Model Reference Adaptive Control for Plants with Unknown Relative Degree
Gang Tao and Petros A. Ioannou

Abstract—One of the basic assumptions in stable model reference adaptive control (MRAC) that the relative degree \( n^* \) of the modeled part of the plant is known exactly and matches that of the reference model is relaxed by the development of a new MRAC scheme which requires only an upper bound \( n^* \) for \( n^* \). This is done at the expense of updating additional parameters and projecting some of the estimated parameters onto appropriate convex sets in the parameter space. These convex sets can be developed without using any \textit{a priori} knowledge about the plant parameters. Furthermore, when a lower bound \( n^* \) in addition to an upper bound for \( n^* \) is known, the number of the estimated parameters can be reduced considerably. The new MRAC scheme guarantees signal boundedness and zero residual tracking error. In addition, it is robust with respect to plant uncertainties provided a modified adaptive law is used to update the controller parameters.

I. INTRODUCTION

In the design and analysis of stable model reference adaptive control (MRAC) schemes the following assumptions [11]-[9] are commonly used for the modeled part of the plant: i) the plant zeros are stable; ii) the plant relative degree \( n^* \) is known exactly and matches that of the reference model; iii) the sign of the high-frequency gain is known; iv) an upper bound for the plant order is known. Assumption ii) has recently been relaxed in [10] by using a Nussbaum gain controller. In addition, it has been shown in [10], [11] that assumption ii) can be relaxed for plants whose relative degree \( n^* \) satisfies \( 1 \leq n^* \leq 3 \).

In this role, we propose a new MRAC scheme different from that proposed in [10], which requires only the upper bound \( n^* \) for the relative degree \( n^* \) of the plant. The new scheme employs a feedforward dynamic block, rather than the static one used in the known \( n^* \) case, whose numerator coefficient vector and a related parameter vector are updated and the latter is constrained to be in a certain convex region \( S_p \). In contrast to the schemes presented in [10], [11] our new scheme is applicable to plants with \( n^* \geq 1 \) and is not of high gain. Closed-loop stability and zero residual tracking error are achieved at the expense of updating some additional parameters and projecting some of the estimated parameters onto \( S_p \). The region \( S_p \) can be developed without using any \textit{a priori} knowledge about the plant parameters, therefore, assumption ii) is relaxed at the expense of no additional assumptions for the unknown plant. Moreover, if we also know a lower bound \( n^* \) for \( n^* \), then the number of the estimated parameters can be reduced and the implementation of the MRAC scheme becomes less complex.

The note is organized as follows: in Section II, a new parameterization scheme is presented for model reference control which guarantees stable model-plant transfer function matching. The new scheme has a linear dynamic feedforward block whose numerator and denominator should be stable polynomials for closed-loop stability. In Section III, an adaptive law is designed to update the controller parameters in the feedback loop and the coefficients of the numerator of the feedforward block as well as some additional parameters. Since for stability, the inverse of the polynomial differential operator defined by some of the estimated parameters has to be stable the adaptive law for those parameters is modified using a projection technique. The stability analysis of the proposed MRAC scheme is given in Section IV. In Section V, the robustness of the MRAC scheme with respect to unmodeled dynamics is established.

II. PROBLEM STATEMENT AND A NEW CONTROL STRUCTURE

Let us consider the following single-input, single-output linear time-invariant plant

\[ y(t) = G_d(s)[u(t), G_o(s) = k_p \frac{Z_o(s)}{R_o(s)} \] (2.1)

where \( G_d(s) \) is the plant transfer function which is strictly proper with \( Z_o(s), R_o(s) \) being monic polynomials, \( y(t) \) and \( u(t) \) are the plant output and input, respectively.

The control objective may be stated as follows: Given a reference model

\[ y_m(t) = W_m(s)[r(t) \] (2.2)

where \( W_m(s) \) is a stable transfer function, \( r(t) \) is piecewise continuous and uniformly bounded, find the feedback control \( u(t) \) such that the closed-loop plant is globally stable in the sense that all signals are uniformly bounded and \( y(t) \) tracks \( y_m(t) \) as close as possible.

In order to meet the control objective we make the following assumptions for the plant:

A1) \( Z_o(s) \) is a Hurwitz polynomial; A2) an upper bound \( \bar{n} \) for the unknown degree \( n \) of \( R_o(s) \) is known; A3) the sign of \( k_p \) is known; A4) the relative degree \( n^* \) of \( G_d(s) \) satisfies \( n^*_l \leq n^* \leq n^*_u \), where \( n^*_l \) and \( n^*_u \) are known.

It follows that \( n^*_l \geq 1 \) and \( n^*_u \leq \bar{n} \) because \( G_d(s) \) is strictly proper and \( n^* \leq n \).

We choose the reference model as: \( W_m(s) = (k_m/D_m(s)) \), where \( k_m \in R^+ \) and \( D_m(s) \) is a monic Hurwitz polynomial of degree \( n^*_m \), and propose to use the following controller structure:

\[ u(t) = \theta^T_0 \omega_0(t) + \theta^T_1 \omega_1(t) + \theta_2 y(t) + \theta_3 y(t) \] (2.3)

\[ \omega_0(t) = \frac{a(s)}{\Lambda(s)}[u(t)], \quad \omega_1(t) = \frac{a(s)}{\Lambda(s)}[y(t)], \quad \omega_2(t) = \frac{b(s)}{D_m(s)}[r(t)] \] (2.4)

where \( a(s) = (1, s, \cdots, s^{n^*_m-1})^T, b(s) = (1, s, \cdots, s^{n^*_m-1})^T, n^*_m = n^*_u - n^*_l \) is the uncertainty in the relative degree \( n^* \), \( \theta_1, \theta_2, \theta_3 \in R^{n^*_m+1}, \theta_3 \in R^{n^*_m+1}, \theta_2 \in R, \) and \( \Lambda(s), n_2(s) \) are arbitrary monic Hurwitz polynomials of degrees \( \bar{n} - 1, n^*_m \), respectively. The difference between the control structure (2.3) and the usual MRAC scheme is the use of the feedforward dynamic term \( \theta^T_0 \omega_0(t) = \theta^T_0 \left[ b(s)m_0(s)/D_0(s)[r(t)] \right] \) rather than the static term \( k_0 r(t) \). When \( n^*_m \) is known exactly we have that \( n^*_m = 0, n^*_l = n^*_u = n^* \) and \( b(s) = 1 \), and for \( n_1(s) = D_m(s) \) the controller structure (2.3) reduces to the one used in the standard MRAC scheme where \( \theta_3 \) is a scalar.

The following lemma ensures the existence of constant parameters \( \theta_1, \theta_2, \theta_20, \theta_3 \) for which the closed-loop plant is stable and the plant-model matching is achieved.
Lemma 2.1: There exist constant vectors $\theta_1^* - \theta_1^*, \theta_2^* - \theta_2^*$ and $\theta_3 = \theta_3^*$, and a constant $\theta_{20} = \theta_{20}^*$ such that the closed-loop poles of (2.1) and (2.3) are stable and $\gamma(s)/(r(s))$ is a $W_p(s)$.

Proof: From (2.1), (2.3) we have that

$$T(s) \triangleq \frac{\gamma(s)}{\gamma_m(s)} = \frac{k_pZ_0(s)\Lambda(s)\theta_2^*b(s)n(s)}{(\Lambda(s) - \theta_1^*a(s))R_0(s) - k_pZ_0(s)(\theta_2^*a(s) + \theta_{20}\Lambda(s))}.$$  

(2.5)

We choose $\theta_1^*$ such that $k_p\theta_1^*b(s)$ is a monic Hurwitz polynomial of degree $n^* - n_1^*$ so that all zeros of $T(s)$ in (2.5) are located in $\Re\{s\} < 0$. It follows from [1] that $\theta_1^*, \theta_2^*, \theta_{20}$ exist such that

$$(\Lambda(s) - \theta_1^*a(s))R_0(s) - k_pZ_0(s)(\theta_2^*a(s) + \theta_{20}\Lambda(s)) = k_pZ_0(s)\Lambda(s)\theta_2^*b(s)n(s).$$  

(2.6)

Hence, $T(s) = 1$, i.e., $\gamma(s) = W_p(s)r(s)$, and all closed-loop poles of $\gamma(s)$ are stable.

Remark 2.1: We should note that $\theta_1^*$ is not unique. Once $\theta_1^*$ is chosen however $\theta_2^*, \theta_{20}^*$ are unique provided $\theta_1 = \theta_1^* + a(s)$ and $Z_0(s), R_0(s)$, are coprime. The use of $\theta_3 \in R^{n_1 - 1}$ in (2.3) indicates that the controller structure is overparameterized in order to take care of the uncertainty in $n^*$. The stable plant-model matching requires that all zero-pole cancellations in the closed-loop transfer function occur in the left-half complex plane, which is ensured by choosing $\theta_1^*$ and $n(s)$ such that $\theta_2^*b(s), n(s)$ are Hurwitz.

III. THE NEW MRAC SCHEME

When the plant parameters, i.e., $k_p$ and the coefficients of $Z_0(s)$ and $R_0(s)$, are unknown, the matching equation (2.6) cannot be solved for $\theta_2^*, \theta_{20}^*$, and $\theta_3^*$, and therefore the control law (2.3) cannot be implemented with $\theta_1^*, \theta_2^*, \theta_{20}^*$, and $\theta_3^*$ being replaced by the desired $\theta_1^*, \theta_2^*, \theta_{20}^*$, and $\theta_3^*$, respectively. In this case, we follow the certainty equivalence approach and combine (2.3) with an adaptive law that generates $\theta_1(t), \theta_2(t), \theta_{20}(t)$, and $\theta_3(t)$, the estimates of $\theta_1^*, \theta_2^*, \theta_{20}^*$, and $\theta_3^*$ respectively at time $t$. Such an adaptive law is developed as follows.

From (2.1) and (2.6) it follows that

$$\frac{1}{D_m(s)}[u(t)] = \frac{\theta_1^*a(s)}{D_m(s)\Lambda(s)}[u(t)] + \frac{\theta_2^*a(s)}{D_m(s)\Lambda(s)}[y(t)] + \frac{\theta_{20}^*}{D_m(s)}[y(t)] + \frac{\theta_3^*b(s)n(s)}{D_m(s)}[y(t)].$$  

(3.1)

Define $\omega(t) = (\omega_1^T(t), \omega_2^T(t), y(t), \omega_1^T(t))^T$, $\theta^* = (\theta_1^*, \theta_2^*, \theta_{20}^*, \theta_3^*)^T$, $\zeta(t) = (1/D_m(s))\omega(t)$, $\bar{u} = (1/D_m(s))u(t)$, $\bar{y} = (n(s)/D_m(s))y(t)$, and $\dot{\gamma}_m(t) = (n(s)/D_m(s))\gamma_m(t)$. Then from (3.1) we have

$$\dot{\theta}^*(t) = \theta_1^* + \theta_2^*b(s)[\bar{y} - \bar{u}](t).$$  

(3.2)

Now $\theta_1^*b(s) = \theta_1^*s^n + \cdots + \theta_{20}^*s + \theta_3^*$, where $\theta_1^* = (\theta_{20}^*, \theta_{21}^*, \cdots, \theta_{n_1}^*)^T$, $n^* = n^* - n_1^*$. Since $\theta_1^*$ is such that $k_p\theta_1^*b(s)$ is monic Hurwitz and of degree $n^* - n_1^*$, we have that $k_p\tilde{n}_{10} = 1$, $\tilde{n}_1^* = 0$, for $i = n_1 + 1, \cdots, n^*$, $\text{sign}(\theta_{n_1}^*) = \text{sign}(k_p)$, and

$$\theta_2^*b(s) = \theta_2^* + \theta_{20}^*\theta_2^*b(s).$$  

(3.3)

where $b(s) = (s, s^2, \cdots, s^n)^T$ and $\theta_2^* \in R^{n^*}$. Hence, (3.2) becomes

$$\frac{1}{\theta_{20}^*} \left( \theta_1^* \zeta(t) - \bar{u}(t) \right) + \theta_2^*b(s)[\bar{y} - \bar{u}](t) = -\zeta(t) - \gamma(t).$$  

(3.4)

We define the estimation error $\epsilon(t)$ as:

$$\epsilon(t) = \gamma(t) - \gamma_m(t) = \theta_1^* \zeta(t) - \bar{u}(t) + \theta_2^*b(s)[\bar{y} - \gamma_m(t)] + \rho(t) \theta^*(t) \zeta(t) - \bar{u}(t)$$  

(3.5)

where $\rho(t), \theta(t), \gamma(t)$ are the estimates of $\rho^* = (1/\theta_{n_1}^*)$, $\theta^*$, and $\gamma^*$, respectively.

Define $\xi(t) = \theta_1^* \zeta(t) - 1/(1/D_m(s))\theta_2^*b(s)\bar{y} - \gamma_m(t)$. Then from (3.4) and (3.5) we obtain

$$\epsilon(t) = \psi(t) \xi(t) + \theta_1^* \zeta(t) + \rho^* \phi^*(t) \xi(t)$$  

(3.6)

where $\psi(t) = \rho(t) - \rho^*$, $\phi(t) = \theta(t) - \theta^*$, $\phi_0(t) = \theta_1(t) - \theta_1^*$ are the parameter errors.

Define $P(s, q) = q^2b(s) + 1$, $q \in R^{n^*}$. Then for $\theta_1^*, \theta_2^*, \theta_{20}^*$, and $\theta_3^*$ in (3.6) we have that $P(s, \theta_2^*) = \theta_2^*b(s) + 1 = \theta_{20}^*s^n + \cdots + \theta_{n_1}^*s + 1 = (1/\theta_{n_1}^*) \theta_2^*b(s)$ which is Hurwitz with $\theta_{n_1}^* > 0$, for $i = 1, \cdots, n^* - n_1^*$ and $\theta_{20}^* = 0$, for $i = n^* - n_1^* + 1, \cdots, n^*$. Let $S_\lambda$ be the stability region in the parameter space such that

$$S_\lambda = \{ q \in R^{n^*} | \text{roots of } P(s, q) \}$$  

(3.7)

We define a parameter projection region $S_\delta$ such that $i, \delta \in S_\delta$, and $i \cap S_\delta \cap S^* = \emptyset$, for any unknown $n^*$ such that $n^* \geq n_1^* \geq 1$, where $S^* = \{ q \in R^{n^*} | \text{roots of } P(s, q) = q^2b(s) + 1 \}$. Commence with the degree $n^* - n_1^*$.

We use the following adaptive law to update $\theta(t), \rho(t), \theta_1(t)$:

$$\dot{\theta}(t) = -\frac{\text{sign}(k_p)\Gamma_1 e(t) \xi(t)}{m_2^*(t)}, \Gamma_1 = G_1^2 > 0$$  

(3.8)

$$\dot{\rho}(t) = -\frac{\gamma(t) \xi(t)}{m_2^*(t)}, \gamma > 0$$  

(3.9)

where $m_2^*(t) = \sqrt{1 + \xi^T(t)\xi(t) + \xi^T(t)\chi(t)\chi(t)}$, and

$$\hat{\theta}(t) = \hat{\theta}(t)$$  

(3.10)

otherwise

where $\hat{\theta}(t)$ is the pre-estimate of $\theta^*$ from

$$\dot{\hat{\theta}}(t) = \hat{\theta}(t)$$  

(3.11)

where $\hat{\theta}(t)$ is the estimate of $\theta^*$ from

$$\dot{\hat{\theta}}(t) = \hat{\theta}(t) - \theta_1(t)$$  

(3.11)

where $\hat{\theta}(t)$ is the estimate of $\theta_1(t)$ from (3.11) by using the projection algorithm (3.10). For any $q \in S_\delta$, $q^2b(s) + 1 = 0$ has all roots in $\Re\{s\} \leq -\lambda$. The projection algorithm (3.10) guarantees that $\hat{\theta}(t) \in S_\delta$, $\forall t \geq 0$, i.e., for each time $t$, $q^2b(s) + 1$ is strictly Hurwitz.
(Ω(τ)(b, x)+ 1) is a stable operator pointwise in τ, which together with the "slow" variation of θ(t) (see Section IV and Section V) implies that Ω(τ) is a stable operator, ∀τ ≥ 0, a crucial condition for the global stability of the closed-loop plant.

The definition of S₀ requires no a priori information about the unknown plant parameters or the value of θ or θ* beyond that it is related to parametric characteristics of a Hurwitz polynomial and geometrical properties of the space R^m. Given n⁰ = n⁰ + n⁰ + 1, we examine the polynomial P(s, q) = q² + b₁, s + 1 = q₂ₐ + q₃ₐ + ... + qₘₐ + 1 and search for the convex set Sₜ ∈ R⁰, which is illustrated as follows for different cases.

Case 1: n₁ = 1. When the uncertainty in relative degree is one, i.e., n₁ = n₁ + 1, either n² = n² + 1 or n² = n₁; we have that θ = θ₁, i.e., P(s, q₁) = s + 1. Let Sₜ ∈ R₀ be 0 ≤ q₁ ≤ λ³. Hence, we see that Sₜ is Sₜ satisfies all conditions for Sₜ and therefore can be used in the projection algorithm (3.10). In the case P(s, q₁) = q₂ₐ + q₃ₐ + ... + qₘₐ + 1, with qₘₐ = 0 if n² = n₁, qₘₐ > 0 if n² = n₁ + 1. Sₜ = Sₜ = Sₜ if n² = n₁ + 1 = n₁.

Case 2: n₁ = 2. When the plant relative degree bound difference n² = n² + 1 is two, i.e., n₁ = n₁ + 2, either n² = n² + 1 or n² = n₁ + 1 = n₁ + 1; we have that θ₁ = θ₁(t), θ₂ₐ(t)+1, i.e., P(s, q₁(t)) = s + 1 + λ⁻¹. (12)

In this case, P(s, q₁) = q₂ₐ + q₃ₐ + ... + qₘₐ + 1, where qₘₐ = 0 if n² = n₁, qₘₐ > 0 if n² = n₁ + 1, and qₘₐ > 0 if n² = n₁. Sₜ = Sₜ defined in (3.12) satisfies all conditions for Sₜ and therefore can be used for parameter projection (3.10). Sₜ = Sₜ = (q₁(t), q₂ₐ(t)) ∈ R²(q₁(t), q₂ₐ(t)) = (0, 0) if n² = n₁, Sₜ = Sₜ = (q₁(t), q₂ₐ(t)) ∈ R²(q₁(t), q₂ₐ(t)) if q₁ ≤ λ, q₁ = 0 if n² = n₁ + 1, and Sₜ = Sₜ = n₁ = n₁.

Case 3: n₁ ≥ 3. For the general case when n₁ ≥ 3, the conditions for q₂ₐq₃ₐ + ... + qₘₐ + 1 to have all zeros in R, i.e., qᵢ = 1, ..., n₁, though quite complicated, can be derived by using Routh–Hurwitz stability criterion. Therefore, the stability region Sₜ can be specified in the parameter space R². Sₜ may not be convex when n₁ ≥ 3; however, we can construct the projection set Sₜ = Sₜ by using the following method: specify a convex set Sₜ = Sₜ such that the elements (0, 0, ..., 0)², (0, 0, ..., 0, 0)², (0, 0, ..., 0, q₁)², ..., (0, 0, ..., 0, q₁)²) ∈ Sₜ for some q₁ = R, i = 1, ..., n₁, q₁ ∈ R +, i = 2, ..., m, q₁ ∈ R +, i = n₁, q₁ = R +, i = n₁, q₁ = R +, i = n₁. It is clear that Sₜ contains such elements so that the desired Sₜ can be specified.

IV. STABILITY OF MRAC

In this section, we show that if the MRAC scheme developed in Section III is applied to the plant (2.1) then the closed-loop plant is globally stable and the tracking error e(t) = y(t) − y_m(t) goes to zero asymptotically with time.

We first establish the stability properties of the adaptive law (3.3)–(3.11).

Lemma 4.1: The adaptive law (3.3)–(3.11) guarantees that θ(t), θ(t), θ(t), θ(t) ∈ L², and (e(t)/m₉(t), θ(t), θ(t), θ(t), θ(t) ∈ L₂ ∩ L₂.

Proof: Define the positive definite function

\[ \tilde{V}(t) = \frac{1}{2} (\rho^* \Omega(\tau)(\tau)) \tilde{\phi}(\tau) + \gamma \tilde{h}(\tau) + \tilde{a}_i^2(\tau)(\tau) \tilde{\phi}(\tau) \]

(4.1)

where \( \tilde{\phi}(\tau) = \tilde{\phi}(\tau) - \theta^* \) with \( \tilde{\phi}(\tau) \), the unprojected estimate of \( \phi^* \), being generated from (3.11), from (3.8)–(3.11), (4.1) we have that \( \tilde{V}(t) \tilde{h}(\tau)/m₉(t) = -\epsilon \tilde{h}(\tau)/m₉(t) \). Let \( \Delta t > 0 \) be an infinitesimal time increment at time t, from (3.10), (3.11) we have that \( \tilde{h}(\tau + \Delta t) \tilde{h}(\tau + \Delta t) \tilde{h}(\tau + \Delta t) \), where \( \tilde{h}(\tau) \) denotes the distance between x and y. Therefore, \( \theta(t), \rho(t), \tilde{\phi}(\tau), \tilde{h}(\tau), (\epsilon(t)/m₉(t) \tilde{h}(\tau), \rho(t), \tilde{h}(\tau) \) all belong to \( L₂ \), and \( (\epsilon(t)/m₉(t), \theta(t), \rho(t), \tilde{\phi}(\tau), \tilde{h}(\tau) \) all belong to \( L₂ \). Now we establish the global stability and tracking performance of the MRAC system.

Theorem 4.1: All signals in the closed-loop plant (2.1)–(2.3), (3.8)–(3.11) are uniformly bounded and \( \lim_{t \to \infty} e(t) = 0 \).

Proof: For \( a > a \) sufficiently large but finite, we define the following fictitious filters [9]:

\[ s(t) = 1 - K(t), K(t) = \frac{a_i^2}{(s + a_i)^2}, \]

(4.2)

where \( h(t) \) is the impulse response function of \( H(t) \). Let \( \omega(t) = A \omega(t) + b \omega(t) \), where \( A \in R^{m₉(t)} \), \( b \in R^{m₉(t)} \). Using \( H(t) \) and \( K(t) \) defined in (2.4) and (4.2), we have that

\[ (I - H(t)(A + b \tilde{h}(\tau))) \omega(t) \]

(4.3)

\[ = (G₀^{-1}(s)K(t)(F(s) + H(t)(b \tilde{h}(\tau))F(s)) \]

(4.4)

\[ + H(t)(b \tilde{h}(\tau))) \omega(t) = H(t)(b \tilde{h}(\tau))) \omega(t) \]

(4.5)

Since \( \omega(t) \) and \( H(t) \) satisfies (4.2), there exists \( a_i^2 \in R^+ \) such that \( I - H(t)(A + b \tilde{h}(\tau)) \) is stable and proper for any \( a_i^2 \). Hence, for a finite \( a_i^2 \) it follows from (4.3) that

\[ \omega(t) < G(t)(y(t) + G(t)(y(t))) \]

(4.6)

where the stable and strictly proper operators \( G(t), t, G(t), t \) are defined as:

\[ G(t)(s, t) = (I - H(t)(A + b \tilde{h}(\tau)))^{-1}(K(t)(s)G₀^{-1}(s)F(s)) \]

(4.7)

\[ = (I - H(t)(A + b \tilde{h}(\tau)))^{-1} \]

(4.8)

(4.9)

Hence, it follows from (4.4) and the definition of \( \omega(t) \) that

\[ \omega(t) = G(t)(y(t) + G(t)(y(t))) \]

(4.10)

where \( G(t), t = G(t)(s, t, F(T), t, 0), G(t), t = G(t)(s, t, 0, 0, 0) \).

From (2.3), (2.6) it follows that

\[ y(t) = y_m(t) + \frac{s}{H(t)} \tilde{\phi}(\tau)(H(t) = \theta^* \tilde{h}(t), s) \]

(4.11)
Using (4.8) and $H_2(s), K_2(s)$ defined in (4.2), we have that
\[
y(t) = D_n(s) K_2(s) \frac{1}{D_m(s)} y(t) + H_2(s) \frac{s}{H^*(s)} \left[ \phi^T G(s, \cdot) [y] (t) \right] + H_2(s) \frac{s}{H^*(s)} \left[ \phi^T G(s, \cdot) [r] (t) \right] + H_2(s) \epsilon_m(t).
\]
\[
(4.9)
\]
Since $(s/H^*(s)) \phi^T G(s, t)$ is stable and proper and $H_2(s)$ satisfies (4.2), there exists $a_2 \in R^*$ such that $(1 - H_2(s) \frac{s}{H^*(s)} \phi^T G(s, t))^{-1}$ is stable and proper for any $a_2 > a_2$. Hence, for a finite $a_2 > a_2$, it follows from (4.9) that
\[
y(t) = G_2(s, \cdot) \frac{1}{D_m(s)} y(t) + G_2(s, \cdot) [r] (t)
\]
where the stable and proper operators $G_2(s, t), G_2(s, t)$ are defined as:
\[
G_2(s, t) = \left( 1 - H_2(s) \frac{s}{H^*(s)} \phi^T (t) G_2(s, t) \right)^{-1} D_n(s) K_2(s)
\]
\[
G_2(s, t) = \left( 1 - H_2(s) \frac{s}{H^*(s)} \phi^T (t) G_2(s, t) \right)^{-1}
\]
\[
\cdot H_2(s) \left( \frac{s}{H^*(s)} \phi^T (t) G_2(s, t) + \frac{k_m}{D_m(s)} \right).
\]
\[
(4.11)
\]
Hence, for $\tilde{y}(t) = (1/D_m(s)) y(t)$ it follows from (4.7), (4.10) that
\[
\omega(t) = G_2(s, \cdot) G_2(s, \cdot) [\tilde{y}] (t)
\]
\[
+ (G_2(s, \cdot) G_2(s, \cdot) + G_2(s, \cdot)) [r] (t).
\]
\[
(4.12)
\]
For $\tilde{y}_m(t) = (1/D_m(s)) y_m(t)$, it follows from (2.3), (3.2), the definition of $\xi(t), (3.6)$ that
\[
\rho^* \sigma^*(b(s) n_2(s)) [\tilde{y} - \tilde{y}_m] (t) = \epsilon(t) - \rho(\xi(t)) = \Phi(t).
\]
\[
(4.13)
\]
From (3.3) and the fact that $\rho^* = (1/\theta^*_s), \phi_2(t) = \theta(t) - \theta^*_s$ we have that
\[
\rho^* \sigma^*(b(s) n_2(s) + \phi_2(t) b_2(s) n_2(s) = (1 + \theta^*_s(t) b_2(s)) n_2(s)
\]
\[
= P(s, \theta(t)) n_2(s) \Phi(t).
\]
\[
(4.14)
\]
which is strictly Hurwitz pointwise in $t$. Hence, from (4.14), (4.15) it follows that
\[
\tilde{y}(t) = \frac{1}{D_m(s)} [y_m(t)] + \frac{1}{P(s, \theta(t))} [\epsilon - \rho(\xi)]
\]
\[
(4.16)
\]
where $(1/P(s, \theta(t)))$ is stable and strictly proper because $\theta(t) \in L_2 \cap L_\infty$. Further $n_2(t) = 1 \geq 0$.

Let $(A_m, b_m, c_m)$ be a minimal realization of $(1/D_m(s))$, i.e.,
\[
(1/D_m(s)) = c_m(sI - A_m)^{-1} b_m
define $W(s) = c_m(sI - A_m)^{-1}, W_0(s) = (sI - A_m)^{-1} b_m$, we have [2], [9] that
\[
\tilde{y}(t) = W_0(s)[W(s) [\sigma^*] \tilde{\theta}] (t).
\]
\[
(4.17)
\]
Using the definition of $m_0(t)$, we have that
\[
|\epsilon(t)| \leq m_0(t) (1 + ||\xi(t)|| + ||\xi(t)|| + ||\chi(t)||).
\]
\[
(4.18)
\]
It follows from (4.16) that $|\xi(t)| \leq x(t) + T_1(s, \cdot) \chi_5(s, \cdot) ||\chi|| (t)$, where $x(t), T_1(s, \cdot), T_2(s, \cdot)$ satisfy all of the conditions of Lemma A.1 (see Appendix). Hence, $\tilde{y}(t) \in L_\infty$. It follows from (2.3), (4.10), (4.13) that all signals in the closed-loop plant are uniformly bounded.

Using the usual arguments in adaptive control theory we can establish that $\lim_{t \to \infty} \epsilon(t) = 0, \lim_{t \to \infty} \theta(t) = 0, \lim_{t \to \infty} \xi(t) = 0, \lim_{t \to \infty} y(t) = 0.
\]
\[
(4.19)
\]
Using $H_2(s), K_2(s)$ defined in (4.2), (4.8), we have that
\[
y(t) - y_m(t) = H_2(s) \frac{s}{H^*(s)} \phi^T [w(t)]
\]
\[
+ K_2(s) D_n(s) \frac{1}{D_m(s)} [y - y_m(t)].
\]
\[
(4.20)
\]
Since $(s/H^*(s)) \phi^T [w(t)] \in L_\infty, K_2(s) D_n(s)$ is stable and proper, it follows from (4.19), (4.20) that
\[
|y(t) - y_m(t)| \leq c_\epsilon + c_\epsilon(t)
\]
\[
(4.21)
\]
where $c_\epsilon \in R^$, $c_\epsilon(t) \in L_\infty$ and $\lim_{t \to \infty} c_\epsilon(t) = 0$. Since $a_3 > 0$ can be chosen to be arbitrarily large, it follows from (4.21) that $\lim_{t \to \infty} y(t) = y_m(t) = 0$.

**Remark 4.1:** The above stability analysis reveals two important properties of the MRAC system: 1) the boundedness of the estimated and projected parameters leads to the inequality $|\xi(t)| \leq \xi(t) + T_1(s, \cdot) \chi_5(s, \cdot) ||\chi|| (t)$ which characterizes the "gain structure" of the MRAC system, and 2) the $L_2$ property of $(\epsilon(t)/\epsilon_m(t))$ and $\theta(t)$ ensures that the loop gain of the system is small so that the boundedness of $\tilde{y}(t)$ is guaranteed. These two properties also hold in the robust adaptive control case (see Section V) where the plant to be controlled has certain unmodelled dynamics whose "size" is small.

**V. ROBUST MRAC**

In this section, we analyze the robustness of global stability of the developed MRAC scheme with a modified adaptive law in the presence of unmodeled dynamics.

We consider the following plant:
\[
y(t) = G(s)[u(t)] + G(s) = G(s)[1 + \mu \Delta_m(s)] + \mu \Delta_s(s)
\]
\[
(5.1)
\]
where $G(s)$ is the plant transfer function, $G_m(s) = k_2(Z_k(s)/R_k(s))$ is the plant modeled part which satisfies the assumptions A1–A4 in Section II, $\mu \Delta_m(s), \mu \Delta_s(s)$ are the multiplicative and additive plant unmodeled dynamics respectively rated by $\mu > 0$.

To achieve the control objective in the presence of the unmodeled dynamics $\mu \Delta_m(s), \mu \Delta_s(s)$ which satisfy: A5 there exists $k_2 \in R^*$ such that $\|u_m(t)|| \leq \chi_s k_2, \|u_m(t)|| \leq k_2$ where $u_m(t), u_s(t)$ are the impulse response functions of $sW_m(s), sW_m(s), sW_s(s), sW_s(s)$ respectively, for some known $k_2 \in R^*$, we use the reference model (2.2), controller structure (2.3) and projection (3.10), and define the stable plant-model matching parameters $\theta^*_r, \theta^*_t, \theta^*_s, \theta^*_s$ from (2.6). However we modify
the adaptive law (3.8), (3.9), (3.11) as:

\[
\dot{\theta}(t) = -\frac{\text{sign}(k_{\theta})}{m^2(t)} \epsilon (t) \dot{x}(t) + \Gamma_1 \phi(t) \quad (5.2)
\]

\[
\dot{\rho}(t) = -\frac{\gamma}{m(t)} \dot{x}(t) - \gamma \phi(t) \quad (5.3)
\]

\[
\ddot{\sigma}(t) = \frac{\delta_2(t) \xi(t)}{m^2(t)} - \Gamma_2 \phi(t) \quad (5.4)
\]

where \(\epsilon(t), \xi(t), \dot{\xi}(t), \phi(t), \gamma(t)\) are given in (3.5), (3.6), \(m(t)\) is the normalizing signal [7] defined as:

\[
m(t) = -\delta_1 m(t) + \delta_1 |u(t)| + |y(t)| + 1, \quad m(0) > \frac{\delta_1}{\delta_0} \quad (5.5)
\]

\(\delta_1 \in R^+ \), \(0 < \delta_1 \leq \min\{\delta_2, \delta_3\}, \quad \delta_2 \in R^+ \) such that \(\Lambda(s) - \delta_2\), \(g(s) - \delta_3\) are Hurwitz polynomials, and \(g_i(t), i = 1, 2, 3\), are time-varying functions for each system. The switching-\(\sigma\) modification [7] gives the following \(g_1(t), g_2(t), g_3(t)\):

\[
g_1(t) = \sigma_1(t) \theta(t), \quad \sigma_1(t) = \begin{cases} 0 & \text{if } \|\theta(t)\| < M_1 \\ \sigma_2 \left( \frac{\|\theta(t)\|}{M_1} \right) - 1 & \text{if } M_1 \leq \|\theta(t)\| < 2M_1 \\ \sigma_0 & \text{if } \|\theta(t)\| \geq 2M_1 \end{cases} \quad (5.6)
\]

\[
g_2(t) = \sigma_2(t) \rho(t), \quad \sigma_2(t) = \begin{cases} 0 & \text{if } |\rho(t)| < M_2 \\ \sigma_3 \left( \frac{|\rho(t)|}{M_2} \right) - 1 & \text{if } M_2 \leq \|\rho(t)\| < 2M_2 \\ \sigma_0 & \text{if } \|\rho(t)\| \geq 2M_2 \end{cases} \quad (5.7)
\]

\[
g_3(t) = \sigma_3(t) \phi(t), \quad \sigma_3(t) = \begin{cases} 0 & \text{if } |\phi(t)| < M_3 \\ \sigma_4 \left( \frac{|\phi(t)|}{M_3} \right) - 1 & \text{if } M_3 \leq |\phi(t)| < 2M_3 \\ \sigma_0 & \text{if } |\phi(t)| \geq 2M_3 \end{cases} \quad (5.8)
\]

where \(M_1 > \sup \|\theta(t)\|, M_2 > \sup |\rho(t)|, M_3 > \sup |\phi(t)|\) with sup being taken over \(\theta, \rho, \phi\) such that \(k_{\theta}, k_{\rho}, k_{\phi}\) is monic, of degree \(n^* - n\), and has zeros in \(\mathbb{R}^+\) and \(\sigma_0, \sigma_2, \sigma_3 \in R^+\).

We present the following definition.

**Definition 5.1:** A continuous signal \(x(t) \in B_{L^p}\), where the integer \(p \geq 1\) and the scalar \(\nu \geq 0\), if \(\int_0^t |x(t)|^p \, dt \leq v(t_2 - t_1) + c_0\) for any \(0 \leq t_1 < t_2\) and some \(c_0 \in R^+\), where \(L^p = \mathbb{R}^n\).

Definition 5.1 characterizes a class of signals which are more general than \(L^p\) signals and are important in robust adaptive control problems where some signals are in \(L^p\), but not in \(L^p\).

The following lemma summarizes the robust stability properties of the modified adaptive law.

**Lemma 5.1:** The adaptive law (5.2)–(5.4), (3.10) guarantees that \(\theta(t), \phi(t), \xi(t), \dot{\xi}(t), \epsilon(t), \phi(t), \gamma(t), \rho(t), \dot{\phi}(t), \dot{\xi}(t) \in L_{n_1}(w(t)), w(t) = \dot{w}(t)/m(t), w(t) \in L_{n_2}(w(t)), w(t) \in L_{n_3}(w(t))\) for some \(n_1, n_2, n_3 \in R^+\).

**Proof:** It follows from (5.1), (2.6), (3.5) that

\[
\epsilon(t) = \psi(t) \xi(t) + \phi \xi(t) \dot{x}(t) + \rho \phi \xi(t) + \mu \eta(t) \quad (5.9)
\]

where \(\eta(t) = \Delta(s)[u(t), \Lambda(s), \phi(t), \gamma(t)]\), and

\[
\Delta(s) = \rho \left( \frac{1}{D_m(s)} \right) \left( \frac{1 - \theta^T \phi \xi(t)}{\Lambda(s)} \right) (\Delta_m(s) + G_0^0 \Delta(s))
\]

\[
\rho \left( \frac{1 - \theta^T \phi \xi(t)}{\Lambda(s)} \right) \left( \frac{1}{D_m(s)} \right) \Delta_m(s)
\]

\[
+ \rho \left( \frac{1 - \theta^T \phi \xi(t)}{\Lambda(s)} \right) \left( \frac{\phi}{D_m(s)} \right) \Delta_m(s)
\]

\[
(5.10)
\]

Consider \(\tilde{V}(t)\) defined in (4.1). Using (5.9) in (5.2)–(5.4), we have that

\[
\tilde{V}(t) = \frac{\epsilon^T(t)}{m^2(t)} + \mu \frac{\epsilon(t)}{m(t)} \phi \theta(t) - \sigma_2(t) \psi \rho(t) - \sigma_3(t) \phi \theta(t) \quad (5.10)
\]

\[
\tilde{V}(t) = -\frac{\epsilon^T(t)}{m^2(t)} + \mu \frac{\epsilon(t)}{m(t)} \phi \theta(t) - \sigma_2(t) \psi \rho(t) - \sigma_3(t) \phi \theta(t) \quad (5.10)
\]

From (5.5), (5.10), it follows [7] that \(|\eta(t)|/m(t) \leq k_1\) for some \(k_1 \in R^+\). Therefore, it follows from (4.1) and (5.11) that \(\theta(t), \rho(t), \phi(t) \in L_{n_1}\). From (5.9) and (5.2)–(5.4) that \(\epsilon(t)/m(t) \leq k_1\) and \(\phi(t), \phi(t) \in L_{n_2}\). From (3.10) that \(\phi(t), \phi(t) \in L_{n_3}\). From (5.11), (5.6)–(5.8) and (5.10) that \(\epsilon(t)/m(t) \leq k_1\), \(\phi(t), \phi(t) \in L_{n_2}\), \(\phi(t), \phi(t) \in L_{n_3}\) for some \(k_0 \in R^+\).

We now present the robustness properties of the modified MRAC scheme.

**Theorem 5.1:** There exists \(\mu > 0\) such that for any \(\mu \in [0, \mu^*]\), all signals in the closed-loop plant are uniformly bounded, and the tracking error \(e(t) = y(t) - y_m(t)\) satisfies \(\lambda(s)/(\eta(t)) \in B_{L^p}\), for some \(\mu \in R^+\).

**Proof:** Using \(H_{e(s)}(s)\), \(K(s)\) defined in (4.2), we have that

\[
\omega(t) = G_{f(s)}[f(t)] + G_{f(s)}[r(t)] + \mu G_{f(s)}[u(t)] \quad (5.12)
\]

where \(G_{f(s)}(s), G_{f(s)}(s)\) are given in (4.5), (4.6) and

\[
G_{f(s)}(s) = -(I - K(s)(A + b\theta^T(t)))^{-1} \Lambda(s) F(s)
\]

\[
(5.13)
\]

It follows from (5.12) and the definition of \(\omega(t)\) that

\[
\omega(t) = G_{f(s)}[f(t)] + G_{f(s)}[r(t)] + \mu G_{f(s)}[u(t)] \quad (5.14)
\]

where \(G_{f(s)}(s), G_{f(s)}(s)\) are the same as in (4.7) and \(G_{f(s)}(s) = G_{f(s)}(s), 0, \ldots, 0\).

Similarly, using \(H_{f(s)}(s)\), \(K(s)\) defined in (4.2), we have that

\[
y(t) = G_{f(s)}[f(t)] + G_{f(s)}[r(t)] + \mu G_{f(s)}[u(t)] \quad (5.15)
\]
where \( G_5(s, t), G_6(s, t) \) are given in (4.11), (4.12) and
\[
G_5(s, t) = \left(1 - \frac{sH_5(s)}{H^*(s)}(\phi^*(t)G_3(s, t)) \right)^{-1}
\]
\[
\cdot \frac{sH_5(s)}{H^*(s)}(\phi^*(t)G_2(s, t)) + (1 - F_1(s))(\Delta(s) + G_0^*(-s)\Delta(s)).
\] (5.16)

Therefore for \( \tilde{y} = (1/D_m(s))\tilde{y}(t) \), it follows from (5.14), (5.15) that
\[
\omega(t) = G_5(s, \cdot)G_6(s, \cdot)[\tilde{y}(t) + (G_2(s, \cdot)G_3(s, \cdot) + G_4(s, \cdot))[\tau(t)] + \mu(G_3(s, \cdot)G_6(s, \cdot) + G_2(s, \cdot))u(t) \]

where \( G_5(s, \cdot), G_6(s, \cdot), G_3(s, t), G_4(s, t) \) are stable and proper, \( G_5(s, \cdot), G_6(s, \cdot), G_3(s, t), G_4(s, t) \) are stable and strictly proper, slowly time-varying (in the sense that \( \theta(t) \in B_{2k} \), \( \mu > 0 \) is small, \( k_5 \in \mathbb{R}^* \)).

It can be shown that for any \( \mu \in [0, \mu_0^*] \) and some \( \mu_0^* > 0 \)
\[
\tilde{y}(t) = \frac{1}{D_m(s)}[\mu_0^*](t) + \frac{1}{\bar{F}(s, \theta)}[(e - \mu \tilde{e})(t)]
\] (5.18)

where \( (1/D_m(s))\tilde{y}(t) \) is stable and strictly proper, and \( \bar{F}(s, \theta(t)) \) is given in (4.15).

Using the definition of \( m(t) \) in (5.5), we have that
\[
|\tilde{e}(t)| \leq \left( \frac{\mu_0^*}{m(t)} \right) \left[ k_5 + \frac{1}{s + \delta_0} [\mu_0^*][t] + \frac{1}{s + \delta_0} [\tilde{e}(t)] \right]
\] (5.19)

for some \( k_5 \in \mathbb{R}^* \). From (2.3), (5.17) we have that
\[
u(t) = G_5(s, \cdot)\phi^*(\cdot)G_6(s, \cdot)G_3(s, \cdot)\tilde{y}(t) + G_4(s, \cdot)\phi^*(\cdot)(G_2(s, \cdot)G_5(s, \cdot) + G_4(s, \cdot))[\tau(t)]
\] (5.20)

where \( G_5(s, t) = (1 - \mu \theta^T(t)(G_5(s, t)G_3(s, t) + G_2(s, t))^{-1} \) is stable and proper for any \( \mu \in [0, \mu_0^*] \) and some \( \mu_0^* > 0 \) because \( \theta(t) \in L_m \) and \( G_5(s, t)G_3(s, t) + G_2(s, t) \) is stable and proper.

From (4.17), (5.17)–(5.20) we have that \( \tilde{y}(t) \leq \mu \tilde{e}(t) + T_\mu(s, \cdot)x_\nu(t) + T_\mu(s, \cdot)y(t) \) \( t \), where \( x_\nu(t), x_\mu(t), T_\mu(s, t), T_\mu(s, t) \) satisfy all conditions of Lemma A.1 (see Appendix). It follows that \( \tilde{y}(t) \in L_m \) for some \( \mu \in [0, \mu_0^*] \) and some \( 0 < \mu_0^* \leq \min (\mu_0^*, \mu_0^*) \).

So all signals in the closed-loop plant are uniformly bounded, and \( (n(s)/D_m(s))\tilde{y} = \mu \tilde{e}(t) \) \( t \in B_{2k} \), for some \( \gamma \in \mathbb{R}^* \) from (5.18). The asymptotic zero tracking error when \( \mu = 0 \) follows from the proof for Theorem 4.1.

Remark 5.1: When there are also output disturbances in the plant to be controlled, i.e., \( y(t) - G(s)u_k(t) + d(t) \), where \( d(t) \) is a uniformly bounded signal due to measurement noises, we can also show global stability of the closed-loop plant with our MRAC scheme.

VI. CONCLUSION

In this note, we relax one of the crucial assumptions in MRAC which involves the exact knowledge of the relative degree \( n^* \) of the plant by developing a new controller structure and an appropriate adaptive law. The controller has a feedforward dynamic block whose numerator coefficients are updated on line. The adaptive law updates the controller parameters as well as an extra parameter vector which is projected onto a certain convex set \( S_\mu \). The set \( S_\mu \) can be specified by using no a priori additional knowledge about the unknown plant except for an upper bound on \( n^* \). The number of the estimated parameters can be reduced when a lower bound on \( n^* \) is also known. The new MRAC scheme guarantees global stability and asymptotic convergence of the tracking error to zero for the closed-loop plant and can be applied, together with a modified adaptive law, to plants with unmodeled dynamics to ensure robustness.

APPENDIX

Recall that a vector signal \( x(t) \) belongs to \( L_p \), if \( \|x(t)\|^{\mu} \leq c_0 \), for some \( c_0 \in \mathbb{R}^* \). A generalized class of signals is the class \( B_{p, \nu}(x(t) \in B_{p, \nu} \) if \( \|x(t)\|^{\mu} \leq c(t) + c_0 \), for any \( t < t_0 \), some \( c_0 \in \mathbb{R}^* \), see Definition 5.1, e.g., \( x(t) \in B_{p, \nu} \) for \( \nu = 0 \) means that \( x(t) \in L_p \).

In the Appendix, we present a general stability result which characterizes the robustness and stability of adaptive control systems.

Lemma A.1: Assume that the signals, \( x(t) \geq 0 \) is continuous, \( x_\nu(t) \in L_m, x_\mu(t) \in B_{p, \nu}, \forall \mu \geq 1 \), the operators \( T_\mu(s, \cdot) \), \( T_\nu(s, \cdot) \) are stable and strictly proper [13], the impulse response function of \( T_\mu(s, \cdot) \) is nonnegative, and that
\[
z(t) \leq x_\nu(t) + T_\nu(s, \cdot)[x_\nu(t), \cdot] \]
then there exists \( \nu^* > 0 \) such that \( z(t) \in L_{\nu^*} \) for any \( \nu \in [0, \nu^*] \).

Proof: Define \( z(t) = T_\nu(s, \cdot)x(t) \), from the lemma conditions, we have that
\[
z(t) = b_1 + b_2 \int_0^t e^{-a(t-r)}(x(t) - z_0(t)) dr
\] (A.2)

for some \( a, b_1, b_2 \in \mathbb{R}^* \). Since \( x_\nu(t) \in B_{p, \nu} \), using the Gronwall lemma [9], we have that
\[
z(t) \leq b_1 + b_2 \int_0^t e^{-a(t-r)}(b_1 e^{b_1 t} + b_2 e^{b_2 t}) e^{b_1(t-r)} dr
\] (A.3)

for some \( b_2 \in \mathbb{R}^* \). Since
\[
f \int_0^t e^{-a(t-r)} + b_2 \int_0^t e^{-a(t-r)}(b_1 e^{b_1 t} + b_2 e^{b_2 t}) e^{b_1(t-r)} dr
\] (A.4)

for any \( \nu \in [0, \nu^*] \), we have that \( z(t) \in L_{\nu^*} \), so \( z(t) \in L_\nu \).

It follows as a special case that \( z(t) \in L_\nu \) if \( x_\nu(t) \in L_{\nu^*} \), or more specially, \( x(t) \in L_{\nu^*} \), as in the case of the robust stability or ideal stability of the adaptive control systems.

ACKNOWLEDGMENTS

The second author would like to acknowledge the hospitality of the Center for Industrial Control Science at the University of Newcastle where most of the work of this note was performed and the numerous discussions with Profs. D. Hill, G. Goodwin, R. Middleton, C. De Souza, and I. Mareels. The authors also would like to thank Prof. S. Morse for many useful discussions on the problem considered in this note.

REFERENCES

Robust Strict Positive Realness: New Results for Interval Plant-Controller Families

Antonio Vicino and Alberto Tesi

Abstract—Several classical results in the systems and control field rely on strict positive realness of suitable transfer functions. Absolute stability of nonlinear systems as well as convergence and performance analysis of adaptive schemes are typical areas where strict positive realness is a key property in many problems. In this note, invariance of this property for families of perturbed transfer functions is studied. Necessary and sufficient conditions are given for strict positive realness of interval and real shifted interval plant-controller families of transfer functions.

I. INTRODUCTION

The Strict Positive Realness (SPR) property of transfer functions and related polynomial positivity tests have been recognized as important topics in different systems and control areas for some time (see [1]–[5]). Reference [1] provided for the first time an algebraic Routh-like test for checking positivity of a polynomial. In recent years, the extensive literature on the robust Hurwitz property of families of polynomials with perturbed coefficients following the theorem of Khartoumov has stimulated research on invariance of SPR property with respect to transfer function coefficient perturbations. Papers [4], [6], [7] are among the first references in the literature addressing the robust SPR problem for families of interval transfer functions, i.e., transfer functions with numerator and denominator coefficients belonging to independent uncertainty intervals. Motivation for these investigations can be found in several research areas. The importance of the robust SPR property in absolute stability of nonlinear systems has been recognized in [4], [5], [8], [9], while [6], [10] focus on the analysis and design of filters for stability and optimal performance of adaptive schemes. In particular, [10] gives necessary and sufficient conditions for the existence of a rational filter making strict positive real a given polytopic family of transfer functions. Reference [9] by extending the results in [7], solves the robust SPR problem for interval families of transfer functions with specific reference to the circle criterion of absolute stability of nonlinear control systems.

The objective of this note is to provide a framework based on simple frequency domain geometric properties, yielding very simple proofs of many of the results in the literature on robust SPR. The fruitfulness of the approach is confirmed by the fact that it allows us to prove several interesting extensions of the available results and some important new results on robust SPR. In particular, with reference to the real shifted SPR property of transfer functions, i.e., SPR of a transfer function added to a fixed real number, which is a very important property both for absolute stability of Lur'e control systems and for analysis and design of algorithms in adaptive identification and control contexts, we are able to further simplify the conditions given in [9]. More importantly, we solve the problem of robust real shifted SPR when a given transfer function, which may be represented by a linear controller in a nonlinear control context or by a suitable linear filter for adaptive algorithms in an adaptive control context, is cascaded to a given interval plant. This result can be readily used to derive a robust version of the circle criterion for stability of Lur'e control systems, which generalizes that provided in [9], in the sense that a linear controller is added in the uncertain nonlinear control system.

The note is structured as follows. Section II introduces basic notation and properties of polynomials and transfer functions. Section III presents easy proofs of some important theorems on robust SPR of interval transfer functions and states new results for the case when an interval plant is cascaded to a fixed transfer function. Section IV gives robust versions of the well-known circle criterion for absolute stability of Lur'e control systems as immediate consequence of the results of Section III. Finally, Section V reports some concluding remarks.

II. PROBLEM STATEMENT AND PRELIMINARY RESULTS

A. Strict Positive Real Transfer Functions

Consider the rational transfer function (t.f.)

\[ F(s) = \frac{N(s)}{D(s)} \]  

where

\[ N(s) = \sum_{i=0}^{m} n_i s^i, \quad D(s) = \sum_{i=0}^{q} d_i s^i, \quad n_i, d_i \in \mathbb{R}. \]  

Definition I: The rational function \( F(s) \) of the complex variable \( s = \sigma + j\omega \) is said to be Strict Positive Real (SPR) if:

- \( F(s) \) is analytic in the closed right-half plane (\( \sigma \geq 0 \));
- \( \text{Re}[F(\sigma + j\omega)] > 0 \), for \( \sigma \geq 0 \).

The following property gives necessary and sufficient conditions for a rational function to be SPR.