters may cause $\theta(t)$ in (4.1) to be TV. In a similar way, TV $\theta(t)$ may result in (4.1) if the functions $F(t)$ are approximately known. Corollary 3.1 is then applicable provided, of course, that the time variations of the $\xi(t)$ are slow.

**Remark 4.2:** Unstructured parameter variations If no a priori information on the structure of the plant parameter variations is available i.e., in (4.1), $\nu = 0$, $f_1(t) = 1$ and $\theta(t)$ are functions of $\nu$ then the MRAC scheme (3.5)-(3.6), (4.2), (4.3)-(4.7) reduces to the one presented in (1). In this case, by Corollary 4.1 with $\nu = \nu_0$, the new MRAC guarantees the boundedness of all signals in the closed loop and the smoothness in the mean of the tracking error for slow plant parameter variations that is, for any $\mu \in [0, \nu^*]$ and for some $\nu^* > 0$. We note that, in contrast to the standard MRAC [4], the new adaptive controller requires no a priori knowledge about the stability margin of the inverse plant.

The significance of Theorem 4.1 and Corollary 4.1 is that it allows a wide class of TV plants which may possess fast TV parameters to be adaptively controlled in a stable manner and with a zero or "small" residual tracking error, at the expense of course of updating additional parameters and requiring some a priori knowledge about the speed and the structure of the time variations of the fast parameters. It should be noted that Theorem 4.1 is based on the fact that there exist $\theta(t), \phi(t)$ so that the TV operator of the closed-loop plant is equal to that of the reference model. Such a property is guaranteed by using the new MRAC structure, but not the standard one and therefore, Theorem 4.1 will not be

1 Taking $\xi(t)$ to be TV will only result to similar but more complicated expressions.
applicable to the MRAC which is based on the standard MRC structure.

5. Conclusion

In this paper we considered the MRAC problem of LTV plants. The plant parameters were assumed to be bounded, smooth functions of time which satisfy the usual assumptions of MRC for LTV plants, extended to the TV case. We then designed a MRAC scheme by using the new MRC structure, proposed in [1], together with a suitable robust adaptive law for adjusting the controller parameters. We showed that, the assumption of slow parameter variations, imposed by the finite speed of adaptation in the case of unstructured parameter variations, can be relaxed in the case where some information about the frequency or form of variations of the fast TV plant parameters is available a priori. Such a priori information can be utilized by an appropriately designed adaptive law so that the stability and tracking performance of the new MRAC is considerably improved for a wider class of TV plants which includes plants with fast parameter variations.

The development of other controller structures, such as pole placement and structures derived as the solution of the linear quadratic problem ([22]) for LVTV plants, are topics for future research.

References


