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# Equilibria in second price auctions with participation costs

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## Abstract

We investigate equilibria of sealed-bid second price auctions with bidder participation costs in the independent private values environment. We focus on equilibria in cutoff strategies (participate and bid the valuation iff it is greater than the cutoff), since if a bidder finds it optimal to participate, she cannot do better than bidding her valuation. When bidders are symmetric, concavity (strict convexity) of the cumulative distribution function from which the valuations are drawn is a sufficient condition for uniqueness (multiplicity) within this class. We also study a special case with asymmetric bidders and show that concavity/convexity plays a similar role.

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## 1. Introduction

In this paper, we investigate (Bayesian–Nash) equilibria of sealed-bid second price, or Vickrey, auctions in the independent private values environment with bidder participation

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costs.<sup>1</sup> We assume that bidders know their valuations when they decide whether to participate in the auction.<sup>2</sup> When the bidders are ex-ante symmetric, i.e., their valuations are drawn from the same distribution, the literature, in general, has focused on the unique symmetric equilibrium in which each bidder bids her valuation if it is greater than a cutoff point common to all bidders, otherwise chooses not to participate.<sup>3</sup> We want to know when, if at all, this is the only equilibrium. We also want to identify sufficient conditions for the existence of asymmetric equilibria.

It is not a weakly dominant strategy for a bidder, as it is the case when there is no participation cost, to always bid her valuation. However, it is still true that if a bidder finds participating optimal, she cannot do better than bidding her valuation. Therefore, in this paper we only consider equilibria in which each bidder uses a *cutoff strategy*, i.e., bids her valuation if it is greater than a certain cutoff point, does not participate otherwise.<sup>4</sup> All of our results about uniqueness or multiplicity of equilibria, then, should be interpreted accordingly.

When the bidders are ex-ante symmetric, there is a unique symmetric equilibrium. We show that if the valuations of the bidders are distributed according to a *concave* cumulative distribution function (cdf), then there is no other equilibrium.<sup>5</sup> When the cdf is *strictly convex*, on the other hand, there will *always* be asymmetric equilibria.<sup>6</sup> In particular, there will always be “two cutoff” equilibria: Arbitrarily divide the set of bidders into two groups. There is an equilibrium in which bidders in the same group use the same cutoff that is different from the one used by the other group. We also show that at most two distinct cutoff points are used in any equilibrium when the cdf is log-concave.

The existence of asymmetric equilibria has important consequences for both the efficiency of the auction mechanism and the seller’s revenue. To begin with, asymmetric equilibria will necessarily be ex-post inefficient: the bidder with the highest valuation does not always get the object. Secondly, Stegeman [10] considers ex-ante efficient mechanisms (maximizing expected total surplus net of participation costs) in the same environment and shows that the second price auction always has an efficient equilibrium, whereas the first price auction has one iff the second price auction has an efficient *symmetric* equilibrium. So, by finding sufficient conditions for uniqueness of equilibrium (necessarily symmetric), we also identify

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<sup>1</sup> In the analysis below, (bidder) *participation cost* can be replaced by *entry fee* (charged by the seller) without any change in the formal model or results. Accordingly, we will use both interpretations in our discussions.

<sup>2</sup> See, for example, Samuelson [9], Matthews [5], Stegeman [10], and Celik and Yilankaya [2] for studies of auctions with participation costs or entry fees and the timing considered here. There is a parallel literature where bidders make costly entry decisions before learning their valuations, see, for example, McAfee and McMillan [7], Harstad [3], McAfee et al. [6], Tan [11], Levin and Smith [4], and McAfee et al. [8]. In both cases entry is endogenously determined.

<sup>3</sup> The only exception that we are aware of is Stegeman [10], where there is also an example with two equilibria.

<sup>4</sup> The only exception is Remark 5 below, where we discuss the effect of a participation cost on the possible existence of equilibria in which participating bidders do not always bid their valuations. See Blume and Heidhues [1] for the characterization of *all* equilibria of the second price auction in the independent private values setting.

<sup>5</sup> Concavity of the cdf may be even more restrictive than it seems. See Remark 2 in Section 3.

<sup>6</sup> See Remark 1 in Section 3 on the effect of a binding reserve price.

environments in which both first and second price auctions are ex-ante efficient mechanisms. Thirdly, consider the entry fee interpretation. The result that, under certain assumptions, the second price auction with appropriately chosen reserve price and entry fee is the seller's optimal mechanism depends on bidders playing the symmetric equilibrium. If the seller uses an entry fee, then her revenue from an asymmetric equilibrium will (generically) be lower than the maximum possible, i.e., revenue from the symmetric equilibrium. Finally, in a repeated auction environment, the existence of asymmetric equilibria of the stage game has an effect on collusive repeated game equilibria. The worst asymmetric stage game equilibrium for any given bidder will work better than the symmetric one in supporting collusive outcomes using Nash reversion strategies.

We also study a special case with asymmetric bidders. We consider two groups of bidders, where bidders are symmetric within groups, and one group is "stronger" than the other, i.e., bidders in this group are more likely to have higher valuations in the sense of first-order stochastic dominance. Besides the literal interpretation, this case may also endogenously arise during the cartel formation process among symmetric bidders, as in Tan and Yilankaya [12]. Given that within-group asymmetries pose similar issues that we analyzed in the symmetric bidders case, we concentrate on equilibria that are symmetric within groups. We show that there always exists an "intuitive" equilibrium, where strong bidders are more likely to participate in the auction, and that there is a unique intuitive equilibrium when both cdfs are concave. When weak bidders' cdf is concave, there is never an equilibrium in which they are more likely to participate in the auction than strong bidders. When it is strictly convex, on the other hand, there will be such counterintuitive equilibria if the participation cost is high enough.

In the next section we briefly describe the setup. We look at the cases of symmetric and asymmetric bidders in Sections 3 and 4, respectively. All the proofs are in the appendix.

## 2. The setup

We consider an independent private values environment. There are  $n \geq 2$  risk-neutral (potential) bidders. The valuation of bidder  $i$  is  $v_i$ , which is private and independently distributed with cdf  $F_i(\cdot)$  that has continuously differentiable density  $f_i(\cdot)$  and full support on  $[0, 1]$ .

The auction format is sealed-bid second price. There is a participation cost, common to all bidders, denoted by  $c \in (0, 1)$ ; bidders must incur  $c$  in order to be able to submit a bid and know their valuations before deciding whether to participate in the auction. Bidders do not know others' participation decisions when they make theirs.

Let the feasible action set for any type of bidder be:  $\{No\} \cup [0, \infty)$ , where "No" denotes not participating; bidder  $i$  incurs the participation cost iff her action is different from "No". Let  $b_i(v_i)$  denote  $i$ 's strategy.

If a bidder finds participating in this second-price auction optimal, she cannot do better than bidding her valuation. Therefore, we naturally restrict our attention to (Bayesian–Nash) equilibria in which each bidder uses a *cutoff strategy*, i.e., she bids her valuation if it

is greater than a cutoff point and does not participate otherwise.<sup>7</sup> That is, for each bidder,

$$b_i(v_i) = \begin{cases} \text{No} & \text{if } v_i \leq v_i^*, \\ v_i & \text{if } v_i > v_i^*, \end{cases} \quad (1)$$

where  $v_i^* \in [0, 1]$ . Notice that  $v_i^* = 1$  means that bidder  $i$  does not participate in the auction whatever her valuation is. From now on we focus exclusively on cutoff points, since they are sufficient to describe equilibria in cutoff strategies that we consider. Cutoff points in turn are characterized by indifference (or, when  $v_i^* = 1$ , no profitable participation) conditions.<sup>8</sup> In particular, if  $v_i^* < 1$ , it must be that bidder  $i$  is indifferent between participating and not participating when her valuation is  $v_i^*$ . Similarly, if  $v_i^* = 1$ , bidder  $i$ 's payoff when  $v_i = 1$  must be nonpositive. Conversely, a cutoff strategy profile given by  $\{v_i^*\}_{i=1}^n$  is an equilibrium if the appropriate indifference (or no profitable participation, if  $v_i^* = 1$ ) conditions are satisfied at these cutoffs. Notice also that in every equilibrium  $v_i^* \geq c$  for all  $i$ .

Unless specified, results below are valid for all  $c \in (0, 1)$ .

### 3. Symmetric bidders

In this section we analyze the case in which bidders' valuations are drawn from the same distribution function, i.e.,  $F_i(\cdot) = F(\cdot) \forall i$ . It is well known that there is a unique symmetric equilibrium; we include this result here for completeness.

**Proposition 1.** *There exists a unique symmetric equilibrium with  $v_i^* = v^s \in (c, 1) \forall i$ , where*

$$v^s F(v^s)^{n-1} - c = 0. \quad (2)$$

When  $F(\cdot)$  is concave this is the only equilibrium: No asymmetric equilibrium exists.

**Proposition 2.** *If  $F(\cdot)$  is concave, then no asymmetric equilibrium exists.*

On the other hand, when  $F(\cdot)$  is strictly convex there will always be asymmetric equilibria. In particular, if we partition the set of bidders into two groups in an arbitrary way, then there will be an equilibrium where all the bidders within a group use the same cutoff that is different than the cutoff used by bidders in the other group. Furthermore, when  $F(\cdot)$  is log-concave there is no equilibria in which three or more cutoff points are used. Thus, when  $F(\cdot)$  is both strictly convex and log-concave, the set of equilibria is characterized by Propositions 1 and 3i.

<sup>7</sup> Given that each participating bidder bids her valuation, a bidder's payoff will be strictly increasing in her valuation. This implies that each bidder's payoff will be zero for at most one type where she will be indifferent between participating and not participating. The participation decisions of these zero measure types are inconsequential.

<sup>8</sup> See Lemma 1 in the appendix for a complete characterization of equilibria in cutoff strategies.

**Proposition 3.** (i) If  $F(\cdot)$  is strictly convex, then, for any  $k \in \{1, 2, \dots, n - 1\}$ , there exists an asymmetric equilibrium where

$$v_1^* = \dots = v_k^* < v_{k+1}^* = \dots = v_n^*, \tag{3}$$

$$v_1^* F(v_1^*)^{k-1} F(v_n^*)^{n-k} - c = 0, \tag{4}$$

$$F(v_n^*)^{n-k-1} [v_1^* F(v_1^*)^k + \int_{v_1^*}^{v_n^*} F(v)^k dv] - c \leq 0, \tag{5}$$

Eq. (5) holds with equality whenever  $v_n^* < 1$ , and we have  $v_1^* < v^s < v_n^*$ .

(ii) If  $F(\cdot)$  is log-concave, then there is no equilibrium in which three or more distinct cutoff points are used.

To illustrate the role played by concavity/convexity of the cdf on the possible existence of asymmetric equilibria and to gain some intuition, consider the two bidders case. Let, without loss of generality,  $v_2^* \geq v_1^*$ . Expected net-payoff of the first bidder must be zero when her type is  $v_1^*$ :

$$v_1^* F(v_2^*) - c = 0. \tag{6}$$

Similarly, for the second bidder the following must be true (with equality whenever  $v_2^* < 1$ ):

$$v_2^* F(v_2^*) - \int_{v_1^*}^{v_2^*} v f(v) dv - c \leq 0, \tag{7}$$

where the first term on the left-hand side is the expected gross payoff of type- $v_2^*$  bidder 2 and the second term is her expected payment to the seller. Obviously, (6) and (7) always admit a symmetric solution which gives us the symmetric equilibrium with  $v_1^* = v_2^* = v^s \in (c, 1)$ , where  $v^s F(v^s) - c = 0$ .

To see whether there exists an asymmetric equilibrium with  $v_2^* > v_1^*$ , we shall compare the payoffs of the bidders when their valuations are equal to their respective cutoffs in a proposed equilibrium, given by the left-hand sides of (6) and (7). Note that both bidders have the same probability of winning,  $F(v_2^*)$ . Bidder 2, therefore, has higher expected gross payoff than bidder 1, i.e.,  $v_2^* F(v_2^*) > v_1^* F(v_2^*)$ . On the other hand, bidder 2 pays the seller (bidder 1's valuation, whenever it is between  $v_1^*$  and  $v_2^*$ ) while bidder 1 does not. So, type- $v_2^*$  bidder 2's payoff exceeds type- $v_1^*$  bidder 1's payoff by

$$(v_2^* - v_1^*) F(v_2^*) - \int_{v_1^*}^{v_2^*} v f(v) dv. \tag{8}$$

We need this payoff difference to be zero (or nonpositive if  $v_2^* = 1$ ) in an equilibrium.

If  $F(\cdot)$  is concave, then (8) is strictly positive whenever  $v_2^* > v_1^*$ . Bidder 2 does not have to pay much to the seller compared to her expected gross payoff since relatively more bidder 1 types are at the low end of the support. Thus, there is no asymmetric equilibrium when  $F(\cdot)$  is concave. On the other hand, if types are more skewed toward the high end of the support, the expected payment can be large. When  $F(\cdot)$  is strictly convex, for any  $v_1^*$ , it

is always possible to find a  $v_2^* > v_1^*$  such that the higher type's payoff is not more than the lower type's, i.e., given  $v_1^* < 1$  we can always *either* find a  $v_2^* \in (v_1^*, 1)$  so that (8) equals to zero *or* for this  $v_1^*$ , and  $v_2^* = 1$ , (8) is nonpositive. This, combined with (6), leads to an asymmetric equilibrium.

**Remarks.** (1) Existence of a reserve price does not affect Propositions 1 and 2, except that the equation for the symmetric cutoff  $v^s$  becomes:

$$(v^s - r)F(v^s)^{n-1} - c = 0, \quad (9)$$

where  $r \in [0, 1 - c]$  denotes the reserve price.<sup>9</sup> However, with a reserve price, strict convexity of  $F(\cdot)$  is no longer sufficient for existence of multiple equilibria specified in Proposition 3. It is not difficult to show that, for any given  $r$  and  $c$ , a sufficient condition for the existence of asymmetric equilibria is

$$F(v^s) - (v^s - r)f(v^s) < 0, \quad (10)$$

where  $v^s$  is the symmetric cutoff defined in (9). When there is no reserve price, a sufficient condition for the existence of an asymmetric equilibrium, for any given  $c$ , is just (10) with  $r = 0$  (note that  $c$  affects  $v^s$ ). We used strict convexity in Proposition 3 instead so that the result holds for every  $c$ : If  $F(\cdot)$  strictly convex, then (10), with  $r = 0$ , will always be satisfied independent of  $v^s$ , and hence  $c$ .

(2) It is crucial that the lowest possible valuation of a bidder is zero for the uniqueness result in Proposition 2 to be meaningful. When it is greater than zero, this will introduce convexity (even when  $F(\cdot)$  is concave when its domain is restricted to its support), and so there will be asymmetric equilibria. Stegeman's [10] example belongs to this case.

(3) When  $F(\cdot)$  is neither concave nor convex, the existence of asymmetric equilibria may depend on the magnitude of the participation cost. For example, suppose there are two bidders whose valuations are distributed according to  $F(v) = v^3 - v^2 + v$ . (Notice that  $F''(\cdot) > 0$  iff  $v > \frac{1}{3}$ .) It is not difficult to show that there exists an asymmetric equilibrium iff  $c > 0.1875$ . More generally, if  $F(\cdot)$  is not concave, then we can always find a participation cost for which asymmetric equilibria do exist.<sup>10</sup>

(4) When  $F(\cdot)$  is convex, but not log-concave, there may exist equilibria with three or more cutoff points. Let  $n = 3$ ,  $F(v) = \frac{v+v^5}{2}$  ( $\frac{f(\cdot)}{F(\cdot)}$  is increasing iff  $v \gtrsim 0.57$ ), and  $c = 0.15$ . There is an equilibrium in which bidders use (approximately) 0.3938, 0.8034, and 0.8622 as cutoffs.

(5) There is an equilibrium (in weakly dominated strategies) of the second price auction with  $r = c = 0$  in which one of the bidders bids 1, and all others bid 0. Blume and

<sup>9</sup> We illustrate for  $n = 2$  that Proposition 2 still holds when there is a reserve price. Suppose there is an asymmetric equilibrium with  $v_2^* > v_1^*$ . The following conditions must be satisfied:  $(v_1^* - r)F(v_2^*) - c = 0$  and  $(v_2^* - r)F(v_1^*) + \int_{v_1^*}^{v_2^*} (v_2^* - v) dF(v) - c \leq 0$ . Combining these:  $v_1^* F(v_2^*) > v_2^* F(v_1^*) + r(F(v_2^*) - F(v_1^*))$ . This inequality cannot hold if  $F(\cdot)$  is concave. Notice that, everything else being equal, a positive reserve price makes this last inequality less likely to be satisfied, thus works against the existence of asymmetric equilibria.

<sup>10</sup> To see this, note that if  $F(\cdot)$  is not concave, then  $\exists v' \in (0, 1)$  such that  $F(v') - v'f(v') < 0$ . It follows that, if the symmetric cutoff,  $v^s$ , equals to  $v'$ , then asymmetric equilibria exist. Now, it is clear from (2) that we can find  $c' \in (0, 1)$  such that the symmetric cutoff given  $c'$  is  $v'$ .

Heidhues [1] show that, when  $r > 0$  (and  $c = 0$ ), there is essentially a unique equilibrium in which each bidder bids her valuation whenever it is greater than the reserve price, as long as there are at least three bidders. When  $c > 0$ , equilibria in weakly dominated strategies may exist, and, interestingly enough, the concavity/convexity of the cdf plays a similar role. Suppose  $r = 0$  and  $c > 0$ , and consider the following strategy profile: One bidder bids 1 if her valuation is greater than  $c$ , otherwise she does not participate; other bidders never participate. For this profile to be an equilibrium, we only need to check that the highest possible payoff of nonparticipating bidders is nonpositive, i.e.,  $F(c) - c \leq 0$ . It follows that, for any  $c$ , when  $F(\cdot)$  is convex this strategy profile is an equilibrium, and when  $F(\cdot)$  is strictly concave, it is not.<sup>11</sup>

#### 4. Asymmetric bidders

In this section we study the case in which there are two groups of bidders. In particular, there are  $s$  “strong” bidders whose valuations are distributed with  $G(\cdot)$ , and  $n - s$  “weak” bidders whose valuations are distributed according to  $F(\cdot)$ , where  $s \in \{1, 2, \dots, n - 1\}$  and  $G(v) < F(v)$  for all  $v \in (0, 1)$ .

We concentrate on equilibria that are symmetric within groups, i.e., every strong (respectively, weak) bidder uses the same cutoff point.<sup>12</sup> Denote the strong bidders’ cutoff by  $a$ , and the weak bidders’ by  $b$ .

We first show that the “intuitive” equilibrium, where the strong bidders are more likely to participate in the auction, always exists. Moreover, if both cdfs are concave, then there is a unique intuitive equilibrium.

**Proposition 4.** *There always exists an intuitive equilibrium (where  $b > a$ ). If  $F(\cdot)$  and  $G(\cdot)$  are concave, then there is a unique intuitive equilibrium.*

When  $F(\cdot)$  is concave, there is never an equilibrium in which weak bidders are more likely to participate in the auction. However, when  $F(\cdot)$  is strictly convex, as long as the participation cost is high enough, there will be such counterintuitive equilibria.

**Proposition 5.** (i) *If  $F(\cdot)$  is concave, then there is no equilibrium with  $a > b$ .*

(ii) *If  $F(\cdot)$  is strictly convex, then there exists  $c^* < 1$  such that there is an equilibrium with  $a > b$  whenever  $c > c^*$ .*

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<sup>11</sup> When  $0 < r, c, r + c < 1$  consider the following profile: One bidder bids 1 if her valuation is greater than  $r + c$ , otherwise she does not participate; other bidders never participate. Again, this is an equilibrium iff the highest possible payoff of nonparticipating bidders is nonpositive, i.e.,  $F(r + c)(1 - r) - c \leq 0$ . When  $F(\cdot)$  is concave, this is never the case, when  $F(\cdot)$  is strictly convex it depends on the magnitude of  $r$  and  $c$ .

<sup>12</sup> The issue of within-group asymmetry is similar to that of the previous section, and thus ignored.

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### Appendix

We will first give a characterization of all equilibria in cutoff strategies that we will use in our proofs of Propositions 1–5.<sup>13</sup>

**Lemma 1.** *Let  $v_i^* \in [0, 1]$  be the cutoff used by bidder  $i$  where, without loss of generality,  $v_1^* \leq v_2^* \leq \dots \leq v_n^*$ . Define  $v_0^* = 0$  and  $v_{n+1}^* = 1$  for notational convenience. Cutoff strategies given by  $\{v_i^*\}_{i=1}^n$  constitute a Bayesian–Nash equilibrium iff for each bidder  $i$*

$$\sum_{j=1}^i \left\{ \left[ \prod_{\substack{k=j \\ k \neq i}}^{n+1} F(v_k^*) \right] \int_{v_{j-1}^*}^{v_j^*} F(v)^{j-1} dv \right\} - c \leq 0, \tag{11}$$

satisfied with equality for those with  $v_i^* < 1$ .

**Proof.** First of all, notice that, due to the usual second-price auction reasoning, if a bidder finds it optimal to participate, then bidding her valuation is an optimal bid. Now, suppose all bidders but  $i$  use cutoff strategies. Clearly,  $i$ 's payoff if she participates (and bids her valuation) is strictly increasing in her valuation  $v_i$ . Thus, her participation decision will be given by a cutoff rule. Her cutoff,  $v_i^*$ , is equal to 1 (no participation independent of her valuation) iff her payoff is nonpositive when her valuation is 1 and she participates (and hence strictly negative for all other valuations). Otherwise,  $v_i^* < 1$ , and her payoff if she participates is zero when her valuation is  $v_i^*$  (and hence strictly negative for  $v_i < v_i^*$  and strictly positive for  $v_i > v_i^*$ ). The only thing left to show is that the left-hand side of (11) is bidder  $i$ 's payoff when her valuation is  $v_i^*$  and she participates, which is given by

$$\begin{aligned} & \prod_{k=i+1}^{n+1} F(v_k^*) \left\{ v_i^* \prod_{k=1}^{i-1} F(v_k^*) + \left[ \prod_{k=2}^{i-1} F(v_k^*) \right] \int_{v_1^*}^{v_2^*} (v_i^* - v) dF(v) \right. \\ & + \dots + \left[ \prod_{k=j}^{i-1} F(v_k^*) \right] \int_{v_{j-1}^*}^{v_j^*} (v_i^* - v) dF(v)^{j-1} \\ & \left. + \dots + \int_{v_{i-1}^*}^{v_i^*} (v_i^* - v) dF(v)^{i-1} \right\} - c. \end{aligned}$$

Using integration by parts and arranging gives the left-hand side of (11).  $\square$

<sup>13</sup> We do this for the case of symmetric bidders to keep the notation simpler. When bidders are asymmetric, the only change will be in (11) below to accommodate each bidder's potentially different cdfs.

**Proof of Proposition 1.** If we impose symmetry, i.e.,  $v_i^* = v^s \forall i$ , then all the indifference conditions (11) collapse into

$$v^s F(v^s)^{n-1} - c \leq 0. \tag{12}$$

Now, since  $vF(v)^{n-1}$  is continuous, strictly increasing, and

$$cF(c)^{n-1} < c < 1F(1)^{n-1} = 1,$$

a unique symmetric equilibrium exists with  $v^s \in (c, 1)$  satisfying (12) with equality.  $\square$

**Proof of Proposition 2.** The proof is by contradiction. Suppose there is an asymmetric equilibrium. Then there must exist an  $i \in \{1, \dots, n - 1\}$  with  $v_{i+1}^* > v_i^*$ .<sup>14</sup> Choose the smallest of such  $i$ 's, i.e., we have

$$v_1^* = v_2^* = \dots = v_i^* < v_{i+1}^* \leq 1.$$

It must be the case that (writing (11) for  $i$  and  $i + 1$ , respectively)

$$\sum_{j=1}^i \left\{ \left[ \prod_{\substack{k=j \\ k \neq i}}^{n+1} F(v_k^*) \right] \int_{v_{j-1}^*}^{v_j^*} F(v)^{j-1} dv \right\} - c = 0$$

and

$$\begin{aligned} & \sum_{j=1}^i \left\{ \left[ \prod_{\substack{k=j \\ k \neq i+1}}^{n+1} F(v_k^*) \right] \int_{v_{j-1}^*}^{v_j^*} F(v)^{j-1} dv \right\} \\ & + \left[ \prod_{k=i+2}^{n+1} F(v_k^*) \right] \int_{v_i^*}^{v_{i+1}^*} F(v)^i dv - c \leq 0. \end{aligned}$$

Using  $v_j^* = v_i^*$  for  $j < i$ , these, respectively, become

$$v_i^* \prod_{\substack{k=1 \\ k \neq i}}^{n+1} F(v_k^*) - c = 0$$

and

$$v_i^* \prod_{\substack{k=1 \\ k \neq i+1}}^{n+1} F(v_k^*) + \left[ \prod_{k=i+2}^{n+1} F(v_k^*) \right] \int_{v_i^*}^{v_{i+1}^*} F(v)^i dv - c \leq 0.$$

<sup>14</sup> We will use the ordering  $v_1^* \leq v_2^* \leq \dots \leq v_n^*$  throughout.

Combining these

$$v_i^* F(v_i^*)^{i-1} F(v_{i+1}^*) \geq v_i^* F(v_i^*)^i + \int_{v_i^*}^{v_{i+1}^*} F(v)^i dv > v_i^* F(v_i^*)^i + (v_{i+1}^* - v_i^*) F(v_i^*)^i,$$

or

$$\frac{F(v_{i+1}^*)}{v_{i+1}^*} > \frac{F(v_i^*)}{v_i^*},$$

which cannot be true if  $F(\cdot)$  is concave.  $\square$

**Proof of Proposition 3.** (i) Notice that (4) and (5) are just indifference conditions from Lemma 1, i.e., (11), when  $v_1^* = \dots = v_k^* < v_{k+1}^* = \dots = v_n^*$ . So, we only need to show that if  $F(\cdot)$  is strictly convex, then there exist  $v_1^*$  and  $v_n^*$ , with  $v_1^* < v_n^* \leq 1$ , that satisfy (4) and (5) (and (5) must hold with equality if  $v_n^* < 1$ ). Define  $x = \phi(y)$  implicitly from

$$x F(x)^{k-1} F(y)^{n-k} - c = 0, \tag{13}$$

where  $y \in [v^s, 1]$ . Notice that  $\phi(\cdot)$  is continuously differentiable and strictly decreasing, and that  $\phi(v^s) = v^s$ . Define

$$h(y) = F(y)^{n-k-1} \left[ \phi(y) F(\phi(y))^k + \int_{\phi(y)}^y F(v)^k dv \right] - c,$$

for  $y \in [v^s, 1]$ , and notice that  $h(\cdot)$  is continuously differentiable with  $h(v^s) = 0$ . Comparing (13) with (4) and  $h(y)$  with (5), we only need to show that *either* there exists a  $y^* \in (v^s, 1)$  such that  $h(y^*) = 0$  (in which case  $v_n^* = y^*$  and  $v_1^* = \phi(y^*) < y^*$ ) or  $h(1) \leq 0$  (in which case  $v_n^* = 1$  and  $v_1^* = \phi(1) < 1$ ). If  $h(1) \leq 0$ , then we are done, so, suppose  $h(1) > 0$ . Now, since  $h(\cdot)$  is continuous with  $h(v^s) = 0$  and  $h(1) > 0$ , if  $h(\cdot)$  is decreasing at  $v^s$ , then we are done, i.e., there exists a  $y^* \in (v^s, 1)$  such that  $h(y^*) = 0$ . We have

$$h'(y) = (n - k - 1) F(y)^{n-k-2} f(y) \left[ \phi(y) F(\phi(y))^k + \int_{\phi(y)}^y F(v)^k dv \right] + F(y)^{n-k-1} [k \phi(y) F(\phi(y))^{k-1} f(\phi(y)) \phi'(y) + F(y)^k],$$

and thus

$$h'(v^s) = F(v^s)^{n-2} [(n - k - 1) v^s f(v^s) + k v^s f(v^s) \phi'(v^s) + F(v^s)],$$

so that  $h'(v^s) < 0$  iff

$$\frac{(n - k) v^s f(v^s)}{(k - 1) v^s f(v^s) + F(v^s)} > \frac{(n - k - 1) v^s f(v^s) + F(v^s)}{k v^s f(v^s)},$$

where the left-hand side is the absolute value of  $\phi'(v^s)$ . Now, when  $F(\cdot)$  is strictly convex,  $v^s f(v^s) > F(v^s)$ , and hence  $h'(v^s) < 0$ .

Finally,  $v_1^* < v^s < v_n^*$  follows from comparing (4) and (2).

(ii) We prove it by contradiction. If  $n = 2$ , then the claim is vacuously true. Let  $n \geq 3$ . Suppose there exists an equilibrium in which three or more distinct cutoff points are used, i.e., there exist  $i, j, l$  such that  $1 \leq i < j < l \leq n$  and

$$v_1^* = \dots = v_i^* < v_{i+1}^* = \dots = v_j^* < v_{j+1}^* = \dots = v_l^* \leq 1. \tag{14}$$

It must be the case that (writing (11) for 1,  $i + 1$ , and  $j + 1$ , respectively)

$$v_1^* \prod_{k=2}^{n+1} F(v_k^*) - c = 0, \tag{15}$$

$$v_1^* \prod_{\substack{k=1 \\ k \neq i+1}}^{n+1} F(v_k^*) + \left[ \prod_{k=i+2}^{n+1} F(v_k^*) \right] \int_{v_i^*}^{v_{i+1}^*} F(v)^i dv - c = 0, \tag{16}$$

and

$$v_1^* \prod_{\substack{k=1 \\ k \neq j+1}}^{n+1} F(v_k^*) + \left[ \prod_{\substack{k=i+1 \\ k \neq j+1}}^{n+1} F(v_k^*) \right] \int_{v_i^*}^{v_{i+1}^*} F(v)^i dv + \left[ \prod_{k=j+2}^{n+1} F(v_k^*) \right] \int_{v_j^*}^{v_{j+1}^*} F(v)^j dv - c \leq 0. \tag{17}$$

Combining (15) and (16), and using (14), yields

$$\frac{F(v_{i+1}^*)^{j-i} \left[ \prod_{k=j+1}^{n+1} F(v_k^*) \right] \int_{v_i^*}^{v_{i+1}^*} F(v)^i dv}{F(v_{i+1}^*) - F(v_i^*)} = c.$$

Similarly, combining (16) and (17) yields

$$\frac{\left[ \prod_{k=j+1}^{n+1} F(v_k^*) \right] \int_{v_{i+1}^*}^{v_{j+1}^*} F(v)^j dv}{F(v_{j+1}^*) - F(v_{i+1}^*)} \leq c.$$

Since  $F(v) > F(v_{i+1}^*)$  for  $v > v_{i+1}^*$ , it follows that

$$\frac{\int_{v_{i+1}^*}^{v_{j+1}^*} F(v)^i dv}{F(v_{j+1}^*) - F(v_{i+1}^*)} < \frac{\int_{v_i^*}^{v_{i+1}^*} F(v)^i dv}{F(v_{i+1}^*) - F(v_i^*)},$$

or, equivalently,

$$\frac{\varphi(z_{j+1}) - \varphi(z_{i+1})}{z_{j+1} - z_{i+1}} < \frac{\varphi(z_{i+1}) - \varphi(z_i)}{z_{i+1} - z_i}, \tag{18}$$

where  $z_i = F(v_i^*) \forall i$  and

$$\varphi(z) = \int_0^{F^{-1}(z)} F(v)^i dv.$$

Note that

$$\varphi'(z) = z^{i-1} \frac{F(F^{-1}(z))}{f(F^{-1}(z))}.$$

Since  $\varphi'(\cdot) > 0$ ,  $\varphi(\cdot)$  is strictly increasing. Since  $F^{-1}(\cdot)$  is strictly increasing and  $\frac{F(\cdot)}{f(\cdot)}$  is increasing ( $F(\cdot)$  is log-concave by assumption),  $\varphi''(\cdot) \geq 0$ . However, if  $\varphi(\cdot)$  is strictly increasing and convex, then (18) cannot hold, a contradiction.  $\square$

**Proof of Proposition 4.** Consider the following equation and inequality:

$$xG(x)^{s-1}F(y)^{n-s} - c = 0, \tag{19}$$

$$F(y)^{n-s-1} \left[ xG(x)^s + \int_x^y G(v)^s dv \right] - c \leq 0 \tag{20}$$

with  $c \leq x \leq y \leq 1$ , and (20) must be satisfied with equality whenever  $y < 1$ . It follows from Lemma 1 that the cutoffs given by  $a$  and  $b$ , where  $b \geq a$ , are equilibrium cutoffs iff  $(x, y) = (a, b)$  satisfy these conditions. First, we will show that such  $a$  and  $b$  always exist (with  $b > a$ ), i.e., an intuitive equilibrium always exists. Let  $\bar{b}$  satisfy  $\bar{b}G(\bar{b})^{s-1}F(\bar{b})^{n-s} - c = 0$ . Note that  $\bar{b} < 1$ . For  $y \in [\bar{b}, 1]$ , define  $x = \phi(y)$  from (19), and let

$$h(y) = F(y)^{n-s-1} \left[ \phi(y)G(\phi(y))^s + \int_{\phi(y)}^y G(v)^s dv \right] - c.$$

Notice that  $h(y)$  is continuous,

$$h(\bar{b}) = \bar{b}G(\bar{b})^s F(\bar{b})^{n-s-1} - c = \frac{G(\bar{b})}{F(\bar{b})}c - c < 0,$$

and

$$h(1) = \phi(1)G(\phi(1))^s + \int_{\phi(1)}^1 G(v)^s dv - c.$$

Now, if  $h(1) > 0$ , then there exists a  $b \in (\bar{b}, 1)$  such that  $h(b) = 0$  so that there is an equilibrium with cutoffs  $a = \phi(b)$  and  $b$  where  $\phi(b) < b$  (since  $\phi(\cdot)$  is strictly decreasing and  $\phi(\bar{b}) = \bar{b}$ ). If  $h(1) \leq 0$ , then there is an equilibrium with cutoffs  $a = \phi(1)$  and  $b = 1$ , where  $\phi(1) < 1$ .

We will now show that this intuitive equilibrium we found above is the unique intuitive equilibrium if  $F(\cdot)$  and  $G(\cdot)$  are concave. Suppose  $y < 1$ . Substituting for  $c$  from (19), and dividing by  $F(y)^{n-s-1}$ , (20) becomes

$$xG(x)^s + \int_x^y G(v)^s dv - xG(x)^{s-1}F(y) = 0. \tag{21}$$

We will show below that (21) implicitly defines  $x$  as a strictly increasing function of  $y$ . Consequently, it either has a unique intersection with  $x = \phi(y)$  (which is strictly decreasing), or it does not intersect with  $x = \phi(y)$ , in which case the unique equilibrium is given by  $a = \phi(1)$  and  $b = 1$ .

For (21) we have

$$\frac{dx}{dy} = \frac{G(y)^s - xG(x)^{s-1}f(y)}{-sxG(x)^{s-1}g(x) + F(y)[(s-1)xG(x)^{s-2}g(x) + G(x)^{s-1}]}$$

For  $y > x$ , the denominator is strictly positive, since  $F(y) \geq G(y) > G(x) \geq xg(x)$ , where the last inequality follows from the concavity of  $G(\cdot)$ . To see that the numerator is strictly positive as well, note that

$$G(y)^s - xG(x)^{s-1}f(y) \geq G(y)^s - xG(x)^{s-1}\frac{F(y)}{y}, \tag{22}$$

since  $F(\cdot)$  is concave. So, it suffices to show that the right-hand side of (22) is strictly positive, or equivalently,

$$yG(y)^s - xG(x)^{s-1}F(y) > 0.$$

This is indeed the case, since

$$\begin{aligned} yG(y)^s - xG(x)^{s-1}F(y) &= yG(y)^s - xG(x)^s - \int_x^y G(v)^s dv \\ &> yG(y)^s - xG(x)^s - (y-x)G(y)^s \\ &= x[G(y)^s - G(x)^s] > 0, \end{aligned}$$

where the first equality follows from (21). The claim follows.  $\square$

**Proof of Proposition 5.** (i) Suppose there is such an equilibrium. Then, it must be the case that

$$\begin{aligned} bF(b)^{n-s-1}G(a)^s - c &= 0, \\ G(a)^{s-1}[bF(b)^{n-s} + \int_b^a F(y)^{n-s} dy] - c &\leq 0. \end{aligned}$$

Combining,

$$bF(b)^{n-s-1}G(a) - bF(b)^{n-s} \geq \int_b^a F(y)^{n-s} dy > (a-b)F(b)^{n-s},$$

or

$$\frac{G(a)}{a} > \frac{F(b)}{b},$$

which is a contradiction, since concavity of  $F(\cdot)$  implies that

$$\frac{F(b)}{b} \geq \frac{F(a)}{a} \geq \frac{G(a)}{a}.$$

(ii) Let  $b_1$  and  $b_2$  satisfy

$$b_1 F(b_1)^{n-s-1} - c = 0 \text{ and } b_2 F(b_2)^{n-s-1} G(b_2)^s - c = 0.$$

Note that  $b_1 < b_2$ . For  $y \in [b_1, b_2]$ , define  $x = \phi(y)$  satisfying

$$y F(y)^{n-s-1} G(x)^s - c = 0.$$

Note that  $\phi(b_1) = 1$  and  $\phi(b_2) = b_2$ . Let

$$h(y) = G(\phi(y))^{s-1} \left[ y F(y)^{n-s} + \int_y^{\phi(y)} F(v)^{n-s} dv \right] - c.$$

We have the required equilibrium if  $\exists b \in [b_1, b_2]$  with  $h(b) = 0$ . We know that

$$h(b_2) = b_2 F(b_2)^{n-s} G(b_2)^{s-1} - c > 0,$$

since  $F(b_2) > G(b_2)$ . Since  $h(y)$  is continuous, we only need

$$h(b_1) = b_1 F(b_1)^{n-s} + \int_{b_1}^1 F(v)^{n-s} dv - c < 0.$$

From its definition,  $b_1$  is an increasing function of  $c$ ,  $b_1(1) = 1$  and

$$b_1'(1) = \frac{1}{1 + (n-s-1)f(1)}.$$

It suffices to show

$$\tilde{h}(c) = b_1(c) F(b_1(c))^{n-s} + \int_{b_1(c)}^1 F(v)^{n-s} dv - c < 0.$$

Now,  $\tilde{h}(1) = 0$  and

$$\tilde{h}'(1) = \frac{f(1) - 1}{1 + (n-s-1)f(1)} > 0$$

when  $F(\cdot)$  is strictly convex. Hence,  $\exists c^* < 1$  s.t.  $\tilde{h}(c) < 0$  whenever  $c > c^*$ . The claim follows.  $\square$

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