ADAPTIVE CONTROL OF LINEAR TIME-VARYING PLANTS

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ABSTRACT

In this paper we consider the model reference adaptive control (MRAC) problem of a class of linear time-varying (LV) plants. The plant parameters are assumed to be smooth, bounded functions of time which satisfy the usual assumptions of MRAC for time-invariant plants, at each frozen time instant. We first show that if the plant parameters are sufficiently slowly-varying with time, a controller parameter vector with smooth elements exists, such that the closed loop plant behaves almost linearly time invariant reference model. We then use the robust adaptive law proposed in Ioannou and Tatsaklis (1995) to adjust the controller parameters and establish boundedness for all signals in the adaptive loop for any bounded initial conditions. The bound for the residual tracking error depends on the speed of the plant parameter variations in such a way that as these parameters become constant the bound reduces to zero.

KEYWORDS


1. INTRODUCTION

One of the major advantages of adaptive controllers over fixed designs, stems from their ability to adjust their parameters on-line. It should be expected that such a property would be beneficial for the control of plants that are slowly time-varying (TV) or can be considered as such, e.g., nonlinear systems whose operating conditions change slowly with time. However, many hard results in adaptive control theory deal with plants that are unknown or linear Time Invariant (LTI) [Landau (1979); Egerett (1980), Goodwin and Sin (1984)]. One of the difficulties of applying most of the adaptive control schemes designed for LTI plants to TV ones is their lack of robustness with respect to perturbations such as bounded disturbances, unmodeled dynamics, etc. [Egerett (1980); Peterson and Narendra (1982); Ioannou and Kokotovic (1993)]. Since parameter variations can be shown to introduce similar types of perturbations the stability of most of the schemes cannot be guaranteed in a time varying environment. Despite this difficulty, several results have been reported in the literature on the adaptive control problem of TV plants. Anderson and Johnstone (1983) used the concept of a time-varying state parameter vector to obtain a formulation of the adaptive control problem of a slowly TV discrete-time plant. Under the assumption that certain signals in the adaptive loop are persistently exciting (PE) they showed the existence of a region of attraction from which all signals are bounded. Following similar steps Goodwin and Teoh (1983) gave a new PE condition for discrete LTI plants in the presence of possibly unbounded signals and investigated qualitatively the results of Anderson and Johnstone (1983) for jump and slowly varying parameters. Martin-Sanchez (1985) also gave a solution to the adaptive control problem of discrete-time plants whose parameters have a finite number of bounded changes. In Chen and Caines (1986) the parameter variations are assumed to be perturbations of some nominal fixed parameters, which are small in the norm, and are modeled as a convergent martingale process with bounded covariance. For this class of parameter variations they established global convergence for a stochastic adaptive control algorithm applied to a multi-input multi-output discrete-time plant. Goodwin, Hill and Xian (1989) showed that a standard stochastic adaptive control algorithm designed for TV plants can be directly applied to a TV discrete-time plant provided the parameter variations decay to zero exponentially fast.

In this paper we considered the MRAC problem of continuous-time LTV plants. In our formulation we assume a separation between the time scale of the plant states and the time scale. Subsequently, we decomposed the problem into the two parts. In the first and by far the most important part, we show that for the same controller structure as in the LTI case [Narendra and Valavani (1976)] there exists a slowly TV control parameter vector $\theta(t)$ with smooth elements, such that the closed loop input-output (I/O) operator is equal to the I/O operator of the LTI reference model within an $O(\epsilon)$ approximation, provided that the plant parameters are smooth bounded functions of time $\mu$ and $\mu$ is sufficiently small. In the second part we use the Robust Adaptive Controller (RAC) of Ioannou and Tatsaklis (1995) to adjust the controller parameter vector $\Delta(t)$ and establish stability. Due to the perturbations introduced by the plant parameter variations in the adaptive control system the stability of most adaptive control schemes cannot be guaranteed [Ioannou and Kokotovic (1983)] and therefore the use of robust adaptive control is essential. Under the assumption that the frozen plant at any time satisfies the usual assumptions for time invariant MRAC we show that, for sufficiently small $\epsilon$, the RAC of Ioannou and Tatsaklis (1995) guarantees boundedness of all signals for any bounded initial conditions.

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Furthermore, the tracking error e converges to a residual bounded set, which depends on w in such a way that as e(0) → 0, e(t) reduces to zero asymptotically. The only additional a priori information required by the SAC is a lower bound on the stability margin of the frozen inverse plant.

2. MATHEMATICAL PRELIMINARIES

In this section we establish certain properties of time varying polynomial differential operators which are used in the sections to follow.

Definition 2.1. A linear time varying polynomial operator of degree n is defined by

\[ P(s,t) = a_n(t)s^n + a_{n-1}(t)s^{n-1} + \cdots + a_1(t)s + a_0(t) \]

where \( \frac{d}{dt} \) is \( \frac{\partial}{\partial t} \); \( a_i(t) \), \( i = 1,2, \ldots, n \) are bounded functions of time. When \( a_n(t) = 1, \forall t \geq 0 \) \( P(s,t) \) is referred to as a monic polynomial operator.

Lemma 2.1. Let \( P(s,t) \) be a monic time varying polynomial operator of degree n. Then \( P^{-1}(s,t) \) exists and is defined by

\[ y(t) = P^{-1}(s,t)x(t) = \int_0^t P(s,t)u(s) \, ds \]

Equation (2.2) shows that the operator \( P^{-1}(s,t) \) introduced in (2.2) is well defined. It remains to show that \( P(s,t)P^{-1}(s,t) = I \) for zero initial conditions.

Proof. From (2.2) we have that

\[ y(t) = \sum_{i=0}^{n} a_i(t)x(t) \]

which is a linear time varying differential equation whose solution exists, is unique (Vidyasagar 1978) and is given by

\[ y(t) = \mathbf{c}^T(t) \mathbf{x}(t) \]

where \( \mathbf{c} \) is a constant vector, \( \mathbf{s}(t+\mathbf{t}) \) is a state transition matrix that corresponds to \( \mathbf{c} \), and \( \mathbf{y}(t) \) is a vector of initial conditions \( \mathbf{y}(0) \). Equation (2.4) shows that the operator \( P^{-1}(s,t) \) introduced in (2.2) is well defined.

Lemma 2.2. Let \( P(s), Q(s) \) be polynomial operators of the form

\[ P(s) = \sum_{i=0}^{n} \beta_i(s) s^i, \quad Q(s) = \sum_{i=0}^{m} \beta_i(s) s^i \]

where \( \beta_i(s), \beta_i(s) \) are \( \mathbb{R}^n \times \mathbb{R}^n \) matrices that are \( \mathbb{R}^n \times \mathbb{R}^n \) matrices with respect to the parameter \( \mathbb{R} \), \( \mathbb{R} = \max(n,m) \) and \( \mathbb{R} = \max(n,m) \) then

\[ P(s) + Q(s) = Q(s) + P(s) \]

\[ P(s)x(t) = x(t)P(s) \]

where \( P(s) = \sum_{i=0}^{n} \beta_i(s) s^i \) and \( x(t) = \sum_{i=0}^{n} x_i(t) s^i \) for \( x_i(t), \beta_i(s) \) are n-dimensional state vector and input vector respectively.

3. MODEL FOLLOWING FOR TIME-VARYING PLANTS

3.1 The Plant and the Control Objective

The single-input single-output plant \( P \) to be controlled is completely represented by the input-output pair \( u(t), y(t) \) that satisfies the following linear ordinary differential equation
\[ \frac{d}{dt} \beta(t) = u_p(t) \]
\[ y_p(t) = k_p(n_p(s)) \beta(t) \]

where

- \( s \) is the differential operator \( \frac{d}{dx} \)
- \( d_p(s) \), \( n_p(s) \) are monic polynomials of degree \( p \) and \( n \) respectively, and their coefficients are functions of \( \beta \), i.e.,
  \[ d_p(s) = \sum_{i=0}^{n_p} d_{pi} s^{n_p-i}, \quad n_p(s) = \sum_{i=0}^{n} n_{ni} s^{n-i} \]

We will make the following assumptions for the plant:

1. \( n \) and \( m \) are known and constant;
2. \( d_p(s) \), \( n_p(s) \) are bounded and they possess at least \( 2n-1 \) bounded derivatives with respect to \( \beta \);
3. \( d_p(s) \), \( n_p(s) \) are strongly coprime polynomials \( \forall \beta \in \mathbb{R}^n \);
4. The sign of \( k_p(s) \) is known and constant (without loss of generality we will assume that \( k_p(s) = 1 \) \( \forall \beta > 0 \)). The range of \( k_p(s) \) is a subset of a closed interval on the real line that does not contain 0.
5. \( n_p(s) \) is a Hurwitz polynomial of \( s \) for all \( \beta > 0 \) and an upper bound \( \lambda \) is known.

The adaptive control problem can be stated as follows: Given the reference model

\[ \frac{d}{dt} \alpha(s) = \beta(s) \]

where \( d_p(s) \) is a monic Hurwitz polynomial of degree \( n = n_p + n_m \) with time-invariant coefficients and \( \beta(t) \) is a uniformly bounded reference signal, design an adaptive controller so that for some \( \lambda > 0 \) and all \( \beta(t) \) the resulting closed-loop plant is stable and the plant output \( y_p(t) \) tracks the output \( \alpha(s) \) as closely as possible.

Remark 3.1: The assumed representation of the plant in (3.1) is equivalent to the phase variable canonical form. As shown in Silverman (1966), a LTV system can be transformed to the form (3.1) if it is uniformly controllable. Hence, our plant representation is not restrictive.

3.2 The Structure of the Controller

The input \( u_p \) and the output \( y_p \) are used to generate a \( (2n-1) \)-dimensional auxiliary vector as

\[ \begin{bmatrix} u_p(t) \\ y_p(t) \end{bmatrix} = \begin{bmatrix} F(t) & G(t) \end{bmatrix} \begin{bmatrix} \alpha(t) \\ \beta(t) \end{bmatrix} \]

where \( F \) is a stable matrix \( \in \mathbb{R}^{(n+1) \times (n+1)} \), \( (F,G) \) is a controllable pair, and \( \tilde{u} = [u_p(t), y_p(t)] \). The input to the plant is taken as

\[ u_p = \tilde{u}^T(t) \alpha(t) + c_0(t) \beta(t) \]

where \( \tilde{c}(t) = [c_1(t), c_2(t), \ldots, c_{2n-1}(t)] \) is a \( (2n-1) \)-dimensional control parameter vector and \( c_0(t) \) is a feedforward scalar parameter. This controller is the same as the one used in Narendra and Valavani (1979) and it is the outcome of a standard approach to the design of MRAC which involves two steps. The first step is to assume that the plant is known and design a linear controller, parameterized by a parameter vector \( \theta \), which will make the closed-loop input-output operator equal to that of a reference model. The second step is to design an estimator which will estimate the parameters of the desired controller (Direct Adaptive Control). While this approach has produced a solution to the MRAC problem when the plant is linear but time-varying.

3.3 The Closed-Loop System and the Existence of \( \alpha(s) \)

In this section we will give a solution to the first part of the design of a MRAC scheme. The main idea of our analysis is to express the closed-loop 1/0 operator in terms of the controller parameter vector \( \theta(t) \) and then, to select \( \theta(t) \) in order to satisfy a certain matching condition. Observing that the 1/0 operator of the filters (3.4), (3.5) can be written as

\[ F_1(s) = s^2 \alpha_1^T(t)(sI - F_1^{-1})^{-1} \alpha_1(t) = i, 1, 2 \]

where the coefficients of \( n_1(s) \) depend on \( \phi_1(t) \),

\[ d(s) = \det(sI - F_1) \]

\[ n_1(s) = \lambda n_2(s) = n_2(t) \]

and the operators of the cascade and the feedback compensators become (dropping the arguments of the polynomials for convenience)

\[ F_2 = \gamma_3^{-1} \quad \alpha_2(t) = d^{-1} \gamma \]

Then, the closed loop 1/0 operator \( G_c \) becomes

\[ G_c = \gamma_3^{-1} \gamma, \quad c_0 = k_p \theta \]

and the model following requirement \( G_c = \gamma \)

becomes

\[ k_p \theta = \gamma_3^{-1} \gamma - \gamma_3^{-1} k_p \theta \]

(3.9)

It is now apparent that due to the non-commutative algebra of the time-varying operators, the term \( \gamma_3^{-1} \gamma \) cannot be factored out of the LHS of (3.9) and the solution for \( n_1, n_2 \) cannot be found simply as the solution of a Bezout equation as in the LTI case. The cause of this problem is the choice of the structure of the compensators which gives a controller operator of the form \( d^{-1} \gamma_3^{-1} \gamma \).

Unlike the time-invariant case, we are not able to obtain, in general, a left coprime factorization of this controller, which is a crucial step in the parameterization of the stabilizing compensators for a given plant [Benson et al. (1980)]. Therefore, exact matching of the closed-loop plant with the LTI reference model cannot be obtained by using this approach. Instead, in Appendix A.1 we show that if we choose \( n_1^*, n_2^*, c_0^* \) being equal to \( n_1^*, n_2^*, c_0^* \).
obtained by a pointwise solution of (3.9) (i.e., considering the frozen plant at each time instant) the closed loop 1/0 operator becomes

$$G_c = \sum_k u_k$$  \hspace{1cm} (3.10)

where L is a strictly proper operator which is exponentially stable for sufficiently small \( u \). Furthermore, according to this choice, if the assumptions (4.2) hold, then the coefficients of \( n_0^s \), \( n_2^s \) as well as \( c_0^s \) are smooth and bounded functions of \( u \) possessing \( n \) bounded derivatives. From (3.7) we can write

$$n_i^s((u_d)\cdot d^i(s) = \sum_{k=0}^i q_k(s) \cdot n_k^s(s) + q_{i+1}(s)\cdot n_1^s(s); \hspace{1cm} i=1,2$$  \hspace{1cm} (3.11)

where \( q_k(s), j \leq i \) are polynomials of \( s \) with constant coefficients. If we let \( L \) be the matrix of the coefficients of \( q_k(s) \) and \( n_k^s(s) \) the vector of the coefficients of \( n_k^s(s) \) (given by (3.7)) as linear functions of \( n_1^s, n_2^s \) we get

$$\sigma^T(0) - n_0^s(0) = 0$$  \hspace{1cm} (3.12)

Since \( f, q \) is a controllable pair, \( q^{-1} \) exists and, hence, all the properties of the coefficients of \( n_k^s(0) \) hold for \( n_k^s(0), c_0^s \) as well.

Remark 3.2. The choice of \( \sigma^s(0) \) as the one that satisfies the matching condition pointwise (that is, for the frozen plane) implies that if we consider the frozen closed loop system at a time instant \( t \), its eigenvalues will be the same as if the plane were LTI, i.e., the roots of \( d(s), q_0(s) \) and \( n_0^s(s) \) for every \( t \geq 0 \).

Remark 3.3. Equation (4.8) shows that the frozen mismatch operator expressed in the state space has eigenvalues \( u \)-close to the roots of \( d(s), q(s) \) and \( n_0^s(s) \) for every \( t \geq 0 \). If we let \( \Delta u \) be the maximum root of \( d(s), q(s) \) then for every \( \epsilon \leq 0, \min (q(s)) \) there exists \( u(q) > 0 \) such that for all \( u \in (\Delta u) \) the rate of exponential decay of \( L(s) \) is at least \([\min (\Delta u) - \epsilon] \)

(Desoer and Vidyasagar [1975]). In the following sections we assume that \( u \in (\Delta u) \) so that \( L \) is a stable operator.

4. MRAC FOR TIME-VARYING PLANTS

4.1 The Adaptive Laws

In section 3 we established the existence of a slowly time varying controller parameter vector \( \Theta^s(0) \) that satisfies the matching condition (3.9) and gave the properties of the mismatch operator \( L \). The problem of designing an MRAC for slowly time-varying plants is thus well posed. In this section we will study the stability properties of the closed loop when the adaptive law, introduced in Ioannou and Tsakalis [1985], is used to update the controller parameters. The complexity of the adaptive law for adjusting the controller parameters \( \Theta(t) \) is determined by the prior knowledge of the gain \( \kappa_0^s \) of the plant. We first deal with the simple case, when \( \kappa_0^s \) is constant and known, and later consider the case when \( \kappa_0^s \) is unknown and varying with \( \sigma \). For clarity of presentation, the proof of stability is also given separately for the two cases.

Case I (\( \kappa_0^s \) Constant and Known). With no loss of generality we can assume that \( \kappa_0^s = \kappa_1 = 1 \). This implies that \( \kappa_0^s = 1 \) in (3.2). The equation for the adaptive law to adjust the parameter vector \( \theta \) is

$$\dot{\theta} = -\frac{\tau^e + \tau^c}{\tau} \cdot \theta + \theta_0, \quad \tau^e = \begin{cases} 0 & \text{if } |\theta| < \theta_0 \\ \frac{|\theta|}{\theta_0^2} & \text{if } |\theta| > \theta_0 \end{cases}$$  \hspace{1cm} (4.1)

where \( \tau > 0 \)

$$\tau^e \cdot \theta_0 = \tau^c \cdot \theta_0 \cdot \theta_0 \cdot \theta_0 \cdot \theta_0 \cdot \theta_0 \cdot \theta_0$$  \hspace{1cm} (4.2)

$$\dot{\Theta} = -\Delta \theta + \frac{1}{\tau} \cdot \dot{\Theta}$$  \hspace{1cm} (4.3)

Case II (\( \kappa_0^s \) Unknown and Slowly Time-Varying). The equation for the adaptive law now given by

$$\dot{\theta} = -\frac{\tau^e + \tau^c}{\tau} \cdot \theta + \theta_0$$  \hspace{1cm} (4.4)

where \( \tau^e \) is as defined in (4.1) with \( \theta_0 \), \( \theta_0 \) replaced by \( \theta_0 \), \( \theta_0 \) respectively.

$$\tau^e = \begin{cases} 0 & \text{if } |\theta_0^s| < \theta_0 \hspace{1cm} \tau^c = \begin{cases} 0 & \text{if } |\theta_0^s| < \theta_0 \\ \frac{|\theta_0^s|}{\theta_0^2} & \text{if } |\theta_0^s| > \theta_0 \end{cases} \end{cases}$$  \hspace{1cm} (4.5)

and the signals, \( m, v, \varphi \) are as given in (4.2), (4.3). In this case two additional parameters \( \Theta_0 \), \( \Theta_0 \), need to be adjusted otherwise, the form of the adaptive laws is the same as in case 1.

In (4.1)-(4.5) \( M_0, M_0, \varphi_1, \varphi_0, \theta_0, \theta_0 \) are positive design parameters, \( M_0 > \Theta^s(0), M_0 > \Theta^s(0) \). The parameter \( \epsilon_0 \) is designed so that

$$\epsilon_0 \cdot \epsilon_0 - \epsilon_0 \cdot \epsilon_0 \leq \min (\epsilon_0, \epsilon_0)$$  \hspace{1cm} (4.6)

where \( \epsilon_0 > 0 \) is such that the poles of \( M_0(s) \) and the eigenvalues of \( F + \theta_0 \) are stable, \( \alpha \) as defined in (5.5) and \( \epsilon_0 \) is a positive constant. Since \( \kappa_0^s(s) \) and \( \Theta(s) \) have stable poles and are to be designed (4.6) can always be satisfied if \( \alpha \) is known.

4.2 Stability Analysis

Our main stability result is given by the following theorem where a class of slowly time varying plants is defined for which the controller (3.4) -(3.5), (4.1)-(4.5) guarantees the boundedness of all signals in the closed loop, and the smallness in the mean of the residual tracking error. This class is defined in terms of the time scale ratio.
for TV plants described by (3.1) and satisfying the assumptions (5.1) - (5.5).

Theorem 4.1. Assume that $r_0, r$ are bounded. Then there exists a $u > 0$ such that for each $u \in (0, u^*)$ all the signals in the closed-loop plant (3.1) with the controller (3.5), (3.6), (4.1), (4.2) are bounded for any bounded initial conditions. Furthermore, there exists a constant $\gamma > 0$ and a small constant $\zeta$ such that the tracking error $e_1 = x_p - x_m$ belongs to the residual set

$$D = \left\{ e_1, u, \mu, e_1 \geq \frac{e_1^0}{T_0} \right\}$$

(4.7)

Corollary 4.1. If the parameters of the plant are constant for some finite time, i.e., $u = 0$, the algorithm (3.1) - (3.6) guarantees boundedness as well as convergence of the tracking error $e_1$ to zero.

Outline of the Proof (1) The proof of Theorem 4.1 can be decomposed into two steps:

(i) By using the properties of the normalizing signal $m$ (Ioannou and Tsakalis (1985), Lemma A.1) and the expression for the augmented error from Appendix A.2, we can show that the parameter error $A_N m(t)$ is uniformly bounded. Furthermore, the signals $\dot{q}_1, \dot{q}_2, \dot{q}_3, \dot{q}_4, \dot{q}_5, \dot{q}_6$ are bounded and small in the mean for small $u$, irrespectively of the boundedness of $m, \dot{m}, u$ or any other signal in the adaptive loop.

(ii) Considering the nominal state representation of the closed-loop given in Appendix A.3, the boundedness of all signals in the closed loop can be established for sufficiently small $u$.

This result is an application of the Bellman–Gronwall Lemma (Deser and Videgård (1975)) on a weighted sum of $k_1, m$ where $k_1$ is the nominal state in A.17. It is, then, concluded that the smallness in the mean of $\dot{q}_1, \dot{q}_2$ implies the smallness in the mean of $\dot{q}_1, \dot{q}_2$ and $e_1$.

When $k_1$ is unknown and slowly TV, Theorem 4.1 can be easily derived as an extension of the case of $k_1 = 1$. The proof of Corollary 4.1 follows from Corollary 4.1 in Ioannou and Tsakalis (1985), since for $u = 0$ the class of LTI plants described by (3.1) is identical to the class of nominal plants considered in that reference.

5. CONCLUSIONS

In this paper we have shown that a controller $C(s)$ with the structure given by (3.1), (3.6) can be adaptively adjusted to control slow- time-varying and pointwise minimum phase linear plants. For such a controller there exists a bounded smooth time-varying vector $\theta(t)$ which makes the closed loop plant together with the controller $C(s)$ behave almost like an LTI reference model. We have also shown that for sufficiently slow time variations of the plant parameters (u sufficiently small), the mismatch between the closed-loop plant and the reference model is exponentially stable and "small." By using normalized signals [Praly (1983)] to perform the adaptation of the controller parameters, together with a switching $w$-modification we can guarantee boundedness of all signals in the loop and small in the mean tracking error. The additional a priori information needed for the design of the adaptive algorithm is a knowledge of an upper bound on $\theta$ and a lower bound on the rate of exponential decay of the frozen inverse plant. It is worth mentioning that the constraint of the boundedness of the time-varying plant parameters does not imply the existence of a nominal LTI plant. Furthermore, if the plant is LTI or becomes LTI after some finite time, it is shown that the tracking error will go to zero.

6. REFERENCES


APPENDIX

A.1 The Closed Loop Operator: Existence of $y$, $x$ and Characterization of the Mismatch

Using the properties of the polynomial operators from section 2 and dropping the arguments $x, t$, the closed loop I/O operator becomes

$$G_c = k_p(s, n_1, n_2, n_3)$$

A.1.1

Due to space limitation only an outline of the proof of Theorem 4.1 will be given here. The complete proof is included in the full version of the paper.
From Lemma 2.2 ([13]) \( d^{-1} \alpha d = d^{-1} y d^{-1} - u d^{-1} x d^{-1} \), 
\( d^{-1} \mu d = \kappa d^{-1} - \kappa d^{-1} \), where the maximum degree of \( \lambda, \mu \) is 2n-3, m+n-2 respectively. 
Moreover, denoting by (7) the pointwise multiplication of two operators as in the proof of 
Lemma 2.2 ([13]), we have \( \pi \in (d^{-1} + \kappa d^{-1} + u d^{-1}) \), 
\( \pi \in (d^{-1} + \kappa d^{-1} + u d^{-1}) \) where the maximum degree of \( \lambda, \mu \) is 2n-3, m+n-2 respectively. 
Hence

\[
\pi = k \pi d^{-1} (\mu d^{-1} + \kappa d^{-1} + u d^{-1}) - x d^{-1} y d^{-1} 
\]

(A.2)

Let us first concentrate on the term \( \pi = d^{-1} + \kappa d^{-1} + u d^{-1} \).

The diophantine equation

\[
\pi = d^{-1} + \kappa d^{-1} + u d^{-1} = 0 
\]

(A.3)

where \( \pi \) is an arbitrary monic polynomial of degree 2n-1 has a solution for \( \pi_1, \pi_2 \) provided that \( \pi_1, \pi_2 \) are coprime [Narendra and Valavani (1973)]. In such a case the coefficients of \( \pi_1, \pi_2 \) can be found as the solution of an algebraic system of linear equations \( x = b \), \( x = b \) where \( x \) is the vector of the unknown coefficients, \( A \) is the symmetric matrix of \( b, \kappa b, \pi b \) and \( b \) is a vector containing linear combinations of the coefficients of \( A, d^{-1} + \kappa d^{-1} + u d^{-1} \). The strong coprimeness of \( \pi_1, \pi_2 \) guarantees that \( \det A \geq c \) for all times, which implies that the coefficients of \( \pi_1, \pi_2 \) will be \( \text{OE} \) \[ \text{OE} \] \( b \). Hence, by choosing \( D = \frac{\pi}{\pi b} \), \( \pi_1, \pi_2 \) obtained by solving (A.3), are bounded functions of \( \alpha \) possessing at least \( n \) bounded derivatives. Setting \( \pi = d^{-1} + \kappa d^{-1} + u d^{-1} \), \( \pi \) becomes

\[
\pi = d^{-1} + \kappa d^{-1} + u d^{-1} 
\]

(A.4)

Applying the matrix inversion lemma, which holds for algebras with commutative addition and non- 
commutative multiplication, on (A.4) we get

\[
\pi = \pi_1 + u \pi_2 
\]

(A.5)

\[
L = [d^{-1} + \kappa d^{-1} + u d^{-1}]^{-1} 
\]

(A.6)

\[
L = [d^{-1} + \kappa d^{-1} + u d^{-1}]^{-1} 
\]

(A.7)

Hence, from (A.5) we have that the closed loop \( I/O \) operator is equal to the model operator 
perturbed by a mismatch operator multiplied by \( u \). It remains to be shown that the mismatch operator 
is stable and strictly proper \( (i.e., L(s) \) does not contain \( \mu \) or \( \kappa \) or \( u \) or \( d \) or \( \alpha \) or \( \beta \) or \( \gamma \)). 
For the forgot part of \( L \) can be written as

\[
L = [d^{-1} + \kappa d^{-1} + u d^{-1}]^{-1} 
\]

(A.8)

From (A.6) - (A.8) the mismatch operator \( L \) can be realized as a strictly proper slowly TV system 
whose pointwise eigenvalues are \( \mu \)-close to the roots of \( d(s), \kappa(s), \mu(s), \pi(s) \) for all times.

Since \( d(s), \kappa(s), \mu(s), \pi(s) \) are Hurwitz polynomials, it can be concluded [Desoer and Vidyasagar (1975)] that \( L \) is exponentially stable for sufficiently small \( u \).

A.2 The Dependence of \( c, \kappa \) on the Parameter Error

From the definition of \( c, \kappa \) of the vector that satisfies the matching condition (A.5) and the 
decomposition \( \alpha = \beta + c \), we have

\[
\pi = \pi_1 + \kappa d^{-1} + u d^{-1} = \pi_1 + \kappa d^{-1} + u d^{-1} = \pi_1 + \kappa d^{-1} + u d^{-1} 
\]

(A.9)

Moreover, from the definition of \( d^{-1} + \kappa d^{-1} + u d^{-1} \) we get

\[
\pi = \pi_1 + \kappa d^{-1} + u d^{-1} = \pi_1 + \kappa d^{-1} + u d^{-1} 
\]

(A.10)

The last two terms in (A.10) can be written as

\[
\pi = \pi_1 + \kappa d^{-1} + u d^{-1} = \pi_1 + \kappa d^{-1} + u d^{-1} 
\]

(A.11)

Letting \( \pi = \pi_1 + \kappa d^{-1} + u d^{-1} \) and using the differentiability properties of \( d(s) \) and \( \kappa(s) \), the operator \( d^{-1} + \kappa d^{-1} + u d^{-1} \) can be expressed as

\[
\pi = \pi_1 + \kappa d^{-1} + u d^{-1} = \pi_1 + \kappa d^{-1} + u d^{-1} 
\]

(A.12)

where \( \pi_1 \) is an \( (n-1) \) order polynomial operator with bounded coefficients since, from section 3.3, 
\( d^{-1} + \kappa d^{-1} + u d^{-1} \) are bounded for \( \pi = \pi_1 \). Combining (A.11) and (A.12) we get

\[
\pi = \pi_1 + \kappa d^{-1} + u d^{-1} = \pi_1 + \kappa d^{-1} + u d^{-1} 
\]

(A.13)

\[
\pi = \pi_1 + \kappa d^{-1} + u d^{-1} = \pi_1 + \kappa d^{-1} + u d^{-1} 
\]

(A.14)

From Appendix A.1, Remark 3.3 and 4.6, when \( u \) is sufficiently small, \( \pi \) can be viewed as the output 
of a strictly proper exponentially stable operator with rate of decay at least \( \delta u \) with input bounded
by \( \|u\| \). 

A.3 A Nonminimal State Representation for the 
Closed Loop \( (K, \pi) \)

In the state space (3.1) can be written as

\[
\pi = d^{-1} + \kappa d^{-1} + u d^{-1} = \pi_1 + \kappa d^{-1} + u d^{-1} = \pi_1 + \kappa d^{-1} + u d^{-1} 
\]

(A.15)

where \( d(s), \kappa(s), \mu(s), \pi(s) \) are composed by the coefficients of \( d(s), \kappa(s), \mu(s), \pi(s) \). Combining \( d(s), \kappa(s), \mu(s), \pi(s) \) as in [Narendra and Valavani (1973)], we obtain

\[
\pi = \pi_1 + \kappa d^{-1} + u d^{-1} = \pi_1 + \kappa d^{-1} + u d^{-1} 
\]

(A.16)

and \( \pi = \pi_1 + \kappa d^{-1} + u d^{-1} = \pi_1 + \kappa d^{-1} + u d^{-1} \). However, from Appendix A.1, (3.1) is the solution \( \pi \) that corresponds to the closed loop \( \pi \), where \( \pi \) is also.

Therefore, \( \pi \) is slowly TV system with eigenvalues the roots of \( d(s), \kappa(s), \mu(s), \pi(s) \) for \( \pi = \pi_1 \) 
[Narendra and Valavani (1973)].