THE HOMOTOPY FIXED POINTS OF THE CIRCLE ACTION ON HOCHSCHILD HOMOLOGY

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Abstract. We show that Connes’ $B$-operator on a cyclic differential graded $k$-module $M$ is a model for the canonical circle action on the geometric realization of $M$. This implies that the negative cyclic homology and the periodic cyclic homology of a differential graded category can be identified with the homotopy fixed points and the Tate fixed points of the circle action on its Hochschild complex.

Let $k$ be a commutative ring and let $A$ be a flat associative $k$-algebra. The Hochschild complex $\text{HH}(A)$ of $A$ with coefficients in itself is defined as the normalization of a simplicial $k$-module $A^\otimes(n+1)$.

The simplicial $k$-module $A^\otimes$ is in fact a cyclic $k$-module: it extends to a contravariant functor on Connes’ cyclic category $\Lambda$. As we will see below, it follows that the chain complex $\text{HH}(A)$ acquires a canonical action of the circle group $T$. The cyclic homology, negative cyclic homology, and periodic cyclic homology of $A$ over $k$ are classically defined by means of explicit bicomplexes. The goal of this note is to show that:

1. The cyclic homology $\text{HC}(A)$ coincides with the homotopy orbits of the $T$-action on $\text{HH}(A)$.
2. The negative cyclic homology $\text{HN}(A)$ coincides with the homotopy fixed points of the $T$-action on $\text{HH}(A)$.
3. The periodic cyclic homology $\text{HP}(A)$ coincides with the Tate fixed points of the $T$-action on $\text{HH}(A)$.

The first statement is due to Kassel [Kas87, Proposition A.5]. The second statement is well-known to experts, but a proof seems to be missing from the literature. This gap was mentioned in the introductions to [TV11] and [TV15], and it was partially filled in [TV11], where (2) is proved at the level of connected components for $A$ a smooth commutative $k$-algebra and $Q \subset k$.

We will proceed as follows:

• In §1, we recall abstract definitions of cyclic, negative cyclic, and periodic cyclic homology in a more general context, namely for $\infty$-categories enriched in a symmetric monoidal $\infty$-category.
• In §2, we show that in the special case of differential graded categories over a commutative ring, the abstract definitions recover the classical ones.

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1. Cyclic homology of enriched $\infty$-categories

Let $\mathcal{E}$ be a presentably symmetric monoidal $\infty$-category, for instance the $\infty$-category $\text{Mod}_k$ for $k$ an $E_\infty$-ring. We denote by $\text{Cat}(\mathcal{E})$ the $\infty$-category of categorical algebras in $\mathcal{E}$ in the sense of [GH15], i.e., $\mathcal{E}$-enriched $\infty$-categories with a specified space of objects $\text{ob}(\mathcal{E})$. Let $\Lambda$ be Connes’ cyclic category [Con83], with objects $[n]$ for $n \in \mathbb{N}$. We can associate to every $\mathcal{E} \in \text{Cat}(\mathcal{E})$ a cyclic object $\mathcal{E}^\otimes: \Lambda^{\text{op}} \to \mathcal{E}$ with

$$\mathcal{E}^\otimes_n = \text{colim}_{a_0, \ldots, a_n \in \text{ob}(\mathcal{E})} \mathcal{E}(a_0, a_1) \otimes \cdots \otimes \mathcal{E}(a_{n-1}, a_n) \otimes \mathcal{E}(a_n, a_0).$$

The cyclic category $\Lambda$ contains the simplex category $\Delta$ as a subcategory, and the Hochschild homology of $\mathcal{E}$ (with coefficients in itself) is the geometric realization of the restriction of $\mathcal{E}^\otimes$ to $\Delta^{\text{op}}$:

$$\text{HH}(\mathcal{E}) = \text{colim}_{n \in \Delta^{\text{op}}} \mathcal{E}^\otimes_n \in \mathcal{E}.$$
We refer to [AMGR] for a precise construction of $\mathcal{E}^\natural$ in this context, and for a proof that $\text{HH}(\mathcal{E})$ depends only on the $\mathcal{E}$-enriched $\infty$-category presented by $\mathcal{E}$ (i.e., the functor $\text{HH}$ inverts fully faithful essentially surjective morphisms).

We now recall the canonical circle action on $\text{HH}(\mathcal{E})$. Let $\Lambda \to \tilde{\Lambda}$ denote the $\infty$-groupoid completion of $\Lambda$, and let $T$ be the automorphism $\infty$-group of $[0]$ in $\tilde{\Lambda}$. Since $\Lambda$ is connected, there is a canonical equivalence $B_T \simeq \tilde{\Lambda}$. Let $\text{PSh}(\Lambda, \mathcal{E})$ be the $\infty$-category of $\mathcal{E}$-valued presheaves on $\Lambda$, and let $\text{PSh}_\simeq(\Lambda, \mathcal{E}) \subset \text{PSh}(\Lambda, \mathcal{E})$ be the full subcategory of presheaves sending all morphisms of $\Lambda$ to equivalences. We have an obvious equivalence

$$\text{PSh}_\simeq(\Lambda, \mathcal{E}) \simeq \text{PSh}(B_T, \mathcal{E}).$$

Since $\mathcal{E}$ is presentable and $\Lambda$ is small, $\text{PSh}_\simeq(\Lambda, \mathcal{E})$ is a reflective subcategory of $\text{PSh}(\Lambda, \mathcal{E})$. We denote by $[-]: \text{PSh}(\Lambda, \mathcal{E}) \to \text{PSh}_\simeq(\Lambda, \mathcal{E}) \simeq \text{PSh}(B_T, \mathcal{E})$ the left adjoint to the inclusion. The morphisms

$$* \xleftarrow{i} B_T \xrightarrow{p} *$$

each induce three functors between the categories of presheaves. We will write $u_T = i^*$, $(-)_{hT} = p_!$, $(-)^{hT} = p_*$ for the forgetful functor, the $T$-orbit functor, and the $T$-fixed point functor, respectively.

**Proposition 1.1.** Let $\mathcal{E}$ be a presentable $\infty$-category and let $X \in \text{PSh}(\Lambda, \mathcal{E})$ be a cyclic object. There is a natural equivalence

$$u_T [X] \simeq \text{colim}_{[n] \in \Delta^\op} X([n]).$$

**Proof.** Let $j: \Delta \to \Lambda$ be the inclusion. Let $X \in \text{PSh}_\simeq(\Delta, \mathcal{E})$ and let $j_*(X) \in \text{PSh}(\Lambda, \mathcal{E})$ be the right Kan extension of $X$. We claim that $j_*(X)$ inverts all morphisms in $\Lambda$. By the formula for right Kan extension, it suffices to show that following: for every $[n]$, the functor of comma categories $\Delta \times_\Lambda \Lambda /[n] \to \Delta \times_\Lambda \Lambda /[0]$ induced by the unique map $[n] \to [0]$ in $\Delta$ is a weak equivalence. Recall that every morphism in $\Lambda$ can be written uniquely as a composite $h \circ t$ where $t$ is an automorphism and $h$ is in $\Delta$ [Lod92, Theorem 6.1.3]. If $\Gamma = \Delta \times_\Lambda \Lambda /[0]$, $\{[n] \to [0], t \in \text{Aut}_\Lambda([m]) = C_{m+1}, \text{a morphism } ([m], t) \to ([m'], t') \}$ is a map $h: [m] \to [m']$ in $\Delta$ such that $t'ht^{-1}$ is in $\Delta$. It is then clear that the functor

$$\Gamma \to \Delta, \quad ([n], t) \mapsto [n], \quad h \mapsto t'ht^{-1},$$

is a cartesian fibration (whose fibers are sets). Moreover, the functor $\Delta \times_\Lambda \Lambda /[n] \to \Delta \times_\Lambda \Lambda /[0]$ can be identified with the projection $\Gamma \times_\Delta \Delta /[n] \to \Gamma$. Since $\Delta$ is cosifted, the forgetful functor $\Delta /[n] \to \Delta$ is coinitial. The pullback of a coinitial functor along a cartesian fibration is still coinitial [Lur17, Proposition 4.1.2.15], so $\Gamma \times_\Delta \Delta /[n] \to \Gamma$ is coinitial and in particular a weak equivalence, as desired.

Thus, we have a commuting square

$$\begin{array}{ccc}
\text{PSh}_\simeq(\Delta, \mathcal{E}) & \xleftarrow{\text{j}_*} & \text{PSh}(\Delta, \mathcal{E}) \\
\downarrow \text{j}_* & & \downarrow \text{j}_*
\text{PSh}_\simeq(\Lambda, \mathcal{E}) & \xleftarrow{\text{j}_*} & \text{PSh}(\Lambda, \mathcal{E}).
\end{array}$$

Since $\Delta^\op$ is weakly contractible, evaluation at $[0]$ is an equivalence $\text{PSh}_\simeq(\Delta, \mathcal{E}) \simeq \mathcal{E}$. The left adjoint square, followed by evaluation at $[0]$, says that $u_T [-] \simeq \text{colim} j^*[\cdot]$, as desired. \(\square\)

It follows from Proposition 1.1 that $u_T [\mathcal{E}] \simeq \text{HH}(\mathcal{E})$, so that $\text{HH}(\mathcal{E})$ acquires a canonical action of the $\infty$-group $T$. As another corollary, we recover the following computation of Connes [Con83, Théorème 10]:

**Corollary 1.2.** $\tilde{\Lambda} \simeq K(\mathbb{Z}, 2)$.

**Proof.** If $X \in \text{PSh}_\simeq(\Lambda)$, then, by Yoneda, $\text{Map}(\Lambda^0, X) \simeq \text{Map}(\tilde{\Lambda}^0, X)$. In other words, the canonical map $\Lambda^0 \to \tilde{\Lambda}^0$ induces an equivalence $|\Lambda^0| \simeq |\tilde{\Lambda}^0|$, and hence $u_T |\Lambda^0| \simeq T$. On the other hand, the underlying simplicial set of $\Lambda^0$ is $\Delta^1/\partial\Delta^1$, so $u_T |\Lambda^0| \simeq K(\mathbb{Z}, 1)$ by Proposition 1.1. Thus, $T$ is a $K(\mathbb{Z}, 1)$, and hence $B_T \simeq \tilde{\Lambda}$ is a $K(\mathbb{Z}, 2)$.
In particular, $\mathcal{T}$ is equivalent to the circle as an $\infty$-group, which justifies the notation. If $\mathcal{E}$ is stable, Atiyah duality for the circle provides the norm map $\nu_T: \Sigma^3(-)^{h\mathcal{T}} \to (-)^{h\mathcal{T}}$, where $t$ is the Lie algebra of $\mathcal{T}$ and $\Sigma^3$ is suspension by its one-point compactification. Explicitly, if $E \in \mathcal{E}$ has an action of $\mathcal{T}$, the norm is induced by the $(\mathcal{T} \times \mathcal{T})$-equivariant composition

$$\Sigma^1 E \to \Sigma^1 \text{Hom}(\Sigma^3 \mathcal{T}, E) \simeq \Sigma^1(\Sigma^3 \mathcal{T})^\vee \otimes E \simeq \Sigma^3 \mathcal{T} \otimes E \to E,$$

where the first map is the diagonal, the third is Atiyah duality, and the last is the action. The cofiber of $\nu_T$ is the Tate fixed point functor $(-)^{T\mathcal{T}}$.

**Definition 1.3.** Let $\mathcal{E}$ be a presentably symmetric monoidal $\infty$-category and let $\mathcal{C} \in \text{Cat}(\mathcal{E})$.

1. The **cyclic homology** of $\mathcal{C}$ is

$$\text{HC}(\mathcal{C}) = [\mathcal{C}]_{T\mathcal{T}} \in \mathcal{E}.$$

2. The **negative cyclic homology** of $\mathcal{C}$ is

$$\text{HN}(\mathcal{C}) = [\mathcal{C}]^{h\mathcal{T}} \in \mathcal{E}.$$

3. If $\mathcal{E}$ is stable, the **periodic cyclic homology** of $\mathcal{C}$ is

$$\text{HP}(\mathcal{C}) = ([\mathcal{C}]^{T\mathcal{T}}) \in \mathcal{E}.$$

Note that $\text{HC}(\mathcal{C})$ is simply the colimit of $\mathcal{C}^n: \Lambda_\mathcal{C}^\text{op} \to \mathcal{E}$. Note also that $\text{HC}(\mathcal{C}), \text{HN}(\mathcal{C}),$ and $\text{HP}(\mathcal{C})$ depend only on the $\mathcal{E}$-enriched $\infty$-category presented by $\mathcal{C}$, since this is the case for $\text{HH}(\mathcal{C})$.

**Remark 1.4.** There are several interesting refinements of the above definitions. By definition, the invariants $\text{HC}(\mathcal{C}), \text{HN}(\mathcal{C}),$ and $\text{HP}(\mathcal{C})$ depend only on the circle action on $\text{HH}(\mathcal{C})$. The **topological cyclic homology** of $\mathcal{C}$ is a refinement of negative cyclic homology, defined when $\mathcal{E}$ is the $\infty$-category of spectra, which uses some finer structure on $\text{HH}(\mathcal{C})$. In another direction, additional structure on $\mathcal{C}$ can lead to $\text{HH}(\mathcal{C})$ being acted on by more complicated $\infty$-groups. For example, if $\mathcal{C}$ has a duality $\dagger$, then $\mathcal{C}$ extends to the dihedral category whose classifying space is $B\text{O}(2)$. The coinvariants $[[\mathcal{C}]^n]_{T\mathcal{O}(2)}$ are called the **dihedral homology** of $(\mathcal{E}, \dagger)$.

The previous definitions apply in particular when $\mathcal{C}$ has a unique object, in which case we may identify it with an $A_\infty$-algebra in $\mathcal{E}$. If $A$ is an $E_\infty$-algebra in $\mathcal{E}$, there is a more direct description of $[A^\natural]$. In this case, $A^\natural$ is the underlying cyclic object of the cyclic $E_\infty$-algebra $A^0 \otimes A \in \text{PSh}(\Lambda, \text{CAlg}(\mathcal{E}))$, where $\Lambda^0$ is the cyclic set represented by $[0] \in \Lambda$ and $\otimes$ is the canonical action of the $\infty$-category $S$ of spaces on the presentable $\infty$-category $\text{CAlg}(\mathcal{E})$. For any cyclic space $K \in \text{PSh}(\Lambda)$, we clearly have $[K \otimes A] \simeq [K] \otimes A$. It follows that $[A^\natural] \in \text{PSh}(BT, \mathcal{E})$ is the underlying object of the $E_\infty$-algebra

$$[\Lambda^0] \otimes A \simeq T \otimes A \in \text{PSh}(BT, \text{CAlg}(\mathcal{E})).$$

In particular, $\text{HH}(A), \text{HN}(A),$ and $\text{HP}(A)$ inherit $E_\infty$-algebra structures from $A$. Their geometric interpretation is the following: if $X = \text{Spec} A$, then $\text{Spec} \text{HH}(A)$ is the **free loop space** of $X$ and $\text{Spec} \text{HN}(A)$ is the space of circles in $X$.

### 2. Comparison with the classical definitions

Let $k$ be a discrete commutative ring and let $A$ be an $A_\infty$-algebra over $k$. The cyclic and negative cyclic homology of $A$ over $k$ are classically defined via explicit bicomplexes. Let us start by recalling these definitions, following [Lod92, §5.1].

Let $M_\bullet$ be a cyclic object in an additive category $A$. The usual presentation of $\Lambda$ provides the face and degeneracy operators $d_i: M_n \to M_{n-1}$ and $s_i: M_n \to M_{n+1}$ ($0 \leq i \leq n$), as well as the cyclic operator $c: M_n \to M_n$ of order $n + 1$. We define the additional operators

$$b: M_n \to M_{n-1}, \quad b = \sum_{i=0}^{n} (-1)^i d_i,$$

$$s_{-1}: M_n \to M_{n+1}, \quad s_{-1} = cs_n,$$

$$t: M_n \to M_n, \quad t = (-1)^n c,$$

$$N: M_n \to M_n, \quad N = \sum_{i=0}^{n} t^i,$$

$$B: M_n \to M_{n+1}, \quad B = (\text{id} - t)s_{-1}N.$$
We easily verify that \( b^2 = 0, B^2 = 0, \) and \( bB + Bb = 0. \) In particular, \((M, b)\) is a chain complex in \( \mathcal{A}. \) We now take \( \mathcal{A} \) to be the category \( \text{Ch}_k \) of chain complexes of \( k \)-modules. Then \((M, b)\) is a (commuting) bicomplex and we denote by \((C_*(M), b)\) the total chain complex with
\[
C_n(M) = \bigoplus_{p+q=n} M_{p,q}, \quad b = b + (-1)^*d.
\]
We then form the (anticommuting) periodic cyclic bicomplex \( \text{BP}(M): \)
\[
\cdots \leftarrow C_2(M) \leftarrow C_1(M) \leftarrow C_0(M) \leftarrow \cdots \bigg\downarrow \bigg\downarrow \bigg\downarrow \\
\downarrow b \quad \downarrow b \quad \downarrow b \quad \downarrow b \quad \cdots \\
\cdots \leftarrow C_1(M) \leftarrow C_0(M) \leftarrow C_{-1}(M) \leftarrow \cdots \bigg\downarrow \bigg\downarrow \bigg\downarrow \\
\downarrow b \quad \downarrow b \quad \downarrow b \quad \downarrow b \quad \cdots \\
\cdots \leftarrow C_0(M) \leftarrow C_{-1}(M) \leftarrow C_{-2}(M) \leftarrow \cdots \\
\downarrow \downarrow \downarrow \downarrow \downarrow \cdots \\
\cdots \cdots \cdots \cdots \cdots \\
\text{with } \text{BP}(M)_{p,q} = C_{q-p}(M). \]
Removing all the negatively graded columns, we obtain the cyclic bicomplex \( \text{BC}(M); \) removing all the positively graded columns, we obtain the negative cyclic bicomplex \( \text{BN}(M). \) Finally, we form the total complexes
\[
\text{Tot BC}, \text{Tot BN}, \text{Tot BP}: \text{PSh}(\Lambda, \text{Ch}_k) \to \text{Ch}_k,
\]
where
\[
\text{Tot}(B)_n = \colim_{r \to \infty} \prod_{p \leq r} B_{p,n-p}.
\]
These functors clearly preserve quasi-isomorphisms and hence induce functors
\[
\text{CC}, \text{CN}, \text{CP}: \text{PSh}(\Lambda, \text{Mod}_k) \to \text{Mod}_k,
\]
where \( \text{Mod}_k \) is the stable \( \infty \)-category of \( k \)-modules. There is a cofiber sequence
\[
\text{CC}[1] \xrightarrow{B} \text{CN} \to \text{CP},
\]
where the map “\( B \)” is induced by the degree \((0,1)\) map of bicomplexes \( \text{BC}(M) \to \text{BN}(M) \) whose nonzero components are \( B: C_{i-1}(M) \to C_i(M). \)

**Theorem 2.1.** Let \( k \) be a discrete commutative ring and \( M \in \text{PSh}(\Lambda, \text{Mod}_k) \) a cyclic \( k \)-module. Then there are natural equivalences
\[
|M|_{hT} \simeq \text{CC}(M), \quad |M|^{hT} \simeq \text{CN}(M), \quad \text{and } |M|^T \simeq \text{CP}(M).
\]
In particular, if \( \mathcal{E} \) is a \( k \)-linear \( \infty \)-category, then
\[
\text{HC}(\mathcal{E}) \simeq \text{CC}(\mathcal{E}), \quad \text{HN}(\mathcal{E}) \simeq \text{CN}(\mathcal{E}), \quad \text{and } \text{HP}(\mathcal{E}) \simeq \text{CP}(\mathcal{E}).
\]

We first rephrase the classical definitions in terms of *mixed complexes*, following Kassel [Kas87]. We let \( k[\epsilon] \) be the differential graded \( k \)-algebra
\[
\cdots \to 0 \to k\epsilon \xrightarrow{\epsilon} k \to 0 \to \cdots,
\]
which is nonzero in degrees 1 and 0. The \( \infty \)-category \( \text{Mod}_{k[\epsilon]} \) is the localization of the category of differential graded \( k[\epsilon] \)-modules, also called mixed complexes, at the quasi-isomorphisms. The functors
\[
k \otimes_{k[\epsilon]} (-), \quad \text{Hom}_{k[\epsilon]}(k, -): \text{Mod}_{k[\epsilon]} \to \text{Mod}_k
\]
are related by a *norm map*
\[
\nu_\epsilon: k[1] \otimes_{k[\epsilon]} (-) \to \text{Hom}_{k[\epsilon]}(k, -),
\]
induced by the \( k[\epsilon] \)-linear map \( \epsilon: k[1] \to k[\epsilon]. \)
We denote by
\[ K : \text{PSh}(\Lambda, \text{Mod}_k) \to \text{Mod}_{k[\epsilon]} \]
the functor induced by sending a cyclic chain complex \( M \) to the mixed complex \( (C_*(M), b, B) \).

**Lemma 2.2.** Let \( M \in \text{PSh}(\Lambda, \text{Mod}_k) \). Then
\[
\text{CC}(M) \cong k \otimes_{k[\epsilon]} K(M) \quad \text{and} \quad \text{CN}(M) \cong \text{Hom}_{k[\epsilon]}(k, K(M)),
\]
and the norm map \( \nu : k[1] \otimes_{k[\epsilon]} K(M) \to \text{Hom}_{k[\epsilon]}(k, K(M)) \) is identified with \( B : \text{CC}(M)[1] \to \text{CN}(M) \).

**Proof.** We work at the level of complexes. Let \( Qk \) be the nonnegatively graded mixed complex
\[
\cdots \xrightarrow{\epsilon} ke \xrightarrow{0} k \xrightarrow{\epsilon} ke \xrightarrow{0} k.
\]
There is an obvious morphism \( Qk \to k \) which is a cofibrant resolution of \( k \) for the projective model structure on mixed complexes. By inspection, we have isomorphisms of chain complexes
\[
\text{Tot BC}(M) \cong Qk \otimes_{k[\epsilon]} (C_*(M), b, B) \quad \text{and} \quad \text{Tot BN}(M) \cong \text{Hom}_{k[\epsilon]}(Qk, (C_*(M), b, B)).
\]
This proves the first claim. For \( C \) a mixed complex, the norm map \( \nu \) is modeled by the composition
\[
Qk[1] \otimes_{k[\epsilon]} C \to C/\text{im}(\epsilon)[1] \xrightarrow{\epsilon} \ker(\epsilon)(C) \leftrightarrow \text{Hom}_{k[\epsilon]}(Qk, C).
\]
The last claim is then obvious. \( \square \)

Let \( k[\mathbb{T}] \) be the \( A_\infty \)-ring \( k \otimes \Sigma_+^\infty \mathbb{T} \). There is an obvious equivalence of \( \infty \)-categories
\[
\text{PSh}(B\mathbb{T}, \text{Mod}_k) \cong \text{Mod}_{k[\mathbb{T}]}\]
that makes the following squares commute:
\[
\begin{array}{ccc}
\text{PSh}(B\mathbb{T}, \text{Mod}_k) & \xrightarrow{(-)_{k\mathbb{T}}} & \text{Mod}_k \\
\cong & \downarrow & \cong \\
\text{Mod}_{k[\mathbb{T}]} & \xrightarrow{k \otimes_{k[\mathbb{T}]} -} & \text{Mod}_k.
\end{array}
\]

Since \( \mathbb{T} \) is equivalent to the circle, \( H_1(\mathbb{T}, \mathbb{Z}) \) is an infinite cyclic group. Let \( \gamma \in H_1(\mathbb{T}, \mathbb{Z}) \) be a generator. Sending \( \epsilon \) to \( \gamma \) defines an equivalence of augmented \( A_\infty \)-\( k \)-algebras \( \gamma : k[\epsilon] \cong k[\mathbb{T}] \), whence an equivalence of \( \infty \)-categories
\[
\gamma^* : \text{Mod}_{k[\mathbb{T}]} \cong \text{Mod}_{k[\epsilon]}.
\]
Moreover, \( \gamma^* \) identifies the norm maps \( \nu : \Sigma.png \xrightarrow{} (-)^{h\mathbb{T}} \) and \( \nu : k[1] \otimes_{k[\epsilon]} (-) \to \text{Hom}_{k[\epsilon]}(k, -) \), provided that \( \Sigma \) is identified with \( \Sigma \) using the orientation of \( \mathbb{T} \) given by \( \gamma \) (since then the Poincaré duality isomorphism \( k \cong H_1(\mathbb{T}, k) \) sends 1 to \( \gamma \)).

The main result of this note is that \( K(M) \) is a model for \( |M| \). More precisely:

**Theorem 2.3.** There exists a generator \( \gamma \in H_1(\mathbb{T}, \mathbb{Z}) \) such that the following triangle commutes:
\[
\begin{array}{ccc}
\text{PSh}(\Lambda, \text{Mod}_k) & \xrightarrow{|-|} & \text{PSh}(B\mathbb{T}, \text{Mod}_k) \\
\downarrow & & \downarrow \gamma^* \\
\text{Mod}_{k[\epsilon]} & \cong & \text{Mod}_{k[\epsilon]}.
\end{array}
\]

Theorem 2.1 follows from Theorem 2.3 and Lemma 2.2. As an immediate corollary, we recover the following result of Dwyer and Kan [DK85, Remark 6.7]:

**Corollary 2.4.** The functor \( K : \text{PSh}(\Lambda, \text{Mod}_k) \to \text{Mod}_{k[\epsilon]} \) induces an equivalence of \( \infty \)-categories
\[
\text{PSh}_{\infty}(\Lambda, \text{Mod}_k) \cong \text{Mod}_{k[\epsilon]}.
\]
To prove Theorem 2.3, we consider the “universal case”, namely the cocyclic cyclic $k$-module $k[\Lambda^\bullet]$. We have a natural equivalence

$$M \simeq k[\Lambda^\bullet] \otimes_A M,$$

where

$$\otimes_A : \text{Fun}(\Lambda, \text{Mod}_k) \times \text{PSh}(\Lambda, \text{Mod}_k) \to \text{Mod}_k$$

is the coend pairing. Similarly, we have

$$|M| \simeq |k[\Lambda^\bullet]| \otimes_A M \quad \text{and} \quad K(M) \simeq K(k[\Lambda^\bullet]) \otimes_A M,$$

since both $|-|$ and $K$ commute with tensoring with constant $k$-modules and with colimits (for $K$, note that colimits in $\text{Mod}_{k[\varepsilon]}$ are detected by the forgetful functor to $\text{Mod}_k$). Thus, it will suffice to produce an equivalence of cocyclic $k[\varepsilon]$-modules

$$(2.5) \quad \gamma^*|k[\Lambda^\bullet]| \simeq K(k[\Lambda^\bullet]).$$

Let $k[u]$ denote the $A_{\infty}$-$k$-coalgebra $k \otimes_{k[\varepsilon]} k$. Note that a $k[u]$-$k$-comodule structure on $M \in \text{Mod}_k$ is the same thing as map $M \to M[2]$. The functor $k \otimes_{k[\varepsilon]} - : \text{Mod}_{k[\varepsilon]} \to \text{Mod}_k$ factors through a fully faithful functor from $k[\varepsilon]$-modules to $k[u]$-comodules:

$$\text{Comod}_{k[u]} \xrightarrow{\text{forget}} \text{Mod}_{k[\varepsilon]} \xrightarrow{k \otimes_{k[\varepsilon]} -} \text{Mod}_k.$$ 

To prove (2.5), it will therefore suffice to produce an equivalence of cocyclic $k[u]$-comodules

$$(2.6) \quad k \otimes_{k[\varepsilon]} \gamma^*|k[\Lambda^\bullet]| \simeq k \otimes_{k[\varepsilon]} K(k[\Lambda^\bullet]).$$

Note that both cocyclic objects send all morphisms in $\Lambda$ to equivalences and hence can be viewed as functors $B_T \to \text{Comod}_{k[u]}$.

Let us first compute the left-hand side of (2.6). The generator $\gamma$ induces an equivalence of coaugmented $A_{\infty}$-$k$-coalgebras $\hat{\gamma} : k[u] \simeq k[B_T]$, whence an equivalence of $\infty$-categories

$$(2.7) \quad \hat{\gamma}^* : \text{Comod}_{k[B_T]} \simeq \text{Comod}_{k[u]}.$$ 

We clearly have

$$k \otimes_{k[\varepsilon]} \gamma^*|k[\Lambda^\bullet]| \simeq \hat{\gamma}^*|k[\Lambda^\bullet]|_{B_T}.$$ 

Now, $|k[\Lambda^\bullet]|_{B_T} \simeq |\Lambda^\bullet|_{B_T}$, where $|\Lambda^\bullet|_{B_T}$ is a $B_T$-comodule in $\text{Fun}_{\infty}(\Lambda, S) \simeq S/B_T$. If $\pi^* : S \to S/B_T$ is the functor $\pi^*X = X \times B_T$, then a $B_T$-comodule structure on $\pi^*X$ is simply a map $\pi^*X \to \pi^*B_T$, i.e., a map $X \times B_T \to B_T$ in $S$. Here, $|\Lambda^\bullet|_{B_T}$ is $\pi^*(*) \in S/B_T$ and its $B_T$-comodule structure $\sigma : \pi^*(*) \to \pi^*(B_T)$ is given by the identity $B_T \to B_T$. Applying $\hat{\gamma}^* k[-]$, we deduce that the left-hand side of (2.6) is the constant cocyclic $k$-module $\check{k}$ with $k[u]$-comodule structure given by the composition

$$(2.7) \quad k \overset{\sigma}{\to} k[B_T] \overset{\check{\gamma}}{\to} \check{k}[u].$$

Note that equivalence classes of $k[u]$-comodule structures on $\check{k}$ are in bijection with

$$[k, k[2]] \simeq H^2(B_T, k).$$

Under this classification, (2.7) comes from an integral cohomology class, namely the image of the identity $B_T \to B_T$ under the isomorphism

$$[B_T, B_T] \overset{\check{\gamma}}{\simeq} H^2(B_T, \mathbb{Z}).$$

In particular, it comes from a generator of the infinite cyclic group $H^2(B_T, \mathbb{Z})$, determined by $\gamma$. We must therefore show that the right-hand side of (2.6) is also equivalent to the constant cocyclic $k$-module $\check{k}$ with $k[u]$-comodule structure classified by a generator of $H^2(B_T, \mathbb{Z})$.

Recall that $K(k[\Lambda^\bullet])$ is the following mixed complex of cocyclic $k$-modules:

$$\cdots \overset{\delta}{\to} k[\Lambda_2] \overset{\delta}{\to} k[\Lambda_1] \overset{\delta}{\to} k[\Lambda_0].$$

Consider the mixed complex $Qk$ from the proof of Lemma 2.2, which can be used to compute $k \otimes_{k[\varepsilon]} -$ at the level of complexes. It comes with an obvious self-map $Qk \to Qk[2]$ which induces the $k[u]$-comodule
structure on $k \otimes k[e] M$ for every mixed complex $M$. Let us write down explicitly the resulting chain complex $Qk \otimes k[e] K(k[\Lambda^*])$ of cocyclic $k[u]$-comodules. It is the total complex of the first-quadrant bicomplex

\[
\begin{array}{ccc}
& & b & & b & & b & \\
k[A_2] & & k[A_1] & & k[A_0] & & & \\
& b & & b & & b & & & \\
k[A_1] & & k[A_0] & & & & & \\
& b & & b & & b & & & \\
k[A_0] & & & & & & & \\
\end{array}
\]

(2.8)

with $k[u]$-comodule structure induced by the obvious degree $(-1, -1)$ endomorphism $\delta$.

**Proposition 2.9.** The bicomplex (2.8) is a resolution of the constant cocyclic $k$-module $k$. Moreover, the endomorphism $\delta$ represents a generator of the invertible $k$-module $[k, k[2]] \simeq H^2(B\mathbb{T}, k)$.

**Proof.** Let $K_{**}$ be the bicomplex (2.8), with the obvious augmentation $K_{**} \rightarrow k$. For $M$ a cyclic object in an additive category, we define the operator $b': M_n \rightarrow M_{n-1}$ by

\[b' = b - (-1)^n d_n = \sum_{i=0}^{n-1} (-1)^i d_i.\]

Let $L_{**}$ be the $(2, 0)$-periodic first-quadrant bicomplex

\[
\begin{array}{ccc}
& & id - t & & id - t & & id - t & \\
k[A_2] & & k[A_1] & & k[A_0] & & & \\
& b & & b & & b & & & \\
k[A_1] & & k[A_0] & & & & & \\
& b & & b & & b & & & \\
k[A_0] & & & & & & & \\
\end{array}
\]

with the obvious augmentation $L_{**} \rightarrow k$, and let $M_{**}$ be the bicomplex obtained from $L_{**}$ by annihilating the even-numbered columns. Let $\phi$: Tot $K_{**} \rightarrow$ Tot $L_{**}$ be the map induced by $(id, s_{-1}N)$: $k[A_n] \rightarrow k[A_n] \oplus k[A_{n+1}]$, and let $\psi$: Tot $L_{**} \rightarrow$ Tot $M_{**}$ be the map induced by $-s_{-1}N + id$: $k[A_n] \oplus k[A_{n+1}] \rightarrow k[A_{n+1}]$. A straightforward computation shows that $\phi$ and $\psi$ are chain maps and that we have a commutative diagram with exact rows

\[
\begin{array}{ccc}
0 & \rightarrow & \text{Tot } K_{**} & \phi & \rightarrow & \text{Tot } L_{**} & \psi & \rightarrow & \text{Tot } M_{**} & \rightarrow & 0 \\
0 & \rightarrow & k & \rightarrow & k & \rightarrow & 0 & \rightarrow & 0.
\end{array}
\]

(2.10)

From the identity $s_{-1} b' + b' s_{-1} = id$, we deduce that each column of $M_{**}$ has zero homology, and hence that $\text{Tot } M_{**} \simeq 0$. Next we show that each row of $L_{**}$ has zero positive homology, so that the homology of Tot $L_{**}$ can be computed as the homology of the zeroth column of horizontal homology of $L_{**}$. This can be proved pointwise, so consider a part of the $n$th row evaluated at $[m]$:

\[
\begin{array}{ccc}
& & k[\Lambda(n, m)]^N & & k[\Lambda(n, m)]^N & & k[\Lambda(n, m)]^N & \\
& & id - t & & id - t & & id - t & \\
n & & k[\Lambda(n, m)] & & k[\Lambda(n, m)] & & k[\Lambda(n, m)] & \\
& n & & n & & n & & & \\
\end{array}
\]

(2.11)

By the structure theorem for $\Lambda$, we have $\Lambda(n, m) = C_{n+1} \times \Delta(n, m)$, where $C_{n+1}$ is the set of automorphisms of $[n]$ in $\Lambda$. Thus, (2.11) is obtained from the complex

\[
\begin{array}{ccc}
& & k[C_{n+1}]^N & & k[C_{n+1}]^N & & k[C_{n+1}]^N & \\
& & id - t & & id - t & & id - t & \\
n & & k[C_{n+1}] & & k[C_{n+1}] & & k[C_{n+1}] & \\
& n & & n & & n & & & \\
\end{array}
\]

(2.12)
by tensoring with the free $k$-module $k[\Delta(n,m)]$, and we need only prove that (2.12) is exact. Let

$$x = \sum_{i=0}^{n} x_i c^i \in k[C_{n+1}].$$

Suppose first that $x(id-t) = 0$; then $x_i = (-1)^n x_0$ and hence $x = x_0 N$. Suppose next that $xN = 0$, i.e., that $\sum_{i=0}^{n} (-1)^n x_{n-i} = 0$; putting $y_0 = x_0$ and $y_i = x_i + (-1)^n y_{i-1}$ for $i > 0$, we find $x = y(id-t)$. This proves the exactness of (2.12), and also that the image of $id - t$ is $k[C_{n+1}]$.

To prove the second statement, we contemplate the complex $\text{Hom}(\text{Tot} K^\ast, k)$: it is the total complex of the bicomplex

$$
\begin{array}{ccc}
& k & \\
\downarrow & & \downarrow \\
k & \leftarrow k & \\
\downarrow & id & \downarrow \\
0 & \leftarrow k & 0 \\
\downarrow & id & \downarrow \\
\vdots & \vdots & \vdots \\
\end{array}
$$

with trivial horizontal differentials and alternating vertical differentials. We immediately check that

$$\text{Tot} K^\ast \delta \rightarrow (\text{Tot} K^\ast)[2] \rightarrow \overline{k}[2]$$

is a cocycle generating the second cohomology module. \hfill \square

It follows from Proposition 2.9 that the right-hand side of (2.6) is the constant cocyclic $k$-module $\overline{k}$ with $k[u]$-comodule structure classified by $\delta: \overline{k} \rightarrow \overline{k}[2]$. Comparing with (2.7) and noting that $\delta$ is natural in $k$, we deduce that Theorem 2.3 holds by choosing $\gamma \in H^1(T, \mathbb{Z})$ to be the generator corresponding to $\delta \in H^2(B^T, \mathbb{Z})$.

References