Strictly Positive Real Matrices and the Lefschetz-Kalman-Yakubovich Lemma

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Abstract—In this note we give necessary and sufficient conditions in the frequency domain for rational matrices to be strictly positive real. Based on this result, the matrix form of the Lefschetz-Kalman-Yakubovich lemma is proved, which gives necessary and sufficient conditions for strictly positive real transfer matrices in the state-space realization form.

I. INTRODUCTION

In most of the literature an \( m \times m \) rational matrix \( H(s) \) is termed to be strictly positive real (SPR) if:

a) all elements of \( H(s) \) are analytic in the closed right-half plane Re\([s]\) \( \geq 0 \);

b) \( H(j\omega) + H^T(-j\omega) > 0 \) \( \forall \omega \in (-\infty, +\infty) \). In some cases, b) is replaced by

\[ b' \quad H(j\omega) + H^T(-j\omega) > 0 \quad \forall \omega \in [-\infty, +\infty]. \]

As stated by Taylor [1] and Narendra and Taylor [2] for the scalar case, a) and b) are only necessary, whereas a') and b') are only sufficient for \( H(s) \) to be SPR. As shown in [2] for the scalar case, conditions a) and b') or (3) are not necessary and sufficient for the Lefschetz-Kalman-Yakubovich (LKY) lemma [1]-[3] to hold.

Motivated from network theory, Narendra and Taylor [2] gave the following definition for SPR matrices.

Definition 1.1: A rational matrix \( H(s) \) is SPR if \( H(s) - s \) is positive real (PR), \( \forall s > 0 \).

In [5], we used Definition 1.1 to develop necessary and sufficient conditions in the frequency domain for a scalar transfer function \( A(s) \) to be SPR. In [6] Definition 1.1 is also indirectly used to establish the interconnection between various time and frequency domain conditions relating SPR systems.

The purpose of this note is to extend the results of [5] and the LKY lemma for scalar transfer functions to the matrix case, and therefore, develop necessary and sufficient conditions in the frequency domain and for state-space realizations of transfer matrices.

II. FREQUENCY DOMAIN CONDITIONS FOR SPR MATRICES

Let us consider the \( m \times m \) rational matrix of the following form:

\[ H(s) = C(s - A)^{-1}B + D + sN \]

(2.1)

where \( A, B, C, D, \) and \( N \) are real constant matrices with appropriate dimensions.

Definition 2.1: An \( m \times m \) rational matrix \( H(s) \) is said to belong to class \( SC(s) \) if \( H(s) + H^T(-s) \) has rank \( m \) almost everywhere in the complex plane.

The following theorem gives necessary and sufficient conditions in the frequency domain for \( H(s) \) in the class \( SC(s) \) to be SPR.

Theorem 2.1: Given an \( m \times m \) rational matrix \( H(s) \in SC(s) \), of the form (2.1), then \( H(s) \) is SPR if:

1) all elements of \( H(s) \) are analytic in Re\([s]\) \( \geq 0 \);

2) \( H(j\omega) + H^T(-j\omega) > 0 \) \( \forall \omega \in R \);

3) \( \lim_{\omega \to \infty} \omega^2 H(j\omega) + H^T(-j\omega) > 0 \) \( \forall \omega \in R \) for some \( \beta > 0 \), if \( D + D^T = 0 \); ii) \( \lim_{\omega \to \infty} \omega H(j\omega) + H^T(-j\omega) > 0 \) if \( \det(D + D^T) \neq 0 \).

4) \( N \geq 0 \).

Proof:

Necessary: If \( H(s) \) is SPR, then \( H(s) - s \) is PR \( \forall \in (0, +\infty) \) for some \( e > 0 \), so all elements of \( H(s) \) are analytic in Re\([s]\) \( \geq 0 \) and \( \lim_{\omega \to \infty} \omega^2 H(j\omega) + H^T(-j\omega) > 0 \) \( \forall \in R \).

Using Youla’s spectral factorization result for rational matrices [4], we have that

\[ H(s) + H^T(-s) = W^T(-s)W(s) \]

(2.2)

where \( W(s) \) is analytic in Re\([s]\) \( \geq 0 \) and has rank \( m \) in Re\([s]\) \( \geq 0 \).

Setting \( s = j\omega + e \) in (2.2), we have that \( H(j\omega) + H^T(-j\omega) = \det(W(j\omega)) \neq 0, \forall \omega \in R \). Since \( H(j\omega) + H^T(-j\omega) > 0 \) \( \forall \omega \in R \), we have that \( H(j\omega) + H^T(-j\omega) > 0 \) \( \forall \omega \in R \).

Case (i) \( \det(D + D^T) = 0 \). Since \( H(s) - s \) is PR \( \forall \omega \in R \), let \( \omega \to +\infty \), we have that \( D + D^T = 0 \). Hence, \( H(s) - s \) is PR for some \( \beta > 0 \), if \( D + D^T = 0 \).

Hence, it follows from (2.3) and (2.4) that \( Kx = 0 \).

For above \( x \), consider the expansion \( H(s) - sN = H(s-\epsilon) - sN \).

\[ \lim_{\epsilon \to 0} \omega^2 H(j\omega) + H^T(-j\omega) = -2s^2B^TPBx < 0. \]

The last inequality comes as follows. If \( Bx = 0 \), then \( x^T(H(j\omega) + H^T(-j\omega))x = 0 \), \( \forall \omega \in R \), which contradicts 2.\( m \). But (2.5) is not true since \( H(s) - s \) is PR. Hence, \( \lim_{\omega \to \infty} \omega^2 H(j\omega) + H^T(-j\omega) > 0 \).

Case (i) \( \det(D + D^T) = 0 \). Since \( N(s) \) and \( H(s) \) are PR, we have that \( D + D^T = 0 \). Hence, \( \lim_{\omega \to \infty} \omega^2 H(j\omega) + H^T(-j\omega) > 0 \).

Sufficiency: All elements of \( H(s) \) are analytic in Re\([s]\) \( \geq 0 \), then there exists \( \epsilon _0 > 0 \) such that all elements of \( H(s) - sN \) are analytic in Re\([s]\) \( \geq 0 \). Hence, \( \lim_{\omega \to \infty} \omega^2 H(j\omega) + H^T(-j\omega) > 0 \).

Hence, it follows from (2.3) and (2.4) that \( Kx = 0 \).

Therefore, \( \lim_{\epsilon \to 0} \omega^2 H(j\omega) + H^T(-j\omega) > 0 \).

For above \( x \), consider the expansion \( H(s) - sN = H(s-\epsilon) - sN \).

\[ \lim_{\epsilon \to 0} \omega^2 H(j\omega) + H^T(-j\omega) = -2s^2B^TPBx < 0. \]

The last inequality comes as follows. If \( Bx = 0 \), then \( x^T(H(j\omega) + H^T(-j\omega))x = 0 \), \( \forall \omega \in R \), which contradicts 2.\( m \). But (2.5) is not true since \( H(s) - s \) is PR. Hence, \( \lim_{\omega \to \infty} \omega^2 H(j\omega) + H^T(-j\omega) > 0 \).

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For above \( x \), consider the expansion \( H(s) - sN = H(s-\epsilon) - sN \).

\[ \lim_{\epsilon \to 0} \omega^2 H(j\omega) + H^T(-j\omega) > 0 \].
positive definite matrix $L$; then real matrices $Q$ and $K$, and a real symmetric positive definite matrix $P$ exist such that
\[ PA + A^TP = -QQ^T - \mu I; PB + C^TQK; K^TK = D^TD + D \]
for some $\delta > 0$ if and only if $H(s)$ is SPR and $\mu$ is sufficiently small.

**Proof of Theorem 3.1:**

**Necessity:** $A$ is stable implies that $H(s) = -\delta I$ and $((s - \lambda)I - A)^{-1}$ are both analytic in $\Re[s] \leq 0$ for some $\lambda > 0$. From (3.4) we have that
\[ H(s) = (s - \lambda)I - A \]
for some $\lambda > 0$. Hence, it follows from Theorem 2.3 that $H(s)$ is SPR and $\mu$ is sufficiently small.

**Sufficiency:** Since $H(s) \in \mathfrak{X}(s)$ is SPR, it follows from Theorem 2.1 that $N \geq 0$, $D + D^T = 0$, and $H(s)$ is SPR. Hence, it follows from Theorem 2.3 that $H(s)$ is SPR and $\mu$ is sufficiently small.

**Remark 3.1:** We should note that $H(s)$ is assumed to belong to class $\mathfrak{X}(s)$ in order to avoid the singularity of $H(s)$ for $s = \pm \infty$.

**III. THE MATRIX FORM OF THE KLY LEMMA**

Let us consider the $m \times m$ rational transfer matrix of the form
\[ H(s) = C(sI - A)^{-1}B + D + snI. \]

When $m = 1$, matrices $B, C, D$, and $n$ are denoted as $b, c, d$, and $n$, respectively. In the scalar case we have the following lemma as the LKY lemma [1]-[3].

**Lemma 3.1:** Given $E = 0, \mu > 0$, a matrix $A$ such that $sI - A$ has a real vector $x$ such that $(A, B)$ is completely controllable, a real vector $c$, a scalar $d$, and an arbitrary real symmetric positive definite matrix $L$, then a real vector $q$ and a real matrix $P$ satisfy
\[ A^TP + PA = -QQ^T - \mu I; PB + C^TQLK; K^TK = D^TD + D \]
for some $\delta > 0$ if and only if $H(s)$ is SPR and $\mu$ is sufficiently small.

For the matrix case, i.e., $m > 1$, we have the following lemma.

**Lemma 3.2 [1]:** Assume that the rational transfer matrix $H(s)$ has poles which lie in $\Re[s] < -\gamma$ where $\gamma > 0$ and $(A, B, C, D)$ is a minimal realization of $H(s)$. Then $H(s) = \gamma I$ if and only if a matrix $P = nI > 0$, and matrices $Q$ and $K$ exist such that
\[ PA + A^TP = -QQ^T - \mu I; PB + C^TQLK; K^TK = D^TD + D \]
for some $\delta > 0$.

Note that if $N = 0$, then (3.3) is not sufficient for $H(s) = \gamma I$ to be PR. If $N = 0$, from Lemma 3.2 and Definition 1 it is clear that the conditions of Lemma 3.2 are necessary and sufficient for $H(s)$ to be SPR. However, Lemma 3.2 is not an extension of Lemma 3.1 to the matrix case since in (3.2) $L > 0$ is arbitrary.

Using Theorem 2.1, the generalization of Lemma 3.1 to the matrix case with $N \neq 0$ is given by the following theorem.

**Theorem 3.1:** Assume that $H(s)$ given in (3.1) belongs to $\mathfrak{X}(s)$, then given $b > 0$, a matrix $A$ such that $(sI - A)$ has only zeros in the open left-hand plane, a real matrix $B$ such that $(A, B)$ is completely controllable, real matrices $C, D, N$, and an arbitrary real symmetric positive definite matrix $L$; then real matrices $Q$ and $K$, and a real symmetric positive definite matrix $P$ exist such that
\[ PA + A^TP = -QQ^T - \mu I; PB + C^TQLK; K^TK = D^TD + D \]
for some $\delta > 0$ if and only if $H(s)$ is SPR and $\mu$ is sufficiently small.

**Proof of Theorem 3.1:**

**Necessity:** $A$ is stable implies that $H(s) = \gamma I$ and $((s - \lambda)I - A)^{-1}1$ are both analytic in $\Re[s] \leq 0$ for some $\lambda > 0$. From (3.4) we have that
\[ H(s) = (s - \lambda)I - A \]
for some $\lambda > 0$. Hence, it follows from Theorem 2.3 that $H(s)$ is SPR and $\mu$ is sufficiently small.
matrix $C$ such that $(C, A)$ is completely controllable, real matrices $B$, $D$, and $N$, and an arbitrary real symmetric positive definite matrix $L$; then real matrices $Q$ and $K$, and a real symmetric positive definite matrix $P$ exist such that

$$PA + A^TP = -QQ^T - \mu L; PB = C^T - QK; K^TK = D + D^T$$

(3.14)

$$N \geq 0; D + D^T \succeq \delta (N + N^T)$$

(3.15)

for some $\delta > 0$ if and only if $H(s)$ is SPR and $\mu$ is sufficiently small.

Proof: Consider the rational matrix $H(s)$; it follows that $H(s) \in \mathcal{H}(s)$ iff $H^s(s) \in \mathcal{H}(s)$. Using the same procedures as above for $H^s(s)$, we show that given $L$, an arbitrary positive definite matrix, there exist a positive definite matrix $P_i$, matrices $Q_1$ and $K$ such that

$$PA_i + A_i^TP_i = -Q_iQ_i^T - \mu L; \quad P_iC_i^T = B - Q_iK; \quad K^TK = D + D^T$$

(3.16)

$$N \geq 0; D + D^T \succeq \delta (N + N^T)$$

(3.17)

for some $\delta > 0$ if and only if $H^s(s)$ is SPR and $\mu$ is sufficiently small. Since $H(s)$ is SPR if and only if $H^s(s)$ is SPR, letting $L = P_i^{-1}L_iP_i^{-1}$, $P = P_i^{-1}$, and $Q = -P_i^{-1}Q_i$ in (3.16), we have Corollary 3.1 for the considered problem.

IV. CONCLUSION

In this note, necessary and sufficient conditions for strictly positive real (SPR) matrices are given in the frequency domain. Furthermore, the Lefschetz–Kalman–Yakubovich lemma for SPR scalar transfer functions is generalized to SPR transfer matrices.

REFERENCES


MTDC of a discrete-time system. Contrary to many existing techniques, no special assumption on the invertibility of the transition matrix is required.

I. INTRODUCTION

The general problem of designing minimum-time deadbeat controllers (MTDC) of linear discrete-time systems was extensively studied in the past. This problem does not generally have a unique solution. In [1], an overparametrization solution was utilized to minimize a quadratic performance criterion and the problem of parametrizing the class of MTDC through the minimum number of parameters was suggested for future research. Several researchers have recently proposed solutions for such a parametrization problem.

Schlegel [2], gave an explicit parametrization of almost the whole class of MTDC for a given system through the minimum-number of parameters. His method is based on considering the set of nonsingular matrices commutative with the Jordan-form matrix of the closed-loop system. His method is, however, restricted to the class of systems having a nonsingular transition matrix.

In [3], the problem was solved by transforming the system to a particular class of phase-variable block companion form. This solution is restricted to a particular class of systems.

Fahmy and O'Reilly [4] treated the problem from the standpoint of closed-loop eigenstructure assignment. Their solution was developed under the assumption that the transition matrix of the system is nonsingular.

Based on the eigenstructure approach, the parametrization of the solution of the general eigenvalue assignment problem has been treated in [12] and [13]. As stated in [12], the given solution is an overparametrized one, while in [13] the solution is minimal parametrized. However, the method in [13] is not applicable to the case where the open-loop and the desired closed-loop eigenvalue sets share some elements. Therefore, the method can be used to solve the considered problem for the class of systems having a nonsingular transition matrix. While this condition is satisfied for sampled data continuous-time systems, it is not necessarily true for more general discrete-time systems.

In the present note, a new approach for parametrizing a class of MTDC through the minimum parameters is presented. This approach is based on the theory of decoupling and the properties of square decouplable systems. The main result is a new compact parametric form for the considered class of MTDC.

Parametrization of a Class of Deadbeat Controllers Via the Theory of Decoupling

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Abstract—This note presents a new approach for parametrizing a class of minimum-time deadbeat controllers (MTDC) through the minimum number of parameters. The approach is based on the theory of decoupling and the properties of square decouplable systems. The main result is a new compact parametric form for the considered class of

$$x(t + 1) = Ax(t) + Bu(t)$$

(1)

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are the state and control input vectors, respectively, at the $t$th control iteration and $A$, $B$ are real constant matrices of appropriate dimensions. It is assumed that matrix $B$ is of full rank $m$ [i.e., there are no redundant inputs in (1)], and the pair $(A, B)$ is controllable with the controllability indexes $\mu = \mu(1) \geq \cdots \geq \mu(m)$, where $\mu$ is its controllability index. It is not necessarily assumed that $A$ is nonsingular.

A. Problem Statement

The considered problem of MTDC design is the following [5]. For the multivariable controllable system (1), find a feedback control law of the form $u(t) = Fx(t)$, such that the closed-loop matrix $\bar{A} = (A + BF)$ is similar to the matrix

$$J = \text{block diag}(J_1, J_2, \ldots, J_m)$$

(2)

where $J_i$ is an $\mu(i) \times \mu(i)$ matrix with ones on the first superdiagonal and

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