Abstract

In this paper, we provide an alternative explanation for why auctioneers often keep the reserve price hidden or secret. We consider a standard independent private values environment in which the buyers are risk-averse and the seller has private information about her valuation of the object to be auctioned. The seller uses a first-price sealed-bid auction mechanism combined with either an announced reserve price or a hidden reserve price. We compare the seller’s ex ante expected profits under these two policies and find that the optimal hidden reserve price policy generates higher expected profits for the seller when the buyers are fairly risk-averse under particular restrictions on buyers’ preferences and the distributions of private values. As the number of the buyers increases, the hidden reserve price is more likely to dominate. Numerical methods are used to demonstrate the generality of our main results. Journal of Economic Literature Classification Number: D44. Key Words: First-price auctions, Hidden reserve price, Risk aversion.
1 Introduction

In this paper, we study a seller’s optimal reserve price policy in an independent private-value auction environment with risk-averse buyers. Matthews (1983) and Maskin and Riley (1984) have analyzed the seller’s optimal mechanism in such an environment. They find that this mechanism takes a complicated form requiring payments from losing bidders who bid low and subsidies of other losing bidders. We focus on two kinds of reserve price policies when the seller uses a first-price sealed-bid auction, namely, announced reserve price and hidden reserve price. Under an announced reserve price policy, the seller publicly commits to a reserve price before the buyers submit their bids. Under a hidden reserve price policy, the seller puts a reserve price in a sealed envelope and opens it to the buyers only after the bids are submitted. The latter policy can be interpreted as a commitment by the seller to use a random reserve price and has been observed in practice. In our model, we also assume that the seller has private information about her valuation of the item to be auctioned. The seller may not want to reveal her information to potential bidders. Our objective is to compare the seller’s ex ante expected profits under the two policies.

The main result in the paper is that when the buyers are sufficiently risk-averse, the optimal hidden reserve price policy can generate higher ex ante expected profits for the seller than the optimal announced reserve price policy. When there is only one buyer, an announced reserve price policy is simply a take-it-or-leave-it offer mechanism offered by the seller. The buyer then takes the offer whenever his valuation of the item exceeds the seller’s offer. Taking this into account, the seller chooses a reserve price to maximize her expected profit. Thus, the seller’s expected profit under the announced reserve price policy is independent of the buyer’s risk attitude. Under a hidden reserve price policy, it is a dominant strategy for the seller to set a reserve price equal to her true reserve price. This suggests that secrecy makes the power of commitment useless. It also implies that the seller does not change her mind after observing the buyer’s bid. However, without observing the seller’s reserve price a more risk averse buyer tends to bid more aggressively. This happens since the seller’s hidden reserve price serves as a competing bid and creates risk. We show that the buyer’s bidding function as well as the seller’s expected profit under the hidden reserve price policy strictly increase as the buyer’s degree of absolute risk aversion uniformly increases. When the degree of risk aversion is sufficiently large, the hidden
reserve price policy dominates the announced reserve price policy.

When there is more than one buyer, the symmetric equilibrium bidding function under each policy is determined by a non-linear ordinary differential equation. The equilibrium bidding functions and the seller’s expected profits under both policies increase with the buyers’ degree of absolute risk aversion and with the number of buyers. A general comparison of the expected profits under the two policies is difficult, however, since non-linear differential equations may not have analytical solutions. Fortunately, we are able to derive explicitly the equilibrium bidding functions when the buyers have constant relative risk aversion preferences and zero initial wealth and when the cumulative distributions of the private values of the buyers and the seller are power functions. Under these conditions, we show that expected profits for the seller under the hidden reserve price exceed those under the announced reserve price when the buyers’ degree of relative risk aversion is sufficiently large. As the number of buyers increases, the hidden reserve price is more likely to dominate. In addition, we find that for general distribution functions, the seller’s optimal (announced) reserve price decreases with the degree of buyers’ risk aversion and with the number of buyers, and it converges to the seller’s true reserve price in the limit. Numerical calculations are used to demonstrate the generality of our main result on the dominance of the hidden reserve price.

There have been a fair number of studies on reserve price policies in the auction literature. Ashenfelter (1989) first observes that in English auctions for wine and arts, the auctioneers usually do not reveal the reserve price and even make it difficult for bidders to infer it. Hendricks, Porter, and Spady (1989) find empirical evidence suggesting that a random reservation price is used in first-price sealed-bid auctions for off-shore oil and gas leases in the U.S., where the federal government often reveals its reservation price after bidding is completed. Elyakime, Laffont, Loisel, and Vuong (1994) analyze data on timber auctions in France where the seller’s reserve price is given to the organizer before the auction begins and is revealed to the buyers only after they have submitted their sealed bids. In a recent paper, Bajari and Hortacsu (2000) study a data set of eBay coin auctions and find that the sellers of items with high book values tend to use a secret reserve price while the sellers of items with low book values use a posted reserve price.

Why would the auctioneers or the sellers often keep the reserve price hidden or secret in practice? One explanation is that a secret reserve price policy might be used to deter collusive bidding behavior (see Ashenfelter, 1989). A
secret reserve price may also encourage the participation of potential bidders in second-price auctions with almost common-values (see Vincent, 1995). In certain cases the seller may have favorable information about the item to be auctioned and find it less costly to communicate that information to potential buyers using a secret reserve price when there is a resale option (see Horstmann and Lacasse, 1997).

In this paper, we provide an alternative, and complementary argument. That is, risk aversion on the part of bidders may favor the hidden reserve price. The intuition is as follows. With an announced reserve price the seller benefits from her pre-commitment. This is a standard strategy of price discrimination. On the other hand, a hidden reserve price by the seller serves as a competing bid since the winning buyer has to out-bid the seller as well as other buyers. As the buyers become more risk-averse, they bid more aggressively under both policies, but even more so when a hidden reserve price is used. We show that the risk aversion effect can dominate the pre-commitment effect. Thus, on average the seller may prefer to keep her reserve price hidden or secret when potential buyers are sufficiently risk-averse. In addition, when competition among the buyers increases, the winning bid is more likely to be high and hence the announced reserve price as a pre-commitment device becomes less effective. Therefore, as the number of the buyers increases, the risk aversion effect becomes dominant and the seller tends to prefer the hidden reserve price.

McAfee and McMillan (1987) and Matthews (1987) obtain similar results regarding the number of bidders. They show that when bidders’ preferences exhibit constant or decreasing absolute risk aversion, the ex ante expected profit for the seller in a first-price auction is strictly higher when the bidders do not know the number of other bidders than when they do. Concealing information about the number of bidders creates extra risk and encourages the risk-averse bidders to compete more aggressively. To the extent that an extra competing bid is more profitable than a simple pre-commitment mechanism, our result also echoes that of Bulow and Klemperer (1996). In a very general setting, they find that a simple ascending auction with \( n + 1 \) symmetric bidders but no reserve price yields higher revenue than any standard auction mechanism with \( n \) bidders and with a final optimal take-it-or-leave-it offer to the winning bidder.

The paper is organized as follows. Section 2 presents the environment. Section 3 analyzes the case of one buyer to illustrate the basic effect of risk aversion on the buyer’s bidding strategies under the two reserve price policies.
Section 4 determines the symmetric equilibrium bidding functions for the case of more than one buyer and compares the seller’s expected profits under the two reserve price policies. We also discuss the effects of buyers’ risk aversion, the buyers’ initial wealth and the number of the buyers on the equilibrium outcomes. We further provide some numerical results in Section 5 and conclude in Section 6. Proofs, except where they contribute to the exposition, are relegated to the Appendix.

2 The Model

We analyze a standard auction model with independent-private values. A risk-neutral seller has a single object to sell. She faces $n$ potential buyers and does not know how much the buyers are willing to pay for the object. Each buyer $i$, where $i = 1, ..., n$, has a maximum willingness to pay for the object, represented by $s_i$, which is his private information. We assume that $s_1, ..., s_n$ are independently drawn from the same cumulative distribution, $F(s)$, with a positive, and continuously differentiable, density function $f(s)$ on the support $[0, 1]$. The hazard rate $[1 - F(s)]/f(s)$ is assumed to be monotonically decreasing on the support.

There are two special features of our model. First, we assume that the seller also has private information about her value (or cost) of the object, which is represented by $t$ and is drawn from a cumulative distribution, $G(t)$, with a positive, and continuously differentiable density function $g(t)$ on the support $[0, 1]$. We assume that $G(t)/g(t)$ increases with $t$ on the support. Note that the support of the seller’s value does not have to be identical to that of the buyers’ values and that our restriction is purely for simplicity. Second, the buyers are assumed to be risk-averse with the same von Neumann-Morgenstern utility function $u(x)$, which is strictly increasing, concave, and continuously differentiable in $x \in [0, +\infty)$. All the buyers have the same initial wealth $w \geq 0$.

The seller uses a first-price sealed-bid auction combined with one of the two reserve price policies: a pre-announced reserve price or a hidden reserve price. We model the hidden reserve price policy as follows: the seller puts a reserve price in a sealed envelope and opens it to the buyers after the bids are submitted. This policy is consistent with the practice in timber auctions in France studied by Elyakime, Laffont, Loisel, and Vuong (1994). It may be interpreted as a commitment by the seller to use a random reserve price.
Under both policies, the winning bid has to be no less than the seller’s reserve price to have a transaction.

Our primary objective in this paper is to compare the seller’s ex ante expected profits under the two reserve price policies. One interpretation of this approach is that the seller selects a mechanism before learning her type. For this reason, we do not address the issue of how the seller should reveal her information when offering the optimal mechanism (see Myerson, 1983, and Maskin and Tirole, 1990). In this regard, our approach is similar to the one taken by McAfee and McMillan (1987), Matthews (1987), and Vincent (1995).

We consider a first-price sealed-bid auction since it generates higher expected revenue for the seller than other standard auction mechanisms in our model of independent private values and risk-averse bidders. In the case of English auction or second-price sealed-bid auction, no matter which reserve price policy the seller uses, bidders’ strategies are not affected by their risk aversion. Therefore, the seller has no incentive to choose a hidden reserve price policy in both English and second-price auctions.

We expect that the effect of bidders’ risk aversion on the seller’s choice of reserve price policies that we study in this paper continues to hold when the seller is also risk averse\(^1\) and when bidders’ values are common or affiliated.\(^2\)

3 The Case of One Buyer

To illustrate our analysis, we first study the performance of the two reserve price policies when there is only one buyer. In this case, the announced reserve price policy is simply the seller’s take-it-or-leave-it mechanism while the hidden reserve price policy corresponds to the buyer’s take-it-or-leave-it mechanism (or the buyer’s bid double auction). Both mechanisms are special cases of sealed-bid \(k\)-double auctions first studied by Chatterjee and Samuelson (1983). When the buyer is risk neutral, it is commonly known that the seller’s take-it-or-leave-it offer mechanism with the optimally chosen reserve price is the best mechanism for the seller (see, for instance, Riley and

\(^1\)When the seller is risk-averse, our analysis of reserve price policies can become more involved. See Waehrer, Harstad and Rothkopf (1998) for a comparision of standard auctions from the perspective of a risk-averse seller when bidders are risk-neutral.

\(^2\)In this case, the seller has extra incentives to use a hidden reserve price (see Vincent, 1995, and Horstmann and Lacasse, 1997).
Samuelson, 1983, and Yilankaya, 1999). When the buyer is sufficiently risk-averse, we will show that the buyer’s bid double auction can generate greater ex ante expected profits for the seller than the seller’s take-it-or-leave-it mechanism.

### 3.1 Announced Reserve Price

Suppose that, in the announced reserve price mechanism, the seller with a type $t$ offers a price $r$. The buyer takes the offer if and only if his valuation $s$ is no less than $r$. This occurs with probability $1 - F(r)$. The seller derives an expected net profit $(r - t)[1 - F(r)]$. The best offer for the seller is then $r^*_A(t) = I^{-1}(t)$, where

$$I(s) = s - \frac{1 - F(s)}{f(s)}$$

is strictly increasing in $s$. Note that $r^*_A(t) > t$ for any $t < 1$ since $I(s) < s$ for $s < 1$. This is a consequence of standard monopoly price discrimination. The seller’s ex-ante expected profit is

$$\Pi_A = \int_0^1 \int_{r^*_A(t)}^1 [r^*_A(t) - t]dF(s)dG(t).$$

Observe that $\Pi_A$ is independent of the buyer’s risk attitude.

### 3.2 Hidden Reserve Price

Under the hidden reserve price mechanism, the seller commits to a reserve price which will not be revealed to the buyer until after the buyer submits his bid. Thus, this policy can be modelled as a Bayesian game in which the buyer and the seller choose their bids simultaneously. We look for a Bayesian Nash equilibrium.

Given the buyer’s bidding strategy $b(s)$, which is assumed to be monotonically increasing, the seller with type $t$ chooses a

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3: This game is a special case of sealed-bid $k$-double auction studied by Chatterjee and Samuelson (1983), Leininger, Linhart, and Radner (1989), Satterthwaite and Williams (1989), among others. It has been noted in this literature that the game has multiple equilibria. We follow Satterthwaite and Williams (1989) and consider equilibria with strictly monotone bidding functions.
reserve price $r$ to maximize her expected profits

$$E[b(s) - t|b(s) \geq r] = \int_{b^{-1}(r)}^{1} [b(s) - t]dF(s).$$

The first-order condition yields $r^*_H(t) = t$. That is, putting her true value in a sealed envelope is the dominant strategy for the seller. This implies that the seller does not have incentives to change her choice of reserve price after observing the buyer’s bid. Thus, the hidden reserve price policy is the same as the buyer’s take-it-or-leave-it mechanism.

Given the seller’s dominant strategy $r^*_H(t) = t$, a buyer of type $s$ chooses $b$ to maximize his expected utility

$$u(s - b + w)G(b) + u(w)[1 - G(b)],$$

where $w$ is the buyer’s initial wealth. The first-order condition for the buyer’s optimization problem is given by

$$\frac{u(s - b + w) - u(w)}{u'(s - b + w)} = \frac{G(b)}{g(b)},$$

which determines the equilibrium bidding strategy $b^*_H(s)$. The seller’s ex-ante expected profit is then given by

$$\Pi_H = \int_{0}^{b^*_H(1)} \int_{b^{-1}_H(t)}^{1} [b^*_H(s) - t]dF(s)dG(t). \quad (2)$$

Note that both the buyer’s equilibrium bidding strategy and $\Pi_H$ depend on the buyer’s utility function.

To determine how the buyer’s degree of risk aversion affects $\Pi_H$, we define

$$\psi(x, w) = \frac{u(x + w) - u(w)}{u'(x + w)}.$$

Observe that $\psi(x, w)$ is increasing in $x$ and that $\psi(0, w) = 0$. The function $\psi$ can be interpreted as the buyer’s loss aversion. That is, it presents the amount that the buyer is willing to pay to avoid a small probability of losing the item (see Maskin and Riley, 1984, pp.1487, Li and Riley, 1999). Intuitively, the more risk-averse the buyer, the larger his loss aversion. To
be precise, where \( A(x) = -u''(x)/u'(x) \) is the buyer’s degree of absolute risk aversion, it is straightforward to show that

\[
\frac{d\psi}{dx} = 1 + \psi(x, w)A(x + w).
\]

Solving this differential equation, with the initial condition \( \psi(0, w) = 0 \), yields

\[
\psi(x, w) = \int_0^x \exp \left( \int_z^x A(y + w)dy \right) dz. \tag{3}
\]

This implies that the absolute risk aversion function completely determines the loss aversion function. Moreover, a uniform increase in \( A(x) \) increases \( \psi(x, w) \). In particular, \( \psi(x, w) \) decreases (resp: increases) with \( w \) if the buyer’s preference displays decreasing (resp: increasing) absolute risk aversion. In the next section, we will show that the loss aversion function also plays an important role in determining the equilibrium bidding function in first-price auctions with risk-averse bidders.

From the first-order condition, the inverse of the buyer’s equilibrium bidding function \( b_H^*(s) \) can be rewritten as

\[
s = b + \psi^{-1}(\frac{G(b)}{g(b)}, w), \tag{4}
\]

where the inverse of \( \psi(x, w) \) is taken with respect to \( x \). Since \( \psi(x, w) \) increases with \( x \), \( \psi(0, w) = 0 \), and we assume that \( G(b)/g(b) \) increases with \( b \), it follows that \( b_H^*(s) \) is monotonically increasing and \( b_H^*(0) = 0 \).

We can compare the bidding functions of two buyers when their absolute risk aversion measures can be compared uniformly in the following sense. Let \( A_j(x) \) be buyer \( j \)’s risk aversion measure for \( j = 1, 2 \). We say that buyer 1 is more risk averse than buyer 2 if \( A_1(x) \geq A_2(x) \) for any \( x > 0 \). From (3) and (4), if \( A_1(x) \geq A_2(x) \) for any \( x > 0 \), then \( b_{H1}^*(s) \geq b_{H2}^*(s) \) for any \( s > 0 \). This implies that a more risk-averse buyer bids more aggressively.\(^4\) The following lemma describes the buyer’s bidding behavior when he is sufficiently risk-averse.

\(^4\)Chatterjee and Samuelson (1983, pp.848) also observe that in sealed-bid \( k \)-double auctions, risk-averse buyers and sellers tend to make offers closer to their true values than their risk-neutral counterparts when they have utility functions displaying constant relative risk aversion with zero initial wealth.
Lemma 1 For any $\varepsilon > 0$, there exists $\alpha > 0$ such that $A(x) > \alpha$ for any $x > 0$ implies $s - b_H^*(s) < \varepsilon$ for any $s$.

Lemma 1 implies that if the buyer is sufficiently risk-averse, his bid is arbitrarily close to his valuation. Consequently, the seller is able to extract almost all the surplus from the buyer. This reasoning suggests that the seller will prefer the hidden reserve price when the buyer is sufficiently risk-averse. In the next subsection, we will formally present our analysis on how the buyer’s risk-aversion affects the seller’s expected revenue.

3.3 Comparing Expected Profits

To compare the seller’s expected profits under the two reserve price policies, we first consider a class of the buyer’s preferences that can be parameterized according to a uniform shift of the degree of absolute risk aversion. We make the following restriction:

(R1): $A(x) \equiv \phi(x, \theta)$, where $\theta \in [0, \infty)$, $\phi(x, 0) = 0$, $\phi(x, \infty) = \infty$, and $\partial \phi(x, \theta)/\partial \theta > 0$ for $x > 0$ and $\theta > 0$.

For the case of constant absolute risk aversion preferences, $\phi(x, \theta) = \theta$. For the case of constant relative risk aversion preferences, $\phi(x, \theta) = \theta/x$.

Proposition 2 Suppose (R1) holds. There exists a unique $\theta^* > 0$, such that $\Pi_H(\theta^*) = \Pi_A(\theta^*)$ and $\Pi_H(\theta) > \Pi_A(\theta)$ if and only if $\theta > \theta^*$. Moreover, if $dA(x)/dx \leq 0$ (resp: $\geq 0$) then $d\theta^*/dw \geq 0$ (resp: $\leq 0$).

Proposition 2 shows that in a class of preferences defined by (R1), when the buyer’s absolute risk aversion is sufficiently large, the seller prefers the hidden reserve price policy. It is interesting to note that if the buyer’s preferences exhibit constant absolute risk aversion, then the critical value, $\theta^*$, is independent of the buyer’s initial wealth. If the buyer’s preferences exhibit decreasing absolute risk aversion, however, the critical value increases with the level of initial wealth. In the latter case, low initial wealth tends to favor the hidden reserve price policy.

As a remark, we also note that under the hidden reserve price, the seller always submits her true type while the buyer submits a bid close to his true type when he is sufficiently risk-averse. Thus, the buyer’s bid double auction
realizes higher expected gain from trade than the seller’s take-it-or-leave-it mechanism when the buyer is sufficiently risk-averse.

The next proposition extends Proposition 2 by relaxing (R1).

**Proposition 3** There exists $\alpha^* > 0$ such that (i) if $A(x) < \alpha^*$ for all $x > 0$, then $\Pi_H < \Pi_A$; and (ii) if $A(x) > \alpha^*$ for all $x > 0$, then $\Pi_H > \Pi_A$.

It should be noted that Lemma 1 and Propositions 2 and 3 also hold when the sufficient conditions are stated in terms of relative risk aversion $R(x)$ instead of absolute risk aversion $A(x)$ since $R(x) = A(x)/x$.

**4 First-Price Sealed-Bid Auctions**

We now turn to the case of $n$ buyers. We first write down the differential equations that determine the symmetric equilibrium bidding functions under the two reserve price policies and then discuss some comparative static properties of these bidding functions. By imposing restrictions on the buyers’ preferences and the distributions of private valuations, we are able to obtain explicit expressions of the bidding functions as well as the seller’s ex ante expected profits, which makes a comparison of the two policies possible.

**4.1 Announced Reserve Price**

Consider a symmetric bidding strategy profile $\{p(s_i)\}_{i=1}^n$ with announced reserve price $r$. We use the direct revelation approach to determine the equilibrium condition. The expected utility for buyer $i$ of type $s$ from reporting $v$ is

$$U_A(s, v) = u(s - p(v) + w)F(v)^{n-1} + u(w)[1 - F(v)^{n-1}].$$

The equilibrium (first-order) condition, $\frac{\partial U_A}{\partial v}|_{v=s} = 0$, yields

$$p'(s) = \frac{(n - 1)f(s)}{F(s)}\psi(s - p(s), w),$$

where $\psi(x, w)$ is the loss aversion function defined in the previous section. This differential equation, together with $p(r) = r$, determines the equilibrium bidding function $p(s, r)$, which is monotonically increasing in $s$. The buyers will bid zero if their types are below $r$. 

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The expected profits for a seller of type $t$ are
\[
\int_0^1 \ldots \int_0^1 \left[ \max\{p(s_i)\} \right]_{i=1}^{n} - t]dF...dF
= \int_r^1 [p(s, r) - t]dF^n(s).
\]
Maximizing this expression by choosing $r$ yields the optimal announced reserve price, $r^*_A(t)$. The ex ante expected profits for the seller are
\[
\Pi_A = \int_0^1 \int_r^1 [p(s, r^*_A(t)) - t]dF^n(s)dG(t). \tag{6}
\]
Observe that, in contrast to the case of one buyer, under the announced reserve price policy the buyer’s equilibrium bidding function, the seller’s optimal reserve price and the seller’s expected profits all depend on the buyers’ risk aversion when there is more than one buyer.

### 4.2 Hidden Reserve Price

Given a bidding strategy profile $\{b_i(s_i)\}_{i=1}^{n}$, a seller of type $t$ chooses a reserve price $r$ to maximize her expected profits
\[
E[\max\{b_i(s_i)\}_{i=1}^{n} - t|\max\{b_i(s_i)\}_{i=1}^{n} \geq r] = \int_r^1 (y - t)dH(y),
\]
where $H(y) = \Pr(\max\{b_i(s_i)\}_{i=1}^{n} \leq y)$. It is clear that the solution is $r^*_H(t) = t$.

Now consider a symmetric bidding strategy $b(s)$ for the buyers. A buyer chooses $v$ to maximize
\[
U_H(s, v) = u(s - b(v) + w)F(v)^{n-1}G(b(v)) + u(w)[1 - F(v)^{n-1}G(b(v))].
\]
The first-order condition, $\frac{\partial U_H}{\partial v}|_{v=s} = 0$, yields the following differential equation
\[
b'(s) = \frac{(n-1)f(s)}{F(s)}\psi(s - b(s), w)
\frac{1 - \frac{g(b(s))}{G(b(s))}\psi(s - b(s))}{1 - \frac{g(b(s))}{G(b(s))}\psi(s - b(s))}, \tag{7}
\]
\]
which, together with the initial condition \( b(0) = 0 \), determines the symmetric equilibrium bidding function. In equilibrium, the ex ante expected profit for the seller can be written as

\[
\Pi_H = \int_0^{b(1)} \int_{b(s) \geq t} [b(s) - t] dF(s) dG(t),
\]

which depends on the buyers’ risk attitudes.

### 4.3 Bidding Behavior

In this subsection, we first establish the existence and uniqueness of the equilibrium bidding functions under the two policies. Since the denominator of the right-hand side of (7) can be zero, the standard existence theorem for ordinary differential equations does not apply directly over the entire domain. We need to consider a narrower domain over which the theorem applies and then extend the solution.

**Proposition 4** There exists a unique symmetric equilibrium bidding function, under both of the reserve price policies.

We next present several lemmas that describe some comparative static properties of the equilibrium bidding functions with respect to the number of buyers, the buyers’ degree of risk aversion, and the buyers’ initial wealth. Lemma 5 shows that buyers bid more aggressively under both policies when the number of buyers is higher and when buyers are more risk-averse. It also shows that if the buyers’ preferences exhibit decreasing (resp: increasing) absolute risk aversion, then they bid less (resp: more) aggressively as their initial wealth increases.

**Lemma 5** (i) Both \( b(s) \) and \( p(s, r) \) increase with \( n \). (ii) If \( A_1(x) \geq A_2(x) \) for any \( x \geq 0 \), then \( b_1(s) \geq b_2(s) \), \( p_1(s, r) \geq p_2(s, r) \) for any \( s \). (iii) If \( A(x) \) decreases (resp: increases) with \( x \) for all \( x \geq 0 \), then both \( b(s) \) and \( p(s, r) \) decrease (resp: increases) with \( w \).

Lemma 6 describes the buyers’ bidding behavior in the limit. It states that when the bidders are sufficiently risk-averse or when the number of bidders is sufficiently large, they bid arbitrarily close to their true valuations under both policies. Consequently, the optimal (announced) reserve approaches the seller’s true value in the limit.
Lemma 6 For any $\varepsilon > 0$, there exist $\alpha > 0$ and $N > 0$ such that either (i) $A(x) > \alpha$ for any $x > 0$ or (ii) $n > N$ implies $s - b(s) < \varepsilon$, for any $s$, and $s - p(s, r) < \varepsilon$, for any $s \geq r$.

Up to this point, we have used the degree of absolute risk aversion as our measure of risk aversion. The following lemma is a parallel result of Lemma 6 using the degree of relative risk aversion $R(x)$ as the measure of risk aversion. In this case, the size of initial wealth matters. Buyers tend to submit a bid close to their true valuation of the item when their degree of relative risk aversion is large and their initial wealth is small.

Lemma 7 Suppose $R(x) \geq 1$ for any $x > 0$. Then for any $\varepsilon > 0$, there exists $w_0 > 0$ such that $w < w_0$ implies $s - b(s) < \varepsilon$ for any $s$, and $s - p(s, r) < \varepsilon$ for any $s \geq r$.

4.4 Comparing Expected Profits

We now examine the comparative static properties of expected profits under the two reserve price policies with respect to three factors: the buyers’ risk aversion and initial wealth and the number of buyers. An implication of Lemma 5 is that profits under both policies increase with the degree of competition among the buyers and with the buyers’ degree of risk aversion measured by a uniform shift in absolute risk aversion.

Define

$$
\Pi(n) = \int_0^1 \int_t^1 (s - t) dF(s) dG(t)
$$

to be the maximum ex ante expected gains from trade between the seller and the buyers. By Lemma 6, when the buyers are sufficiently risk-averse, they bid arbitrarily close to their true values under both reserve price policies. Therefore, both $\Pi_A$ and $\Pi_H$ converge to $\Pi(n)$ as the buyers’ degree of risk aversion increases. This also implies that risk aversion tends to increase efficiency of trade between the seller and buyers. Similarly, Lemma 6 implies that as the number of buyers goes to infinity, $\Pi_A$ and $\Pi_H$ converge to

$$
\lim_{n \to +\infty} \Pi(n) = \int_0^1 (1 - t) dG(t)
$$

which is independent of the buyer’s risk aversion.
Furthermore, Lemma 7 indicates that if the buyers’ preferences display decreasing (resp: increasing) absolute risk aversion, the expected profits under both policies decrease (resp: increase) with the level of the buyers’ initial wealth.

In general, however, it is difficult to rank the expected profits under the two reserve price policies as this involves comparing two non-linear differential equations. To gain some insights, we consider a special class of preferences and a special class of distribution functions. We provide numerical results for more general environments in Section 5 of the paper.

Consider the following restrictions on preferences:

\textbf{(R2): Each buyer has zero initial wealth and preferences exhibiting constant relative risk aversion.}

The buyers’ preferences under (R2) can be described by the following utility function:

\[ u(x) = \begin{cases} 
  (x^{1-\theta} - 1)/(1-\theta), & \theta \neq 1 \\
  \ln(x), & \theta = 1 
\end{cases} \]

where \( \theta \) measures the degree of relative risk aversion. If \( \theta \geq 1 \), Lemma 7 implies that the buyers will bid their true values so that \( \Pi_A = \Pi_H = \Pi \). In the remainder of this section, we will focus on the case where \( \theta \in [0,1) \). In this case, we will show that the hidden reserve price policy tends to dominate when \( \theta \) is close to 1.

Given (R2), the loss aversion function \( \psi(x, w) \) equals \( x/(1-\theta) \), which is linear in \( x \). The first-order condition (5) is reduced to the following linear differential equation

\[ p'(s) = \frac{n-1}{1-\theta} \frac{f(s)}{F(s)} [s - p(s)]. \]

Setting \( p(r) = r \), we solve for the equilibrium bidding function as follows

\[ p(s, r) = s - \int_r^s \frac{F(x)}{F(s)} \frac{1}{r^{\theta-1}} dx. \]

\(^{5}\)This class of preferences has been used previously in the literature on auctions and bargaining (see, for instance, Cox, Smith and Walker, 1982, Chatterjee and Samuelson, 1983). In Cox, Smith and Walker (1982), each buyer’s degree of constant relative risk aversion is assumed to be his private information.
Note that if $r > 1$, then no one submits a bid. The expected profits for a seller of type $t$ can be expressed as

$$
\pi(r, t) = \int_r^1 [p(s, r) - t]dF^n(s)
= \int_r^1 [I(s) - t]dF^n(s).
$$

The second equality above follows from integrating by parts, and

$$
I(s) = \left\{ \begin{array}{ll}
    s - \frac{F(s)}{f(s)} \frac{1-F(s)^{\beta}}{\beta}, & \text{if } \beta \neq 0 \\
    s + \frac{F(s)}{f(s)} \ln F(s), & \text{if } \beta = 0
\end{array} \right.
$$

where $\beta = (n - 1)/(1 - \theta) - n \in [-1, \infty)$. A seller of type $t$ then chooses $r$ to maximize $\pi(r, t)$. The optimal reserve price $r^*_A(t)$ satisfies $I(r^*_A) = t$.

The following lemma summarizes the comparative static properties of the optimal reserve price.

**Lemma 8** Suppose (R2) holds. Then the optimal (announced) reserve price decreases with the number of buyers $n$ and with the buyers’ constant coefficient of relative risk aversion $\theta$, and converges to the seller’s true value in the limit as $n$ goes to $\infty$ or as $\theta$ goes to 1.

**Proof.** Observe that

$$
\frac{\partial \pi(r, t)}{\partial r} = -[I(r) - t]nF^{n-1}(r)f(r).
$$

Since $I(r)$ increases with $\beta$ and since $\beta$ increases with both $n$ and $\theta$, it follows that the cross-partial derivatives of $\pi$ with respect to $(r, n)$ and with respect to $(r, \theta)$ are negative. In other words, $\pi$ is a submodular function in $(r, n)$ and in $(r, \theta)$. This implies that the optimal solution $r^*_A(t)$ decreases with $n$ and $\theta$. Moreover, as $n$ goes to $\infty$ or as $\theta$ goes to 1, $\beta$ diverges to $\infty$ and hence $I(s)$ approaches $s$. Thus, in the limit, $r^*_A(t) = t$. $\blacksquare$

One of the most striking results in auction theory is that in the independent private values environment with risk-neutral bidders, the seller’s optimal announced reserve price is both independent of the number of bidders and above the seller’s true reserve price. When the private signals are affiliated, the optimal reserve price is lower (see Klemperer, 1999) and converges to the
Lemma 8 shows that even in the independent private values environment, if bidders are risk-averse, the optimal reserve price is lower when the bidders are more risk-averse or when there is more bidders. The intuition is that with more competition or with more risk aversion, the winning bid is more likely to be high, so that the seller has less incentive to set a high reserve price to avoid a low winning bid. The optimal reserve price also converges to the seller’s true reserve price in the limit as the number of bidders goes to infinity or as the degree of risk aversion goes to 1.

Given the optimal reserve price, ex ante expected profits for the seller are

\[ \Pi_A(\theta) = \int_0^1 \int_0^{r_A^*(t)} [I(s) - t]dF^m(s)dG(t). \]

Lemma 8 implies that \( r_A^*(t) \) decreases with \( \theta \). Thus, \( \Pi_A(\theta) \) increases with \( \theta \). Moreover, using integration by parts and substitution, we can rewrite \( \Pi_A(\theta) \) as follows

\[ \Pi_A(\theta) = \int_{1^{-1}(0)}^{1} [1 - F^m(s)]G(I(s))I'(s)ds. \quad (9) \]

Therefore, given the buyers’ constant relative risk aversion preferences and zero initial wealth, the expected profits under the optimal announced reserve price can be computed using (9). However, the differential equation (7) associated with the hidden reserve price policy is still non-linear and cannot be solved analytically, except for the following class of distribution functions:

\[ (R3): \quad F(s) = s^\lambda \text{ for } s \in [0, 1], \text{ where } \lambda > 0, \text{ and } G(t) = t^\rho \text{ for } t \in [0, 1], \text{ where } \rho > 0. \]

In this case, it can be easily shown that the differential equation (7) has an explicit solution and that the solution is the following linear function of \( s \),

\[ b(s) = ks, \quad \text{where} \quad k = \frac{(n - 1)\lambda + \rho}{1 - \theta + (n - 1)\lambda + \rho}. \]

It then follows from (8) that

\[ \Pi_H(\theta) = \frac{n\lambda}{(\rho + 1)(n\lambda + \rho + 1)^{\rho + 1}}k^{\rho + 1}. \quad (10) \]
Note that for this class of distribution functions,

\[ I(s) = \begin{cases} 
  s + s^{\frac{\lambda \beta - 1}{\lambda \beta}}, & \text{if } \beta \neq 0 \\
  s + s \ln s, & \text{if } \beta = 0 
\end{cases} \] (11)

Observe that \( I(s) \) is increasing when it is positive, so that the optimal reserve price \( r_A^*(t) = I^{-1}(t) \) at the interior solution. Given the monotonicity of \( I(s) \), \( \Pi_A(\theta) \) can be computed from (9).

Comparing the expressions for expected profits under the two policies yields the following proposition:

**Proposition 9** Suppose (R2) and (R3) hold. There exists \( \theta_0 \in (0, 1) \) such that \( \Pi_H(\theta) > \Pi_A(\theta) \) for any \( \theta \in (\theta_0, 1) \).

Proposition 9 states that when bidders are sufficiently risk-averse, where risk aversion is measured by the degree of relative risk-aversion, the hidden reserve price policy dominates the announced reserve price policy. The intuition behind this result is the same as the one which we discussed in the context of the model with one buyer presented in the previous section. That is, a hidden reserve price creates extra risk to the buyers who therefore tend to bid more aggressively.

Note that Proposition 9 does not provide a complete comparison between the two policies. We pursue two strategies. One strategy is to provide some numerical results in Section 5. Another strategy is to consider a further refinement of restriction (R3) as follows:

(R4): \( F(s) = s \) for \( s \in [0, 1] \) and \( G(t) = t \) for \( t \in [0, 1] \).

Under these restrictions, we are able to establish the uniqueness of the critical point at which the two profit curves cross and its monotonicity with respect to the degree of competition.

**Proposition 10** Suppose (R2) and (R4) hold, then there exists a unique \( \theta^* \in (0, 1) \), such that \( \Pi_H(\theta) > \Pi_A(\theta) \) if and only if \( \theta \in (\theta^*, 1) \). Moreover, \( \theta^* \) decreases with \( n \) and converges to \( \sqrt{5} - 2 \) as \( n \) approaches infinity.

Proposition 10 presents two results in the simple environment where buyers’ preferences exhibit constant relative risk aversion and the private values
of both the buyers and the seller are uniformly distributed. The first result is that, for any fixed number of buyers, the hidden reserve price policy dominates the announced reserve price policy if and only if the buyers are sufficiently risk-averse. The second result shows that the hidden reserve price policy is more likely to be preferred by the seller when the number of buyers is higher. With increased competition, the equilibrium bids are likely to be high. Therefore, adopting an announced reserve price to avoid a low winning bid becomes less significant. However, an announced reserve price is still more effective in driving up the equilibrium bids even in the limit as the number of buyers approaches infinity if the buyers are not sufficiently risk-averse.

As in the case of one buyer, when the buyers are sufficiently risk-averse, the first-price auction with the optimal hidden reserve price generates more expected potential gains from the trade between the seller and the buyers than the auction with the optimal announced reserve price.

5 Numerical Results

In this section, we report a set of numerical results which demonstrate that the results presented in Propositions 9 and 10 continue to hold in more general environments. The numerical results illustrate the empirical relevance of the findings of the previous section and raise a number of questions for future research.

Recall from the previous section that since we are unable to show the uniqueness of the critical point at which the two expected profits are equal, Proposition 9 is a weaker result than Proposition 2. In the model with one buyer, when the only buyer becomes more risk-averse, the seller’s expected profit under the announced reserve price policy is unchanged, but the corresponding profit under the hidden reserve price policy is increased. When multiple bidders become more risk-averse, expected profits under both policies increase. In this case, a general comparison of the expected profits is not straightforward.

Our first set of numerical results illustrates that the critical point $\theta^*$ is generally unique under restrictions (R2) and (R3). Table I lists the critical values $\theta^*$ for various sets of parameters. Column 1 shows the monotonicity of $\theta^*$ with respect to $n$ for the case of uniform distribution $\lambda = \rho = 1$, which is the result stated in Proposition 10. This monotonicity result held for
every value of $\lambda$ and $\rho$ that we tried. Table I provides results for four such experiments.

| Table I: Critical Values, Competition and Distributions |
|-----------------------------------|---|---|---|---|---|
|                                  | $\lambda = 1$ | $\lambda = .5$ | $\lambda = 2$ | $\lambda = .2$ | $\lambda = 8$ |
| $\rho = 1$                       | $\theta^*$   | $\theta^*$   | $\theta^*$   | $\theta^*$   | $\theta^*$   |
| $\rho = 2$                       | $\theta^*$   | $\theta^*$   | $\theta^*$   | $\theta^*$   | $\theta^*$   |
| $\rho = .5$                      | $\theta^*$   | $\theta^*$   | $\theta^*$   | $\theta^*$   | $\theta^*$   |
| $\rho = .2$                      | $\theta^*$   | $\theta^*$   | $\theta^*$   | $\theta^*$   | $\theta^*$   |
| $\rho = 8$                       | $\theta^*$   | $\theta^*$   | $\theta^*$   | $\theta^*$   | $\theta^*$   |
| $n=2$                            | .450         | .462         | .406         | .669         | .382         |
| $n=4$                            | .346         | .328         | .357         | .524         | .310         |
| $n=8$                            | .291         | .278         | .311         | .384         | .273         |
| $n=16$                           | .264         | .253         | .279         | .305         | .254         |
| $n=32$                           | .250         | .233         | .259         | .270         | .245         |
| $n=64$                           | .243         | .190         | .248         | .252         | .240         |
| $n=128$                          | .239         | .167         | .242         | .236         | .232         |
| $\infty$                         | $\approx 0.236$ | $\approx 0.236$ | $\approx 0.236$ | $\approx 0.236$ | $\approx 0.236$ |

Table II shows how critical values vary when the distributions of the seller and buyers values change. When $\lambda$ is small, it means that the buyers’ valuations are concentrated at low values; when $\lambda$ is large, the buyers’ valuations are concentrated at high values. The parameter $\rho$ works similarly for the seller. The first two columns in Table II present the relationship between $\lambda$ and $\theta^*$ given $\rho = 1$. The third and fourth columns present a similar relationship between $\rho$ and $\theta^*$ given $\lambda = 1$. The numbers presented suggest that the hidden reserve price is likely to be preferred when the valuations of the seller or the buyers are relatively high. This result is consistent with Bajari and Hortacsu (2000)’s observation on eBay coin auctions that items with higher book value tend to be sold using a secret reserve price as opposed to a posted reserve price. We have also tried different sets of parameter values and found the same pattern. Note that $\theta^*$ is not close to 1 over a large range of values of $\lambda$ and $\rho$. This suggests that the dominance of the hidden reserve price due to risk-aversion is not sensitive to changes in the distribution of valuations.
Table II: Critical Values and Distribution Parameters, $n = 2$

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\theta^*$</th>
<th>$\rho$</th>
<th>$\theta^*$</th>
</tr>
</thead>
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<td>.474</td>
</tr>
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</table>

We turn our attention next to the effect of buyers’ initial wealth. In Propositions 9 and 10, the buyers’ initial wealth $w$ is assumed to be zero. Our numerical results demonstrate that the uniqueness of the critical point holds with positive initial wealth. Proposition 2 tells us that, in the case of one buyer, the hidden reserve price policy is more likely to dominate when initial wealth is lower. Table III demonstrates that this result still holds in the case of more than one buyer. Table VI illustrates that the uniqueness and monotonicity of the critical point presented in Proposition 10 are robust to positive values of initial wealth. That is, the hidden reserve price policy is more likely to dominate when there are more buyers, even for positive values of initial wealth.

Lastly, we compare the seller’s expected profits when the buyer’s value distributions are truncated normal or exponential while maintaining the seller’s value distribution to be uniform. The results are presented in Table V, suggesting that the main results are not sensitive to the specifications of the value distributions.
### Table III: Profit Comparison and the Effect of Initial Wealth

\[ \lambda = \rho = 1, n = 2 \]

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \Pi_A(\theta) )</th>
<th>( \Pi_H(\theta) )</th>
<th>( \Delta% )</th>
<th>( \Pi_A(\theta) )</th>
<th>( \Pi_H(\theta) )</th>
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<td>.155</td>
<td>.158</td>
<td>-1.4</td>
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**Note:** \( \Delta\% \) is defined as \( 100 \cdot \frac{\Pi_A(\theta) - \Pi_H(\theta)}{\Pi_H(\theta)} \) to represent the percentage difference of the two revenues.

### Table IV: Profit Comparison and the Effect of Competition

\[ \lambda = \rho = 1, n = 2 \]

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \Pi_A(\theta) )</th>
<th>( \Pi_H(\theta) )</th>
<th>( \Delta% )</th>
<th>( \Pi_A(\theta) )</th>
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**Note:** \( \Delta\% \) is defined as \( 100 \cdot \frac{\Pi_A(\theta) - \Pi_H(\theta)}{\Pi_H(\theta)} \) to represent the percentage difference of the two profits.
Table V: Profit Comparison for Other Distributions

<table>
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<th>( \theta )</th>
<th>Normal Buyer Distribution(^a)</th>
<th>( \Pi_A(\theta) )</th>
<th>( \Pi_H(\theta) )</th>
<th>( \Delta% )</th>
<th>Exponential Buyer Distribution</th>
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Notes: \(^a\)We start with a normal buyer distribution with mean .5 and std .3, and an exponential buyer distribution with mean 1, then both are truncated and redefined so that \( F(0) = 0 \) and \( F(1) = 1 \); \(^b\)\( \Delta\% \) is defined as \( 100 \cdot \frac{\Pi_A(\theta) - \Pi_H(\theta)}{\Pi_H(\theta)} \) to represent the percentage difference of the two profits.

6 Conclusion

In this paper, we have made a simple argument that risk aversion on the part of bidders may affect the seller’s choice of reserve price policies in first-price sealed-bid auctions. When bidders are relatively risk-averse, the seller may not want to reveal any information about her valuation of the item to be auctioned. This finding is similar to the result that the seller has incentives to not reveal information about the number of bidders, obtained by McAfee and McMillan (1987) and Matthews (1987). Whether our findings on risk aversion are significant or not is largely an empirical question. Some of the comparative static properties with respect to the number of bidders and the size of bidders’ initial wealth derived in our paper may be tested empirically and in laboratories.

Furthermore, the significance of the wealth effect combined with the risk aversion effect raises a new and interesting issue in auction theory. When bidders are risk-averse and heterogeneous in their initial wealth, how will they behave in sealed-bid auctions and how do we compare the seller’s expected
revenue across standard auction mechanisms? One example of this type of questions is where each bidder has a fixed budget which may be his private information. Further work is needed to explore the implications of the bidders’ heterogeneity in initial wealth and risk aversion.

Appendix

**Proof of Lemma 1:** Since $G(t)/g(t)$ increases with $t \in [0, 1]$, it has an upper bound $\tau = 1/g(1) < \infty$. For any $\varepsilon > 0$, it follows from (4) that it suffices to show that $\psi(\varepsilon, w) > \tau$. Observe that from (3), if $A(x) > \alpha$, then

$$\psi(x, w) > e^{\alpha x} \int_0^x e^{-\alpha z} dz = \frac{e^{\alpha x} - 1}{\alpha}.$$  

It then suffices to show that $(e^{\alpha x} - 1)/\alpha > \tau$. The claim follows since the left-hand side of the inequality strictly increases with $\alpha$ and diverges to $\infty$ as $\alpha$ goes to $\infty$.

**Proof of Proposition 2:** First observe that when $\theta = 0$, the buyer is risk neutral so that the well-known result, $\Pi_A > \Pi_H$, follows. Second, assumption (R1) implies that, for given $s$ and $t$, $b_H^*(s)$ strictly increases and $b_H^{-1}(t)$ strictly decreases with $\theta$. Thus, $\Pi_H(\theta)$ increases with $\theta$ while $\Pi_A$ is independent of $\theta$. Moreover, by Lemma 1, $\lim_{\theta \to \infty} b_H^*(s) = s$. It follows that,

$$\lim_{\theta \to \infty} \Pi_H(\theta) = \int_0^1 \int_t^1 (s - t) dF(s) dG(t)$$

$$> \int_0^1 \int_{r_A(t)}^{r_A(t)} [r_A^*(t) - t] dF(s) dG(t)$$

$$= \Pi_A,$$

where the inequality holds since $\int_t^1 (s - t) dF(s)$ decreases with $t$ faster than $\int_{r_A(t)}^{r_A(t)} [r_A^*(t) - t] dF(s)$ does due to $t < r_A^*(t)$ for all $t < 1$, but both are equal to 0 at $t = 1$. The first part of the proposition follows from the continuity of $\Pi_H(\theta)$ with respect to $\theta$.

To prove the second part, note that from (3), the assumption that $dA(x)/dx \leq 0$ (resp: $\geq 0$) implies that $d\psi(x, w)/dw \leq 0$ (resp: $\geq 0$). It follows from (4)
that $db_H(s)/dw \leq 0$ (resp: $\geq 0$). Thus, $d\Pi_H/dw \leq 0$ (resp: $\geq 0$). The claim follows since $\Pi_A$ is independent of both $\theta$ and $w$ and $\Pi_H$ increases with $\theta$.

**Proof of Proposition 3:** Consider a buyer whose preference exhibits constant absolute risk aversion with degree $\alpha$. By Proposition 2, there exists $\alpha^* > 0$, such that $\Pi_A(\alpha^*) = \Pi_H$. Suppose $A(x) < \alpha^*$ for all $x$. Then, $b(s) < b^*(s)$ for all $s > 0$, where $b(s)$ and $b^*(s)$ are the equilibrium bidding functions under a hidden reserve price for a buyer with $A(x)$ and for a buyer with constant coefficient of absolute risk aversion $\alpha^*$, respectively. It follows that

$$
\Pi_H = \int_0^{b(1)} \int_{b^{-1}(t)}^{1} [b(s) - t]dF(s)dG(t)
< \int_0^{b^*(1)} \int_{b^*^{-1}(t)}^{1} [b^*(s) - t]dF(s)dG(t)
= \Pi_H(\alpha^*) = \Pi_A.
$$

The proof of part (ii) is similar.

**Proof of Proposition 4:** The case of the announced reserve price policy is standard. In what follows, we only prove the existence and uniqueness for the equilibrium bidding function under the hidden reserve price policy.

We look for a solution $b(s)$, $s \in (0, 1]$. Since $F(s)$, $G(b)$, and the denominator of the right-hand side of (7) can be zero, the Cauchy-Peano basic existence theorem does not apply directly in the entire domain (see Kaplan, 1958, Chapter 12). Let $\overline{b}(s)$ satisfy $\psi(s - b, w) = 0$ and $\underline{b}(s)$ satisfy $1 - \frac{\partial(b)}{\partial(b)} \psi(s - b, w) = 0$. It follows that $\overline{b}(s) = s$ and $0 < \underline{b}(s) < \overline{b}(s)$ for $s \in (0, 1]$. Define

$$
D = \{(s, b) : 0 < s \leq 1, \underline{b}(s) < b < \overline{b}(s)\}.
$$

Then the right-hand side of (7) is continuous and has a continuous partial derivative with respect to $b$ in $D$.

For any $(s_0, b_0) \in D$, with initial condition $b(s_0) = b_0$, applying the Cauchy-Peano basic existence theorem, and the uniqueness theorem, there exists a unique solution $b(s)$ in the neighborhood of $s_0$, and it is continuous at the initial value $s_0$. Furthermore, applying the theorem of complete
solutions, such a solution can be extended to the left and right along the $s$-axis, and be defined over a larger interval. If $(s, b(s))$ is the limit of such an extension to the right, it must tend to the boundary of $D$.

To prove that $b(s)$ is defined over $[s_0, 1]$, this boundary must be the right boundary. We need to show that there does not exist $s_1 \in (s_0, 1)$ such that as $s \to s_1^{-}$, $b(s) \to \overline{b}(s)$ or $b(s) \to \overline{b}(s)$. Suppose there exists a $s_1$, as $s \to s_1^{-}$, $b(s) \to \overline{b}(s_1)$. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $s_1 - s < \delta$, $|b(s) - \overline{b}(s_1)| < \varepsilon$. For sufficiently small $\varepsilon$, $b'(s)$ can be arbitrarily large. Therefore, if $s$ moves closer to $s_1$ by a given distance, $b(s)$ can become so large that $|b(s) - \overline{b}(s_1)| < \varepsilon$ does not hold. Next suppose there exists a $s_1$, as $s \to s_1^{-}$, $b(s) \to \overline{b}(s_1) = s_1$. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $s_1 - s < \delta$, $|b(s) - s_1| < \varepsilon$. For sufficiently small $\varepsilon$, $b'(s)$ can be arbitrarily small, say $b'(s) < 1$. Consider a given $\tilde{s}$ in this range, and denote $\tilde{\varepsilon} = b(\tilde{s}) - \tilde{s}$. For any $s > \tilde{s}$ in this range,

$$b(s) = b(\tilde{s}) + \int_{\tilde{s}}^{s} b'(s) ds < b(\tilde{s}) + s - \tilde{s}. $$

That is, $s - b(s) > b(\tilde{s}) - \tilde{s} = \tilde{\varepsilon}$. When $s$ is sufficiently close to $s_1$, this contradicts the condition $|b(s) - s_1| < \varepsilon$ for any $\varepsilon$.

Finally, note that if $s_0 \to 0^+$, $b_0 \to 0^+$. Therefore, the above solution $b(s)$ satisfies the initial condition $b(0^+) = 0^+$.

**Proof of Lemma 5:** To prove the lemma, we consider the following general differential equation

$$\frac{dx}{ds} = \phi(s, x; \tau), \quad (12)$$

with an initial condition $x(0^+) = x_0$, where $\tau$ is a parameter, $x_0$ is independent of $\tau$, and $\phi(s, x; \tau)$ is single-valued, continuous, and continuously differentiable with respect to its arguments. Then there exists a unique continuous function $x(s, \tau)$ that satisfies (12) and the initial condition (see Proposition 4 above and Kaplan, 1958). We want to show that if $\phi(s, x; \tau)$ is increasing in $\tau$, then the solution $x(s, \tau)$ is also increasing in $\tau$.

Observe that $x(0^+, \tau) = x_0$ is independent of $\tau$ and $dx/ds$ at $s = 0^+$ equals $\phi(0^+, x; \tau)$ which increases with $\tau$. Then for any $\tau' > \tau$, $x(s, \tau') > x(s, \tau)$ for any $s$ positive, but small. Otherwise, there exist a $s' > s$ such that $x(s', \tau') \leq x(s', \tau)$. This implies $x(s, \tau')$ cuts across $x(s, \tau)$ from above
somewhere. That is, there exists a \( s' \in [s, s'] \), such that \( x(s'', \tau') = x(s'', \tau) \) and \( dx(s'', \tau')ds < dx(s'', \tau)/ds \). This contradicts (12). The claim follows.

Given this result, the claims in the lemma follow immediately.

**Proof of Lemma 6:** (i) Note that \( p(s, r) \) is determined by (5) with \( p(r, r) = r \). Suppose the claim does not hold. For any \( s \), suppose \( \exists \varepsilon > 0 \), such that for any \( \alpha > 0 \), when \( A(x) > \alpha \), \( s - p(s, r) > \varepsilon \). Since \( p(s, r) \) is a continuous function, \( s - p(s, r) > \varepsilon \) must hold in a certain neighborhood of \( s \), say \( [s - \varepsilon', s + \varepsilon'] \). It follows that within this interval \( \psi(s - p(s, r), w) > \psi(\varepsilon, w) \) and hence

\[
\lim_{\alpha \to \infty} \psi(s - p(s, r), w) \geq \lim_{\alpha \to \infty} \psi(\varepsilon, w) = \infty.
\]

Note that \( p(s, r) = r + \int_r^s p(t, r)dt \), and \( \lim_{\alpha \to \infty} p(s) = \infty \). This contradicts \( p(s, r) \leq s \), which holds from the underlying problem.

Under the hidden reserve price policy, we apply similar arguments to those given above. For any \( s \), as \( \alpha \) increases, \( A(x) > \alpha \), if \( s - b(s) \) does not converge to zero, \( b'(s) \) can be made so large in a certain fixed interval such that \( b(s) = \int_0^s b'(t)dt > s \), but this cannot hold.

(ii) The arguments are the same as in (i).

**Proof of Lemma 7:** Note that \( A(y + w) = R(y + w)/(y + w) \) for any \( y + w > 0 \). From (3), \( R(x) \geq 1 \) for \( x > 0 \) implies that

\[
\psi(x, w) = \int_0^x \exp \left( \int_z^x [R(y + w)/(y + w)]dy \right) dz \\
\geq \int_0^x \exp \left( \int_z^x [1/(y + w)]dy \right) dz \\
= (x + w) \ln(1 + x/w).
\]

The last expression, and hence \( \psi(x, w) \), can be arbitrarily large for sufficiently small \( w \). The claim follows from a similar argument in the proof of Lemma 6.

**Proof of Proposition 9:** From (11), \( \lim_{\theta \to 1} I(s) = s \). It follows from (9) and (10) that

\[
\lim_{\theta \to 1} \Pi_A(\theta) = \frac{n\lambda}{(\rho + 1)(n\lambda + \rho + 1)} = \lim_{\theta \to 1} \Pi_H(\theta).
\]
Furthermore, it can be verified that
\[
\frac{d}{d\theta} \Pi_A(\theta) \bigg|_{\theta=1} = \frac{n}{(n-1)(n\lambda + \rho + 1)},
\]
\[
\frac{d}{d\theta} \Pi_H(\theta) \bigg|_{\theta=1} = \frac{n\lambda}{((n-1)\lambda + \rho)(n\lambda + \rho + 1)}.
\]
Therefore,
\[
\frac{d}{d\theta} (\Pi_A(\theta) - \Pi_H(\theta)) \bigg|_{\theta=1} > 0
\]
if and only if \( \rho > 0 \). The claim follows.

**Proof of Proposition 10:** Given uniform distributions, expected profits under the announced reserve price policy can be computed from (9). It can be verified that if \( \theta > 2/(n+1) \) then \( I^{-1}(0) = 0 \). In this case
\[
\Pi_A(\theta) = \frac{n(n^2 + n^2\theta - 3n\theta - n + 2\theta^2 - 2\theta + 2)}{2(-2\theta + n\theta + n)(n+1-2\theta)(n+2)}.
\]

\( \delta_1(\theta) = \theta^2 + 4\theta - 1 \), \( \delta_2(\theta) = -7\theta^2 + 6\theta - 3 \), and \( \delta_3(\theta) = 2\theta^3 - 4\theta^2 - 2 + 4\theta \).

Since \( \theta < 1 \) and \( n > 1 \), the sign of \( \Pi_H(\theta) - \Pi_A(\theta) \) is the same as that of \( \Delta(\theta, n) \). It is easily verified that for \( n > 1 \), \( \Delta(\theta, n) \) is strictly increasing in \( \theta \) for \( \theta \in (0, 1) \), \( \Delta(0, n) < 0 \) and \( \Delta(1, n) > 0 \). Thus, there exists a unique
\( \theta^* \in (0, 1) \), such that \( \Pi_H(\theta^*) = \Pi_A(\theta^*) \) and \( \Pi_H(\theta) > \Pi_A(\theta) \) if and only if \( \theta \in (\theta^*, 1) \).

Furthermore, note that
\[
\partial \Delta(\theta, n) / \partial n = 2\delta_1(\theta)n + \delta_2(\theta),
\]
which is positive at \( \theta = \theta^* \), since \( (\delta_1(\theta^*)n + \delta_2(\theta^*))n = -\delta_3(\theta^*), \delta_1(\theta) > 0, \) and \( \delta_3(\theta) < 0 \). Therefore, \( \theta^* \) decreases with \( n \).

Note the above computation and hence \( \theta^* \) assumes \( \theta > 2/(n+1) \). Therefore, \( \theta^* \) is only valid if \( \theta^* > 2/(n+1) \), which implies \( n \geq 6 \).

For \( n = 2, 3, 4, 5 \), we established the same result by computing \( \theta^* \) directly.

Finally, as \( n \) goes to infinity, the solution must satisfy \( \delta_1(\theta) = 0 \), which implies that \( \theta^* \) has the limit \( \sqrt{5} - 2 \).

**Computational Issues:** The numerical computations of \( \Pi_A \) and \( \Pi_H \) require solving differential equations (5) and (7) with initial conditions at \( s = 0 \). Because \( F(0) = G(0) = 0 \), both initial slopes are undefined at \( s = 0 \). However, applying L’Hospital’s rule, one can show that these limits exist. For the power distributions, the limits are solved as follows:

\[
p'(0) = \frac{\lambda(n-1)\psi'(0,w)}{1+\lambda(n-1)\psi'(0,w)}, \quad \text{and} \quad b'(0) = \frac{(p + \lambda(n-1))\psi'(0,w)}{1 + (p + \lambda(n-1))\psi'(0,w)}.
\]

For other valuation distributions, such as truncated exponential and normal, the limits of the initial slope can be solved similarly. With the initial condition defined, we solve for \( b(s) \) and \( p(s) \) by applying the Bulirsch-Stoer method, which is quite efficient for smooth functions (see Press et. al., 1992).

To compute \( \Pi_A \) from (6), for a given \( t \) in \([0, 1]\), we pick a \( r \) in \([t, 1]\) to solve for \( p(s) \) (with \( p(r) = r \)), then integrate over \( s \) in (6). Treat this value as the function of \( r \), we next solve for \( r^*_A(t) \) that maximizes the function. Lastly, we integrate over \( t \). If cumulative distributions \( F(.) \) and \( G(.) \) are not analytic, as in the case of the normal distribution, we need to integrate the density functions before performing all the above steps. Overall our computations involve five dimensions: three integrations, one differentiation, and one optimization. To speed up the task in the case of the normal distribution, we interpolate the cumulative distributions by cubic-spline. \( \Pi_H \) can be similarly computed from (8), because \( b(s) \) is independent of \( t \), it is also interpolated by cubic-spline for efficiency.
References


