Robustness of absolute stability

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The stability of a class of single-input, single-output singularly perturbed systems formed by a linear time-invariant feedback block with a sector-bounded time varying feedback is considered. It is shown that if the reduced order `slow' subsystem is absolutely stable and the parasitics are asymptotically stable and sufficiently fast, then the full system is absolutely stable. Bounds on the singular perturbation parameter for uniform asymptotic stability and absolute stability are obtained.

1. Problem formulation

Consider a linear time-invariant SISO (single-input single-output) singularly perturbed system

\[ \dot{x} = A_{11}x + A_{12} \xi + B_{1}u \]
\[ \mu \dot{\xi} = A_{21}x + A_{22} \xi + B_{2}u \]
\[ y = C \dot{x} \]
\[ u = -K(t) y \]

where \( x \in \mathbb{R}^n, \xi \in \mathbb{R}^n, u, y \in \mathbb{R} \) and \( \mu \) is a small positive parameter representing the speed ratio of the slow versus the fast phenomena (Kokotovic et al. 1976). As \( \mu \to 0 \) the fast part of the plant reaches its steady-state instantaneously, that is the plant order reduces to that of its slow part. We neglect the fast parasitics by setting \( \mu = 0 \) in (2), and this gives the following reduced-order `slow' system:

\[ \dot{\bar{x}} = A_{11} \bar{x} + B_{1} \bar{u} \]
\[ \bar{y} = C \bar{x} \]
\[ \bar{u} = -K(t) \bar{y} \]

where \( A_{11} = A_{11} - A_{12} A_{22}^{-1} A_{21} \) and \( B_{1} = B_{1} - A_{12} A_{22}^{-1} B_{2} \).

Let \( G(\xi, s) = \gamma(\mu, s) / p(\mu, s) \) be the transfer function of (1)-(3) and make the following assumptions:

(i) The degree of \( p(\mu, s) \) in \( s \) exceeds that of \( q(\mu, s) \).
(ii) The polynomials \( p(\mu, s) \) and \( q(\mu, s) \) are coprime for all \( 0 < \mu < 1 \).
(iii) \( K(t) \in \mathcal{K} \), i.e. \( a < K(t) \beta \) where \( a, \beta \) are positive constants and \( K(t) \) is Lebesgue integrable over every finite interval.
(iv) The reduced-order system (6)-(7) is absolutely stable, that is it is asymptotically stable for all time-varying gains \( K(t) \in \mathcal{K} \) (Aizerman and Gantmakher 1964).
The questions asked are as follows: If the system is absolutely stable when the parasitics are neglected ($\mu = 0$), will it still be absolutely stable when they are present ($\mu \neq 0$)? Are there parasitics which, however small, will cause the loss of absolute stability? Is there a wider class of parasitics for which absolute stability is lost but uniform asymptotic stability holds for some $K(0) \in \mathcal{K}$? In other words can we use any extra information about some $K(t)$ of the class $K$ to establish the uniform asymptotic stability of the full-order system even though absolute stability is violated by the parasitics? The answers to those questions are given in the following section.

2. Main result

Let $m_1, m_2, a_1, a_2$ be positive numbers satisfying

$$|\phi_{11}(t, t_0)| \leq m_1 \exp \left( -m_2(t - t_0) \right)$$

(8)

$$|\phi_{22}(t, t_0)| \leq a_1 \exp \left( -\frac{(t-t_0)}{\mu} \right)$$

(9)

where $\phi_{11}(t, t_0)$ and $\phi_{22}(t, t_0)$ are the transition matrices of the reduced-order closed loop system (5)-(7) and the fast subsystem $\dot{x}_2 = A_2 x_2$, respectively. The matrix norm is taken as the induced norm defined by

$$|A| = \sup_{|x| = 1} |Ax|$$

where $A$ is a matrix and $x$ is a vector. Let

$$K_2 = |A_{11}|, \quad K_1 = \sup_{t} |A_{12}^{-1}(A_{11} - B_2 C K(t))|, \quad K_2 = \sup_{t} |A_{12} - B_2 C K(t)|$$

then the following theorem and corollary establish the uniform asymptotic stability and absolute stability of the closed loop system (1)-(4), respectively.

**Theorem**

For every gain $K(0) \in \mathcal{K}$ there exists a $\mu_* > 0$ such that for all $\mu \in (0, \mu_*)$ the closed loop system (1)-(4) is uniformly asymptotically stable and $\mu_* = \min \{\mu_1, \mu_2\}$, where

$$\mu_1 = \frac{a_2}{m_2}, \quad \mu_2 = \frac{a_1}{m_1}, \quad \mu_* = \min \left\{ \mu_1, \mu_2, K_1, K_2 \right\}$$

(10)

**Corollary**

There exists a $\mu_* > 0$ such that for all $\mu \in (0, \mu_*)$ the feedback system (1)-(4) is absolutely stable, where

$$\mu_* = \inf_{K(0)} \mu_*$$

(11)

**Proof of Theorem**

The proof is a modification of the proof given by Javid (1978) for a general class of linear time varying singularly perturbed systems. The details of the proof are presented in the Appendix, where it is shown that for $\mu < a_1/m_2$

$$|x(t)| \leq p_1 \exp \left( -\left( m_2 - a_1 \right)(t - t_0) \right)$$

(12)
where

\[ N_1 = m_2 \left[ \frac{n_1 K_2 K_3}{\mu (n_2 \mu - m_2)} + K_4 \right] \]

and \( p_i \) is a constant. Hence for uniform asymptotic stability we need \( \mu < n_2 / m_2 \)
and \( m_2 > N_1 \), and therefore (10) follows.

**Proof of Corollary**

The bound for \( \mu \) for uniform asymptotic stability established by the theorem depends on the particular choice of \( K(t) \) since \( \mu_2 \) depends on the sup \( \|K(t)\| \)
and also \( m_1 \) and \( m_2 \) are functions of \( K(t) \). Such a bound does not guarantee uniform asymptotic stability for all \( K(t) \in K \), and hence does not imply absolute stability. However, we can show that a bound for \( \mu \) exists for absolute stability by taking the infimum over \( m_1 \) and \( m_2 \) obtained for all possible \( K(t) \in K \) in (10).

For any \( K(t) \in K \), \( m_1 \) and \( m_2 \) are finite; therefore \( \mu_2 \) is always positive.

The following example illustrates the robustness of absolute stability with respect to singular perturbations, as well as the conservation of absolute stability compared with uniform asymptotic stability in the presence of parasitics.

3. Example

The system

\[
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{z}
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 \\
0 & -1 & 1 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\begin{bmatrix}
u
\end{bmatrix}
\]

(13)

\[ y = x \]  

(14)

\[ u = -K(t)y \]  

(15)

has the reduced system

\[ \dot{x} = -K(t)x \]  

(16)

and fast subsystem

\[ \dot{x} = -\dot{x} \]  

(17)

It can be shown (Brockett and Lee 1967) that for

\[ 1 < K(t) \]  

(18)

the reduced system (16) is absolutely stable. That is, given \( 1 < K(t) < 100 \) the transfer function \( (1 + s)(100 + s) \) is strictly positive real, and this is a sufficient condition for the absolute stability of (10).

A sufficient condition for system (13)–(15), (18) to be absolutely stable is that the transfer function

\[ H(\mu, s) = \frac{1 + s(\mu s + 1)}{100 + s(\mu s + 1)} \]  

(19)

be strictly positive real (Brockett and Lee 1967). It can be shown that for all \( \mu \in [0, 0.0125] \), \( H(\mu, s) \) is strictly positive real and hence the full-order system is absolutely stable. Now take \( K(t) = 10 + 2 \cos \theta ) \); then from (5) and (6) \( m_1 = 1.388, m_2 = 10, n_1 = 1 \) and \( n_2 = 1 \). Hence \( \mu_2 = 0.1 \) and \( \mu_1 = 0.63 \), and therefore the full-order system is uniformly asymptotically stable for all \( \mu < 0.03 \).
Obviously the full-order system is not absolutely stable for $0.01235 < \mu < 0.03$ but it is uniformly asymptotically stable for this particular choice of $K(t)$. It can be easily shown that a large class of $K(t)$ can be found such that the full-order system is uniformly asymptotically stable without being absolutely stable.

4. Conclusions

Bounds for the singular perturbation parameter $\mu$ have been obtained for uniform asymptotic stability and absolute stability for a class of linear time invariant SISO systems with time varying feedback gain. We have shown that absolute stability is robust with respect to parasitics provided the parasitics are stable and the reduced-order system is absolutely stable. It is clear from the analysis and the example that, in cases where more information about some $K(t); K$ is available, a wider class of parasitics can be allowed to interfere with the system without destroying uniform asymptotic stability.

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Appendix

Details of the proof of the Theorem

System (1)–(4) is rewritten as

\[ \dot{x} = (A_{11} - B_2 CK(t))x + A_{12}\hat{y} + A_{12}\hat{z} \]  
\[ \dot{\mu}z = (A_{21} - B_2 CK(t))x + A_{22}\hat{y} + A_{32}\hat{x} \]  

with the ‘reduced’ system

\[ \dot{z} = (A_0 - B_2 CK(t))x + A_{12}\hat{z} + A_{12}\hat{y} \]  

System (A 1), (A 2) is a special case of a more general class of time-varying singularly perturbed systems whose stability is treated by Klimachev and Krasovskii (1961), Wilde and Kokotovic (1972) and Javid (1978). The extension of the results of these references to our case would require the differentiability of \( K(t) \). However, having restricted \( K(t) \) to a class of differentiable functions we would need to restrict the notion of absolute stability. This restriction of differentiability can be avoided by modifying Javid’s proof. That is, we express (A 1), (A 2) as

\[ \dot{x} = A_1(t)x + A_{12}\hat{y} \]  
\[ \dot{\mu}z = A_{22}(t)x + A_{32}\hat{x} \]  
\[ \dot{\eta} = z + A_{21}(t)x \]  

Applying the variation of constants to (A 4) and (A 5) and using (A 6) we obtain

\[ x(t) = \phi_{A_1}(t, \xi_0)x_0 + \int_{\xi_0}^t \phi_{A_1}(t, \tau)A_{12}\hat{z}(\tau) \, d\tau \]
\[ + \int_{\xi_0}^t \phi_{A_1}(t, \tau)A_{22}(\tau)A_{32}\hat{x}(\tau) \, d\tau \]  
\[ z(t) = \phi_{A_2}(t, \xi_0)\eta_0 + \int_{\xi_0}^t \phi_{A_2}(t, \tau)A_{22}(\tau)\hat{z}(\tau) \, d\tau \]  
\[ + \int_{\xi_0}^t \phi_{A_2}(t, \tau)A_{21}(\tau)\hat{x}(\tau) \, d\tau \]  

Using (A 8) and (A 9)

\[ |z(t)| \leq m_1 \exp (-m_2(t - \xi_0))|x_0| + \int_{\xi_0}^t m_1 \exp (-m_2(t - \tau))K_3|z(\tau)| \, d\tau \]
\[ + \int_{\xi_0}^t m_1 \exp (-m_2(t - \tau))K_4|x(\tau)| \, d\tau \]  
\[ |\hat{z}(t)| \leq m_1 \exp \left\{ -\frac{m_1(t - \xi_0)}{\mu} \right\} |\eta_0| + \frac{1}{\mu} \int_{\xi_0}^t m_1 \exp \left\{ -\frac{m_1(t - \tau)}{\mu} \right\} K_3|\hat{z}(\tau)| \, d\tau \]  
\[ \times \exp \left\{ -\frac{m_1(t - \tau)}{\mu} \right\} K_4|x(\tau)| \, d\tau \]
Substituting for \(|x(t)|\) in (A 9) we have
\[
|x(t)| \leq m_1 \exp \left( -m_2(t-t_0) \right) |x_0| \\
+ \int_{t_0}^{t} \left[ m_1 K_0 a_2 |x_0| \exp \left( -a_2 \frac{(t-t_0)}{\mu} \right) \exp \left( -m_2(t-\tau) \right) d\tau \right. \\
+ \left. \int_{t_0}^{\tau} m_1 K_0 a_2 K_2 \exp \left( -m_2(t-\tau) \right) \right] \exp \left( -a_2 \frac{(\tau-s)}{\mu} \right) |x(s)| ds d\tau \\
+ \int_{t_0}^{\tau} m_1 K_1 \exp \left( -m_2(t-\tau) \right) |x(\tau)| d\tau
\]
(A 11).

After integrating the second term of (A 11) and using integration by parts for the third term, we obtain
\[
|x(t)| \leq m_1 \left[ |x_0| + \frac{a_2 K_0 |x_0|}{(a_2/\mu - m_2)} \right] \exp \left( -m_2(t-t_0) \right) - \frac{m_2 a_2 K_0 |x_0|}{(a_2/\mu - m_2)} \\
\times \exp \left( -a_2(t-t_0)/\mu \right) + \frac{m_1 a_2 K_0 K_2}{\mu(a_2/\mu - m_2)} \int_{t_0}^{t} \exp \left( -m_2(t-\tau) \right) \\
- \exp \left( a_2(t-t_0)/\mu \right) |x(\tau)| d\tau \\
+ m_1 K_1 \int_{t_0}^{t} \exp \left( -m_2(t-\tau) \right) |x(\tau)| d\tau
\]
(A 12)

We have the first bound on \(\mu\) by taking \(m_2 < a_2/\mu\), which implies that
\[
\mu < \frac{a_2}{m_2}
\]
(A 13)

Then from (A 12)
\[
|x(t)| \leq m_1 \left[ |x_0| + \frac{a_2 K_0 |x_0|}{(a_2/\mu - m_2)} \right] \exp \left( -m_2(t-t_0) \right) \\
+ m_1 \left[ \frac{a_2 K_0 K_2}{a_2/\mu - m_2} + K_1 \right] \int_{t_0}^{t} \exp \left( -m_2(t-\tau) \right) |x(\tau)| d\tau
\]
(A 14)

Defining \(\theta(t) = \exp \left( m_2 t \right) |x(t)|\) and using Gronwall’s lemma (Coppel 1965), we obtain
\[
\theta(t) \leq m_1 \left[ |x_0| + \frac{a_2 K_0 |x_0|}{(a_2/\mu - m_2)} \right] \exp \left( m_2 t_0 \right) \exp \left( N_1(t-t_0) \right)
\]
(A 15)

where
\[
N_1 = m_1 \left[ \frac{a_2 K_0 K_2}{a_2/\mu - m_2} + K_1 \right]
\]
and which yields
\[
|x(t)| \leq m_1 \left[ |x_0| + \frac{a_2 K_0 |x_0|}{(a_2/\mu - m_2)} \right] \exp \left( -m_2(N_1(t-t_0)) \right)
\]
(A 16)
Thus for (A 1), (A 2) to be uniformly asymptotically stable, we need
\[ m_2 > N_1, \text{i.e.} \]
\[ m_2 > m_1 \left[ \frac{\alpha_1 K_2 K_3}{\mu (\sigma_2 \sigma_3 - \sigma_3)} + K_1 \right] \quad (A 17) \]

From (A 13) and (A 17) we have \( \alpha_0 > m_1 (\sigma_2 K_2 + \sigma_3 K_3) \), which implies that
\[ \mu < \frac{m_1 \sigma_2}{m_1 (\sigma_2 K_2 + \sigma_3 K_3)} \quad (A 18) \]

Let \( \mu = \sigma_2/\mu_2 \) and \( \mu_2 = m_1 (\sigma_2 K_2 + \sigma_3 K_3) \), and take \( \mu^* = \min \{ \mu_2, \mu_1 \} \).

Then for all \( \mu > 0 \), \( \mu^* \) inequalities (A 13), (A 17) and (A 18) are always satisfied, and therefore (A 1), (A 2) is uniformly asymptotically stable.

**References**


