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SYMPOSIUM

Asymmetric common-value auctions with applications to private-value auctions with resale

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Abstract We study a model of common-value auctions with two bidders in which bidders' private information are independently and asymmetrically distributed. We present sufficient and necessary conditions, respectively, under which the expected revenues from first-price and second-price auctions can be ranked. Using these conditions and a bid-equivalence between common-value auctions and private-value auctions with resale, we extend the revenue-ranking result of Hafalir and Krishna [Am Econ Rev 98, 2008a] and provide necessary conditions for their ranking to hold. In addition, we provide sufficient and necessary conditions for the opposite ranking of revenues, respectively. Our analysis helps clarify the roles of two forms of regularity assumptions (buyer-regularity and seller-regularity) in ranking revenues and illustrates how revenue ranking is linked to submodularity and supermodularity of the common-value function and to a single-crossing property of a function derived from the monopoly or monopsony pricing function in the resale stage.

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1 Introduction

We study revenue ranking of first-price and second-price auctions with asymmetric, independent private-values (AIPV) when resale among the bidders after the auction is permitted. If resale after the auction is not allowed, it is known that either of the two auction formats can yield higher revenue than the other format (Maskin and Riley 2000). Recently, Hafalir and Krishna (2008a) have shown that in the case of two bidders, when resale after the auction is permitted and the resale mechanism is a single-offer monopoly or monopsony mechanism, the expected revenue from a first-price auction exceeds that from a second-price auction, if the probability distributions of the bidders' private-values satisfy a form of regularity, i.e., the virtual value function of each bidder as a buyer is strictly increasing in his true value. Although this regularity condition is commonly used in auction theory, it is indeed quite restrictive. Its use is justified when the condition simplifies the analysis, but is often not needed as in the case of optimal auction theory (Myerson 1981). One may wonder whether the revenue ranking in Hafalir and Krishna (2008a) holds more generally when the regularity assumption is relaxed. Furthermore, any bidder can become a seller or a buyer in the resale market and there are different forms of regularity assumptions associated with different trading mechanisms (Williams 1987; Lebrun 2007), namely buyer-regularity and seller-regularity. It is therefore also important to know whether the seller-regularity implies a similar revenue ranking and whether the conditions required for revenue ranking depend on the types of resale mechanisms.

To better understand the problem of ranking revenues across the two auctions with resale, we take an alternative approach. In particular, we exploit the bid-equivalence between an AIPV auction with resale and a (pure) common-value auction in the sense that the two auction models generate the same bid equilibrium and the same bid distribution.[1] For a simpler exposition, we focus on a pair of weak–strong bidders as in Hafalir and Krishna (2008a), i.e., the distribution of the strong bidder's private-value dominates the distribution of the weak bidder's private-value in the sense of first-order stochastic dominance. In general there might be many common-value auctions that are bid-equivalent to an AIPV auction with resale. We will adopt the common-value auction model in which the signals received are private valuations and the common-value is the optimal monopoly (monopsony) price when the weak (strong) bidder with his (her) private valuation makes offers and believes that the strong (weak)

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[1] For the monopoly or monopsony resale, the bid-equivalence has been observed in Lebrun (2007). A more general bid-equivalence result and its applications to the analysis of private-value auctions with resale is analyzed in Cheng and Tan (2009).

bidder has the upper (lower) bound in valuation given by her (his) signal, based on Bayesian updating of the winning and losing information. The common-value function so defined has an extension that is symmetric in signals. We first examine revenue ranking for asymmetric common-value auctions and then utilize the bid-equivalence and apply these ranking results to study revenue ranking for AIPV auctions with resale.

We provide two sets of sufficient conditions for the revenue ranking results in a model of asymmetric common-value auctions.[2] The first set is based on linear common-value functions. If the common-value is a weighted average of the signals of the weak and strong bidders and if the weight r of the weak bidder is less than or equal to 0.5, then the first-price auction generates higher revenue than the second-price auction. If r exceeds 0.5, on the other hand, the ranking can be reversed. We use this observation to derive sufficient conditions on common-value functions for revenue ranking. When the common-value function is bounded from below by such a linear function with $r \leq 0.5$, then the submodularity property of the common-value function enhances the above ranking since the revenue under the second-price auction utilizes information of the common-value function at the symmetric signal values (on the diagonal) while the revenue under the first-price auction is based mostly on the common-value function evaluated off the diagonal. When the common-value function is bounded from above by a linear function with $r > 0.5$, it is the case of a supermodular common-value function, which may lead to the reversed ranking.

The second set of sufficient conditions is based on a single-crossing property. Considering a weighted sum of a pair of weak–strong distributions and allowing one of the weights to become small, we find that the relative rates of changes in revenues from the two auctions are determined by a single-crossing property of a function involving both the common-value function and the distributions of signals. If this function crosses the horizontal axis once and from below (Condition R), then the first-price auction generates higher revenue. On the other hand, if the function crosses the horizontal axis once and from above (Condition S), the second-price auction yields higher revenue. Conditions (R) is a consequence of submodularity, while a strong form of supermodularity implies Condition (S). In addition, we identify necessary conditions for the two kinds of ranking, which are derived from (R) and (S) for pairs of distributions that are close in C^1 topology.

Our findings on revenue ranking for the common-value auctions can be applied to AIPV auctions with resale. In the case of monopoly resale market, the buyer-regularity implies a unique optimal monopoly price. In the monopsony resale market, however, it is the seller-regularity (i.e., the seller's virtual value is strictly increasing) that helps determine a unique optimal monopsony price. In general, a bidder may potentially become a seller or a buyer in the resale stage, depending on the auction outcome. The assumption of each bidder's distribution satisfying both buyer-regularity and seller-regularity is too strong, however. Instead we assume that the

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[2] Although there is a multiplicity of equilibria in second-price common-value auctions, we select a specific equilibrium and provide several justifications for such a selection. In particular the selected equilibrium corresponds to the one for the AIPV model of auctions with resale.

optimal monopoly and monopsony prices are unique and explore the conditions required for revenue ranking.[3]

Using the ranking results from the common-value auctions, and the bid-equivalence result, we can derive stronger results on revenue ranking in AIPV auctions with resale than that in Hafalir and Krishna (2008a). We offer similar sufficient conditions for the two kinds of ranking results when monopoly and monopsony resale mechanisms are used, respectively. One of the sufficient conditions is implied by the buyer-regularity assumption. In the case of the monopoly resale mechanism, we present an Example D to demonstrate that our sufficient condition is strictly weaker than the buyer-regularity assumption. Therefore, our analysis provides an extension of Hafalir and Krishna (2008a) and is complementary to theirs. In addition, we construct another Example E to illustrate that in the case of the monopsony resale mechanism, the seller-regularity assumption does not lead to a ranking favoring the first-price auction although it implies the uniqueness of the optimal monopsony price by the strong bidder. Therefore, the regularity conditions of the bidders' distribution functions play two roles in private-value auctions with resale. First, the buyer-regularity and seller-regularity are often used to show the uniqueness of the optimal monopoly and monopsony prices in the resale market, respectively, although these are not necessary. Second, the buyer-regularity is closely related to revenue ranking for both cases of monopoly and monopsony resale mechanisms, while the seller-regularity is not.

The characterization of necessity conditions for two kinds of ranking results in common-value auctions translates into similar and simpler necessary conditions for auctions with resale. The condition is on the derivative of the buyer-virtual value of the offer-receiver (the buyer in the monopoly case, or the seller in the monopsony case). Fixing the distribution of the offer-receiver, for all distributions of the offer-maker that are close to the distribution of the offer-receiver (in C^1 topology), the first-price auction dominates the second-price auction if the derivative of the buyer-virtual value of the offer-receiver is equal to or above -2 while the second-price auction dominates the first-price auction if the derivative does not exceed -2.

We note that revenue ranking reversal in auctions with resale has been first illustrated in Gupta and Lebrun (1999). However, in the Gupta and Lebrun (1999) example, it is assumed that there is complete information in the resale stage, which allows the offer-maker to extract all surplus. This assumption is often not realistic, since there is no credible way for bidders to reveal fully their private information in the second stage. In addition, endogenous revelation of information is incompatible with optimal bidding behavior, as each bidder would bid differently if he or she anticipates the loss of all surplus in resale. What is important is not the assumption of complete information, but rather the extraction of all surplus. This idea has a generalization which is treated in Cheng (2008) using the Coase Theorem. In Sect. 4 we use the above

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[3] This assumption is often used in the industrial organization literature. The strict concavity of the monopoly or monopsony profit function is sufficient and we do not need regularity for it. For distribution functions such as power functions with power between 0 and 1, the optimal monopoly and monopsony prices are unique even though none of the regularity conditions holds. When there are multiple optimal prices, we can take a monotone selection, and prove the existence of equilibrium in auctions with resale. Space limitations prevent us from exploring the more general formulation in this paper. Thus the uniqueness assumption of the monopoly or monopsony price is not really a crucial one.

characterization of the ranking results to construct two Examples E and F to illustrate that the ranking reversal can occur when there is incomplete information in the resale stage in either monopoly or monopsony resale. One major insight from our ranking results in common-value auctions is that such ranking reversal is not a rare phenomena and that the direction of ranking is related to the properties of the common-value function (or the price function in the auction with resale) and of the offer-receiver's distribution function of private value.

The rest of the paper is organized as follows. In Sect. 2, we describe the common-value model with asymmetric distributions of private signals and state three conditions regarding the common-value function and the distribution functions. We present the equilibrium and revenues for the first-price and second-price auctions, respectively. In Sect. 3, we provide some intuitive explanations for and formal statements of our main results on revenue ranking. Examples are provided to illustrate the necessity of the conditions for the revenue ranking. We apply our findings on revenue ranking to the model of AIPV auctions with resale in Sect. 4. Some discussions and concluding remarks are given in Sect. 5. All the proofs of lemmas, propositions and theorems in the paper are relegated to an Appendix.

2 The common-value model

In this section, we first lay out the basic model of pure common-value auctions, describe a number of conditions on the model that will be useful for the subsequent analysis, and then present the equilibrium bidding strategies and expected revenue formulas for the first-price and second-price auctions, respectively.

2.1 The model and assumptions

We consider the following pure common-value auction model. There are two risk neutral bidders in an auction for a single object. There is a common valuation for the object; each bidder has partial information or an estimate about the common value. Let $s_i, i = 1, 2$, be the private signal of bidder i. We assume that s_i is independently distributed with a cumulative distribution function $F_i(s_i)$ on support $[0, a_i]$ and that $F_i(s_i)$ is strictly increasing and continuously differentiable with the density function $f_i > 0$ on $(0, a_i)$. The common value is given by $V = w(s_1, s_2)$, where w is assumed to be twice continuously differentiable on $[0, a_1] \times [0, a_2]$ with partial derivatives $w_i > 0$. We normalize the common-value by setting $w(0, 0) = 0$. The asymmetry between the two bidders may be interpreted as follows. In the context of offshore oil and gas lease auctions, for example, signal s_2 may represent bidder 2's estimate of oil and gas reserve on the west side of the tract to be sold in the auction, while s_1 may represent bidder 1's estimate on the east side of the tract. It is common knowledge that bidder 2 is a neighbor firm that previously operated successfully in the neighborhood of the tract while bidder 1 is either a non-neighbor firm or a neighbor firm that previously did not operate successfully, and hence that bidder 2 has a more favorable distribution of private estimates than bidder 1 has.

In this pure common-value model with independently and asymmetrically distributed signals, we focus on first-price and second-price auctions without a reserve price and compare the expected revenues of the two auction formats. Before proceeding, we present a number of alternative restrictions on the two sets of primitives of the model, common-value function and the distribution functions, which will be useful for our analysis of revenue ranking across auction formats in Sect. 3, as well as for our applications to private-value auctions with resale in Sect. 4.

Let $\underline{a} = \min\{a_1, a_2\}$ and $\bar{a} = \max\{a_1, a_2\}$. A pair of weak–strong bidders means that $F_1(s) \geq F_2(s)$ for all $s \in [0, \bar{a}]$, and we refer to bidder 1 as the weak bidder and bidder 2 as the strong bidder.[4] The common-value function w is symmetric if $w(s_1, s_2) = w(s_2, s_1)$ for all (s_1, s_2) with $s_i \leq \underline{a}$. The symmetry property of w means that the common valuation does not depend on who receives which signal as long as the collection of individual signals are the same. Most of the literature deal with symmetric bidders and symmetric common-value auctions. In this paper, w need not be symmetric even if bidders' distributions are.

When applying our analysis of the common-value model to private-value auctions with resale, the following property of the w function turns out to be useful

$$w(s, s) = s \quad \text{for all} \;\; s \leq \underline{a}. \tag{1}$$

It implies that if the two bidders have identical signals which are unbiased estimates of the common value, then the common value should be identical to their estimates.

The common-value function $w(s_1, s_2)$ is submodular if, for all (s_1, s_2) and (s_1', s_2'), $s_1 \leq s_1'$, $s_2 \leq s_2'$, the following holds

$$w(s_1, s_2) + w(s_1', s_2') \leq w(s_1, s_2') + w(s_1', s_2). \tag{2}$$

When the inequality (2) is reversed, w is supermodular. If ϕ is an increasing concave (convex) function then $w(s_1, s_2) = \phi(s_1 + s_2)$ is both symmetric and submodular (supermodular). Submodularity and supermodularity are closely related to several sufficient conditions for revenue ranking, one of which is as follows.

Condition C For all $s_1, s_2 \in [0, \underline{a}]$,

$$w(s_1, s_2) \geq \frac{w(s_1, s_1) + w(s_2, s_2)}{2}. \tag{3}$$

Condition (C) implies that for any pair of signals, the common value is bounded from below by the average of the two common values evaluated at the two symmetric signals, respectively. When w is symmetric, Condition (C) is implied by the submodular property. When (1) holds, Condition (C) takes the following simple form

$$w(s_1, s_2) \geq \frac{s_1 + s_2}{2}, \tag{4}$$

which implies that the common value is not lower than the average of the two individual estimates. Note that if $s_i > \underline{a}$, $w(s_i, s_i)$ need not be properly defined. We can

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[4] This requirement of weak–strong pair in the sense of the first-order stochastic dominance is weaker than conditional stochastic dominance used in Maskin and Riley (2000).

adopt the convention that $w(s_i, s_i)$ in this case is defined to be 0. With this convention, (3) holds for all $s_i \in [0, a_i]$, $i = 1, 2$.

The following is a weaker version of Condition (C).

Condition C′ For all $s_1, s_2 \in [0, \underline{a}]$ satisfying $F_1(s_1) = F_2(s_2)$, (3) holds.

An alternative restriction on w is jointly with the distribution functions. Let $w_i(s_i, s_j)$ be the partial derivative of w with respect to s_i. Define

$$H(s_i, s_j) = 2w_i(s_i, s_j)[1 - F_j(s_j)] - [w_1(s_i, s_i) + w_2(s_i, s_i)][1 - F_j(s_i)] \qquad (5)$$

for $(s_i, s_j) \in [0, a_i] \times [0, a_j]$. Note that $H(s_j, s_j) = 0$. When w is symmetric, we have

$$H(s_i, s_j) = 2w_i(s_i, s_j)[1 - F_j(s_j)] - 2w_i(s_i, s_i)[1 - F_j(s_i)].$$

When (1) holds, we have

$$H(s_i, s_j) = 2w_i(s_i, s_j)[1 - F_j(s_j)] - [1 - F_j(s_i)].$$

Let $S = \{s \in [0, \underline{a}] : F_1(s) = F_2(s)\}$. For given $s_j \notin S$, either $s_i < F_i^{-1}(F_j(s_j))$, or $s_i > F_i^{-1}(F_j(s_j))$. The following condition is a single-crossing property of H in variable s_i from below.

Condition R For some j, and $i \neq j$, and for all s_j, $H(s_i, s_j) > 0$ holds if $s_j < s_i \leq F_i^{-1}(F_j(s_j))$ and $H(s_i, s_j) < 0$ holds if $F_i^{-1}(F_j(s_j)) \leq s_i < s_j$.

The opposite of Condition (R) is the following single-crossing property of H from above:

Condition S For some j, and $i \neq j$, and for all s_j, $H(s_i, s_j) < 0$ holds if $s_j < s_i \leq F_i^{-1}(F_j(s_j))$ and $H(s_i, s_j) > 0$ holds if $F_i^{-1}(F_j(s_j)) \leq s_i < s_j$.

Both Conditions (R) and (S) hold vacuously for $s_j \in S$. Note that for $s_j = a_j$, Condition (R) has no bite, as $H(s_i, a_j) < 0$ for $s_i < a_j$. Similarly for $s_j = 0$, Condition (S) has no bite either as $0 \in S$. Condition (S) in general is false for $s_i < a_j = s_j$ and it can only hold vacuously when $F_i(s_j) = F_j(s_j)$. For s_j near a_j, Condition (S) can hold only in a small range for $s_i < s_j$. Both Conditions (R) and (S) are essentially conditions on the pair w, F_j as F_i is only used to specify a range in which the respective inequalities hold. It is sometimes useful to say that the respective conditions hold for a subset if the respective inequalities hold for s_j in that subset.

Conditions (R) and (S) are sufficient conditions for opposite revenue ranking results in Sect. 3. As we shall explain in Sect. 3.1, the two conditions imply that the difference between the revenues from the first-price and second-price auctions changes monotonically as the asymmetry of the bidders' distributions declines. This property combined with the revenue equivalence for the symmetric common-value model yields a clear ranking of the revenues across the two auction formats for asymmetric common-value models.

For linear common-value functions of the form, $w = rs_1 + (1-r)s_2$, on the region $s_1 \leq s_2$, Condition (C) requires $r \leq 0.5$ which in turn implies Condition (R). When $r > 0.5$, however, Condition (S) holds for all s_i in a neighborhood of $s_j \neq a_j$ and $s_i < s_j$.

We now discuss the relationship between Conditions (R) or (S) and submodularity or supermodularity of w. The following lemma shows that (R) is a consequence of submodularity. If w is strictly supermodular (meaning $w_{ij} > 0$), then (S) holds for s_j in a neighborhood of 0. Assume that w satisfies (1), we say that w is strongly supermodular at s (with respect to F_j) if

$$2w_{ij}(s, s)[1 - F_j(s)] > f_j(s). \tag{6}$$

The following lemma also shows that strong supermodularity at s implies (S) for s_j in a neighborhood of s. The conditions (1) and (6) together imply that $H(s_i, s)$ is strictly decreasing at $s_i = s$.

Lemma 1 *Suppose w is twice continuously differentiable everywhere. Then* (i) *Condition* (R) *holds when w is submodular; and* (ii) *Condition* (S) *holds for* (s_i, s_j) *in a neighborhood of* (s, s) *if w satisfies* (1) *and* (6).

The following example illustrates that (S) may hold for supermodular w if $F_1(s) = F_2(s)$ for s above some number $d < a_2$.

Example A Let $w(s_1, s_2) = (\frac{s_1+s_2}{2})^k$, $k > 1$ and $F_2(s_2) = s_2$ be the uniform distribution on [0, 1]. Clearly, w is symmetric and supermodular. Note that for $j = 2$,

$$H(s_1, s_2) = k\left(\frac{s_1+s_2}{2}\right)^{k-1}(1-s_2) - ks_1^{k-1}(1-s_1).$$

We now show that (S) holds for any F_1 such that $F_1(s) = F_2(s)$ for $s \in [\frac{k-1}{k+1}, 1]$ and $F_1(s) < F_2(s)$ for $s \in [0, \frac{k-1}{k+1})$. Note that

$$H(s_1, s_2)2^{k-1}k^{-1}s_1^{1-k} = \left(1 + \frac{s_2}{s_1}\right)^{k-1}(1-s_2) - 2^{k-1}(1-s_1).$$

Let $g(s_1, s_2)$ be the right-hand side of the above equation and note that

$$\frac{\partial g}{\partial s_1}(s_1, s_2) = -\frac{(k-1)s_2(1-s_2)}{(s_1+s_2)^2}\left(1 + \frac{s_2}{s_1}\right)^k + 2^{k-1}$$

is increasing in s_1. This implies that

$$\frac{\partial g}{\partial s_1}(s_1, s_2) < \frac{\partial g}{\partial s_1}(s_2, s_2)$$

for all $s_1 < s_2$. Meanwhile,

$$\frac{\partial g}{\partial s_1}(s_2, s_2) = 2^{k-2}\left(k + 1 - \frac{k-1}{s_2}\right) < 0 \quad \text{if } s_2 < \frac{k-1}{k+1}.$$

Therefore, $\frac{\partial g}{\partial s_1}(s_1, s_2) < 0$ for $s_1 < s_2 < \frac{k-1}{k+1}$. Since $g(s_2, s_2) = H(s_2, s_2) = 0$, it follows that $H(s_1, s_2) > 0$ for $s_1 < s_2 < \frac{k-1}{k+1}$. Thus, (S) holds.

Before closing this subsection, we note that for the purpose of our exposition, it is often convenient to relabel signals by $t_i = F_i(s_i)$ and $v_i(t_i) = F_i^{-1}(t_i)$. The common-value can then be rewritten as $V = w(v_1(t_1), v_2(t_2))$ and type t_i is independently and uniformly distributed on [0, 1]. Note that v_i is also strictly increasing and continuously differentiable, and can be called a valuation function of bidder i, with $v_1(0) = v_2(0) = 0$ and $v_1(1) = a_1$, $v_2(1) = a_2$.

2.2 First-price and second-price auctions

We now determine the Bayesian Nash equilibrium in first-price and second-price auctions and derive the expected revenues, respectively. The existence and uniqueness of Bayesian Nash equilibrium in first-price common-value auctions have been studied in the literature.[5] Here, we present the derivation of the equilibrium and revenue for the completeness of our analysis.

Let $b_i(t_i)$ be a strictly increasing bidding strategy of bidder i as a function of type t_i in the first-price auction, and $\phi_i(b)$ be its inverse. The following first-order condition is satisfied by the equilibrium bidding strategy

$$\frac{\mathrm{d}\ln\phi_i(b)}{\mathrm{d}b} = \frac{1}{w(v_1(\phi_1(b), v_2(\phi_2(b))) - b}, \tag{7}$$

for $i = 1, 2$, with boundary conditions $\phi_i(0) = 0$ and $b_1(1) = b_2(1)$. The ordinary differential equation system (7) with the boundary conditions determines the equilibrium inverse functions. It implies that $\phi_1(b) = \phi_2(b)$ for all $b \leq b_1(1) = b_2(1)$. This implies that the winning probabilities of the two bidders are identical when they bid the same amount.[6] The symmetric property of the winning probabilities is equivalent to saying that the two bidders have identical bidding strategies as functions of type t, i.e., $b_1(t) = b_2(t)$ for any $t \in [0, 1]$. However, it should be noted that since signals s_1

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[5] The existence of a strictly increasing Bayesian Nash equilibrium has been shown in Rodriguez (2000). The uniqueness of the equilibrium can be found in Lizzeri and Persico (1998) and Rodriguez (2000).

[6] This result can also be found in Engelbrecht-Wiggans et al. (1983) for the Wilson's Drainage Track model and in Parreiras (2006) and Quint (2006) for more general settings.

and s_2 have different distributions, the equilibrium bidding strategies as functions of s_i are not symmetric.

The symmetry property of the equilibrium bidding strategy implies simple formulas for the equilibrium bidding strategy and the corresponding revenue as presented in the following proposition.[7]

Proposition 1 *The equilibrium bidding strategy in the first-price common-value auction is symmetric and given by*

$$b(t) = \frac{1}{t}\int_0^t w(v_1(r), v_2(r))\mathrm{d}r, \quad t \in (0, 1],$$

with the expected revenue given by

$$R^F = 2\int_0^1 (1-t)w(v_1(t), v_2(t))\mathrm{d}t.$$

When w is separable in its two arguments, the expected revenue has a simple expression. The following corollary presents the revenue for the case of symmetric, linear common value function.[8] We will use this simple model to illustrate intuitively revenue ranking results in Sect. 3.1.

Corollary 1 *Suppose $w(s_1, s_2) = (s_1 + s_2)/2$. Then the expected revenue from the first-price auction is given by*

$$R^F = \frac{1}{2}\int_0^1 (1-t)^2\mathrm{d}v_1(t) + \frac{1}{2}\int_0^1 (1-t)^2\mathrm{d}v_2(t).$$

The equilibrium in the second-price, pure common-value auction has been studied by Milgrom (1981). He shows that there is a continuum of equilibria. It turns out that there is no well-established criterion for equilibrium selection in this model. For the purpose of our analysis, we select the following equilibrium

$$b_i(t_i) = w(v_i(t_i), v_i(t_i)), \quad i = 1, 2. \tag{8}$$

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[7] The equilibrium strategy in the first-price common-value auction has been established in the literature (see Parreiras 2006). We would like to thank Jeremy Bulow for pointing out that the bidding formula can also be obtained from the result in Milgrom and Weber (1982) by using symmetric signals but asymmetric common value functions.

[8] This version of the wallet game has been studied by Klemperer (1998) in a number of applications and by Hendricks et al. (2008) on collusive bidding in the context of offshore oil and gas lease auctions. The expression of the expected revenue for asymmetric distributions in the case of discrete signals is derived in Hörner and Jamison (2008, Suppl).

This equilibrium strategy, as a function of signals s_i, is symmetric across the two bidders, but not symmetric in type t_i.

Equilibrium (8) has a number of interesting properties. First, in the symmetric model where the distributions of the bidders are symmetric, the selected equilibrium is symmetric in both signals s_i and in type t_i, and the revenue equivalence between the first-price and second price auctions holds even if w is not symmetric. This is stated in Proposition 3 in the next section. The revenue equivalence does not hold for other equilibria of the second-price auction even when w is symmetric. Second, as with other equilibria, (8) is an ex post equilibrium in the sense that each bidder's optimal choice does not depend on information about the other bidder's types or actions. Indeed, when w is symmetric and when there are two bidders, Theorem 5.4.8 of Milgrom (2004) shows that there is a continuum of ex post equilibria in the second-price auction which includes the one in (8). The price a bidder pays does not depend on his own strategy, and any additional knowledge about the other bidder's type or action only changes the valuation of the object but not the optimal bidding strategy.

Third, when the common-value function has the property that $w(s, s) = s$ for all s, where the common value is equal to each bidder's estimate of the common value when their estimates are identical, $b_i(t_i) = v_i(t_i)$ or equivalently $b_i(s_i) = s_i$. Intuitively, if the common value is equal to the identical estimates of the bidders, then it is reasonable for each bidder to bid its true estimate, much like the truth-bidding equilibrium in an independent private-value setting. Therefore, in this case the selected equilibrium is both truth-telling and an ex post equilibrium.[9] For example, in the modified wallet game two bidders bid for a prize equal to the average of the amounts of money in the two bidders' wallets. In the selected equilibrium, each bidder bids the amount in his own wallet. This remains an optimal strategy even if he learned about the amount of money in the other's wallet. Each bidder makes a positive profit in the bidding, with the strong bidder winning more often than the weak bidder. Other equilibria require certain coordination device while the selected equilibrium does not. Moreover, the special case of the common-value model with $w(s, s) = s$ for all s turns out to be useful in Sect. 4, where the equilibrium (8) corresponds to the bid-your-true-value equilibrium in a two-stage game of the second-price private-value auction with resale. In such a context, the bid-your-true-value equilibrium has a robustness property illustrated by Borgers and McQuade (2007) and Hafalir and Krishna (2008a), which we will discuss in Sect. 4.

The expected revenue from (8) takes the following form.

Proposition 2 *The expected revenue of the second-price auction equilibrium* (8) *is*

$$R^S = \int_0^{\bar{a}} (1 - F_1(s))(1 - F_2(s))\mathrm{d}w(s, s).$$

Note that the revenue of the equilibrium (8) depends on the value of w on the diagonal $s_1 = s_2$ and is not affected by the value of w off the diagonal $s_1 \neq s_2$. On other

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[9] Truth-telling ex post equilibrium has been used by Holmstrom and Myerson (1983) as "uniform incentive compatibility" in the context of mechanism design with private-values.

hand, the expected revenue in the first-price auction depends on the value of w both on and off the diagonal. This contrast helps comparing the two revenue expressions when w is submodular or supermodular.

Moreover, when $w(s, s) = s$, the revenue of the second-price auction equilibrium (8) is simply given by

$$R^S = \int_0^{\bar{a}} (1 - F_1(s))(1 - F_2(s))\mathrm{d}s,$$

which turns out to be identical to the equilibrium revenue of the second-price auction in an independent private-value model.

3 Revenue ranking in common-value auctions

In this section we rank the revenues from first-price and second-price auctions with the equilibrium (8) described in the last section. We proceed by offering a simple proof of the main ranking result for the case of symmetric and separable common-value function, followed by providing an intuitive explanation of the Conditions (C), (R) and (S) needed for our results. We present our main findings on revenue ranking in Sect. 3.2.

3.1 Intuition

Consider a common-value function of the form $w(s_1, s_2) = (s_1 + s_2)/2$. For simplicity, assume that the support of F_i is [0, 1]. By Corollary 1, we have

$$\begin{aligned} R^F &= \frac{1}{2}\int_0^1 (1 - t)^2 \mathrm{d}v_1(t) + \frac{1}{2}\int_0^1 (1 - t)^2 \mathrm{d}v_2(t) \\ &= \frac{1}{2}\int_0^1 (1 - F_1(s_1))^2 \mathrm{d}s_1 + \frac{1}{2}\int_0^1 (1 - F_2(s_2))^2 \mathrm{d}s_2 \\ &> \int_0^1 (1 - F_1(s))(1 - F_2(s))\mathrm{d}s = R^S, \end{aligned}$$

where the strict inequality holds as long as $F_1(s) \neq F_2(s)$ for a subset of [0, 1] with non-zero measure.[10] Therefore, in this case the first-price auction generates higher expected revenue than the second-price auction.

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[10] Note that the ranking result is a straightforward consequence of the revenue formulas and the inequality $A^2 + B^2 \geq 2AB$.

When w is symmetric and strictly submodular, $R^F > R^S$ holds. The intuition is as follows. The revenue R^S utilizes the w function on the diagonal while R^F uses w off the diagonal. For the symmetric linear (and weak submodular) case, the above proof shows that $R^F > R^S$. As w becomes strictly submodular, its value off the diagonal tends to be relatively higher than the value on the diagonal. Aggregation over all possible values implies that $R^F > R^S$ continues to hold for strict submodular function w.

Both Conditions (C) and (R) defined in Sect. 2.1 are weaker than the submodular property. The proof for the ranking result using (C) is similar to the argument shown for the case $w(s_1, s_2) = (s_1 + s_2)/2$. However, the argument for the ranking result under (R) or (S) is quite different. We now use the case of $w(s, s) = s$ to illustrate the implications of (R) and (S) which relate to whether R^F increases slower or faster than R^S as the distributions of the two bidders become more symmetric. This can be done by considering a weighted sum of the two asymmetric distributions and by allowing the weight to change. Note that in this case, (R) for bidder $j = 2$ requires

$$2w_1(s_1, s_2)[1 - F_2(s_2)] < 1 - F_2(s_1) \quad \text{when } s_1 < s_2.$$

Suppose $v_1(t) \leq v_2(t)$ for all $t \in [0, 1]$ and move bidder 1's distribution toward bidder 2's so that $v_1(t)$ approaches $v_2(t)$ pointwise. Then

$$\int_0^1 2(1 - t)w_1(v_1(t), v_2(t))\mathrm{d}t$$

is the rate of change of R^F. On the other hand, changing integration variable, we can rewrite

$$R^S = \int_0^1 (1 - t)(1 - F_2(v_1(t)))\mathrm{d}t.$$

Using integration by parts, we have

$$R^S = \int_0^1 \left[\int_0^{v_1(t)} (1 - F_2(v))\mathrm{d}v \right] \mathrm{d}t.$$

Hence

$$\int_0^1 (1 - F_2(v_1(t)))\mathrm{d}t$$

is the rate of change of R^S. Therefore, the rates of change of the revenue difference, $R^F - R^S$, is given by

$$\int_0^1 [2(1-t)w_1(v_1(t), v_2(t)) - (1 - F_2(v_1(t)))]\mathrm{d}t = \int_0^1 H(v_1(t), v_2(t))\mathrm{d}t,$$

which is negative if Condition (R) holds. In the limit as the two distributions become identical, the revenue equivalence applies, and hence (R) insures that as $v_1(t)$ converges to $v_2(t)$ from below, the difference $R^F - R^S$ decreases to 0, implying $R^F > R^S$. If $v_2(t)$ converges to $v_1(t)$ from above, $R^F - R^S$ also decreases to 0, leading to the same conclusion. Similarly, Condition (S) implies that the difference in revenues increases to 0, leading to the opposite ranking, $R^F < R^S$.

3.2 Main ranking results

When bidders have symmetric signal distributions, the selected equilibrium (8) of the second-price auction is symmetric in signals s as well as in type t. We have the following revenue equivalence result when signal distributions are symmetric. Note that although the revenue equivalence result has been shown for symmetric w when signals are i.i.d. in Milgrom and Weber (1982), the following revenue equivalence result holds for asymmetric w as well.[11]

Proposition 3 *Suppose $F_1(s) = F_2(s)$ for all s on the same support. Then $R^F = R^S$.*

When bidders are not symmetric, revenue ranking depends on the property of the common-value function and the asymmetry of distribution functions. The following theorems summarize our findings.

Theorem 1 *Suppose $F_1 \neq F_2$ for a subset of non-zero measure. Then $R^F > R^S$ if either* (C′) *or* (C) *holds.*

Theorem 2 *Suppose $F_1 \neq F_2$ for a subset of non-zero measure. Then $R^F > R^S$ if* (R) *holds and $R^F < R^S$ if* (S) *holds.*

It should be noted that Theorems 1 and 2 do not reply on the symmetry of w.

Example A with $k > 1$ in Sect. 2.1 illustrates situations in which Condition (S) holds, implying $R^F < R^S$. In the same example with $k \leq 1$, the common-value function is symmetric and submodular, Condition (C) holds, implying $R^F > R^S$. The following example illustrates the case of linear common-value functions which do not satisfy Condition (C), and have the opposite ranking.

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[11] Jeremy Bulow's comment in note 7 applies here as well. Once the revenue formulas are in place, the revenue equivalence result follows through an integration by parts argument as shown in the proof of the Proposition 3 in the Appendix.

Example B Consider the following linear common-value function

$$w = 0.8s_1 + 0.2s_2$$

and let $F_2(s_2) = s_2$. For Condition (S) to hold for $i = 1$, we need

$$H(s_1, s_2) = s_1 - 1.6s_2 + 0.6 > 0, \quad \text{for } s_1 < s_2$$

if $s_1 > 1.6s_2 - 0.6$. Thus, if

$$1.6t - 0.6 \leq v_1(t) \leq t, \quad 0 \leq t \leq 1,$$

then (S) applies to w, $F_1 = v_1^{-1}$, F_2, and hence by Theorem 2, $R^F < R^S$ holds.

Similarly, let $F_1(s_1) = s_1$. For Condition (S) to hold for $i = 2$, we need

$$H(s_2, s_1) = -0.4s_1 + s_2 - 0.6.$$

Thus if

$$t \leq v_2(t) \leq 0.6 + 0.4t, \quad 0 \leq t \leq 1,$$

then Condition (S) applies to w, F_1, $F_2 = v_2^{-1}$, and $R^F < R^S$ holds.

The idea in Example B can be generalized to any linear common-value functions and a weak–strong pair of bidders, as the following theorem summarizes.

Theorem 3 *Suppose that* $F_1(s) \geq F_2(s)$ *for* $s \in [0, \bar{a}]$ *and* $F_1 \neq F_2$ *for a subset of non-zero measure and that* $w(s_1, s_2) = rs_1 + (1-r)s_2$ *for a parameter* $r \in (0, 1)$. *Then* (i) $R^F > R^S$ *if* $r \leq 0.5$; (ii) $R^F < R^S$ *if* $r > 0.5$ *and* $v_2(1 - 2r + 2rt) \leq v_1(t) \leq v_2(t), 0 \leq t \leq 1$; *and* (iii) $R^F < R^S$ *if* $r > 0.5$ *and* $v_1(t) \leq v_2(t) \leq v_1(2r - 1 + 2(1-r)t), 0 \leq t \leq 1$.

Now, consider an arbitrary common-value function w and compare it with the linear common-value function as in Theorem 3. If w is bounded from above by the linear function then the expected revenue from the first-price auction in the former is also bounded from above by that in the latter. If, in addition, the common-value function satisfies $w(s, s) = s$ for all $s \leq \underline{a}$, then the expected revenues from the second-price auction in the two cases remain the same. Theorems 2 and 3 together imply $R^F < R^S$. Similarly using Theorem 1 we can provide a lower bound condition. These findings are summarized in the following corollary.

Corollary 2 *Suppose that* $F_1(s) \geq F_2(s)$ *for* $s \in [0, \bar{a}]$ *and* $F_1 \neq F_2$ *for a subset of non-zero measure and that* w *satisfies* (1). *Then* (i) $R^F > R^S$ *if there exists* $r \in (0, 0.5]$ *such that* $w(s_1, s_2) \geq rs_1 + (1-r)s_2$ *for all* (s_1, s_2); (ii) $R^F < R^S$ *if there exists* $r \in (0.5, 1)$ *such that* $w(s_1, s_2) \leq rs_1 + (1-r)s_2$ *whenever* $F_1(s_1) = F_2(s_2)$ *and* $v_2(1 - 2r + 2rt) \leq v_1(t) \leq v_2(t), 0 \leq t \leq 1$; *and* (iii) $R^F < R^S$ *if there exists a* $r \in (0.5, 1)$ *such that* $w(s_1, s_2) \leq rs_1 + (1-r)s_2$ *whenever* $F_1(s_1) = F_2(s_2)$ *and* $v_1(t) \leq v_2(t) \leq v_1(2r - 1 + 2(1-r)t), 0 \leq t \leq 1$.

Condition (R) is almost necessary for $R^F > R^S$. This is illustrated by the following example, where the two distributions F_1, F_2 differ only in some small interval $[0, \delta]$ for some $\delta > 0$. When $s_j \in [0, \delta]$, (R) fails and (S) holds.

Example C Consider a common-value function $w(s_1, s_2) = (\frac{\sqrt{s_1}+\sqrt{s_2}}{2})^2$ and two bidders represented by

$$\begin{aligned} v_1(t) &= 0.9t + t^2 \quad \text{for } t \leq 0.1, \\ &= t \quad \text{for } t \geq 0.1, \end{aligned}$$

and $v_2(t) = t$ for $t \in [0, 1]$. The two bidders have the same valuation distribution above $t \geq 0.1$, but for $t \leq 0.1$, bidder 2 is slightly stronger. Solving $s = 0.9t + t^2$ and obtain

$$\begin{aligned} F_1(s) &= \frac{-0.9 + \sqrt{0.9^2 + 4s}}{2} \quad \text{for } s \leq 0.1; \\ &= s \quad \text{for } s \in [0.1, 1]. \end{aligned}$$

Note that

$$R^F = 2\int_0^{0.1} (1-t)\left(\frac{\sqrt{t} + \sqrt{0.9t + t^2}}{2}\right)^2 \mathrm{d}t + 2\int_{0.1}^{1} (1-t)t\mathrm{d}t = 0.33317397$$

and

$$\begin{aligned} R^S = &\int_0^{0.1} (1-x)\left(1 - \frac{-0.9 + \sqrt{0.9^2 + 4x}}{2}\right)\mathrm{d}x \\ &+ \int_{0.1}^{1} (1-x)^2\mathrm{d}x = 0.33317483 > R^F. \end{aligned}$$

The idea in the above example can be generalized to provide necessary conditions for revenue ranking as presented in the following theorem. Let $\|F\|_1$ be the C^1 norm of a c.d.f. F. A C^1 neighborhood of F_j is a neighborhood of F_j in this topology.

Theorem 4 *Given any F_j and w, suppose that w is twice continuously differentiable and F_j is continuously differentiable. If $R^F \geq R^S$ holds for all F_i in some C^1 neighborhood of F_j satisfying the assumptions of the model, then for all $s_j \in [0, a_j]$, $H(s_i, s_j)$ is weakly increasing in s_i at $s_i = s_j$. Moreover, if there is some $d < a_j$, such that $R^F \leq R^S$ holds for all F_i in some C^1 neighborhood of F_j satisfying the assumptions of the model and $F_i = F_j$ on $[d, a_j]$, then for all $s_j \in [0, d]$, $H(s_i, s_j)$ is weakly decreasing in s_i at $s_i = s_j$.*

While Condition (R) can be viewed as a global single-crossing property of H for any pair of (s_i, s_j), the necessary condition in Theorem 4 is a local version of the single-crossing property of H. These conditions determine the direction of revenue ranking.

4 Applications to private-value auctions with resale

In this section, we apply our analysis of the common-value auction model in the last two sections to study private-value auctions with independently and asymmetrically distributed signals when post-auction resale is permitted. To simplify our presentation, we adopt the same formulation as the one provided by Hafalir and Krishna (2008a) and focus on two resale mechanisms, namely, the monopoly and monopsony take-it-or-leave-it offer mechanisms. Our analysis provides an alternative approach to private-value auctions with resale which is complementary to the one in Hafalir and Krishna (2008a).

There are two bidders with bidder i's private value x_i being drawn from a cumulative distribution $F_i(x_i)$ on $[0, a_i]$, $i = 1, 2$. We assume that F_i, $i = 1, 2$, is continuously differentiable and has a density function $f_i > 0$ on $(0, a_i]$. To simplify the presentation of our analysis we adopt a pair of weak–strong bidders unless specified otherwise. As in the previous sections, we also use an alternative representation of signals by defining $v_i(t) = F_i^{-1}(t)$ for $t \in [0, 1]$ and $i = 1, 2$.

The first-price auction with resale is a two-stage game. The bidders first participate in a standard sealed-bid first-price auction. In the second stage, either the winner or the loser of the auction may offer to sell the object to or buy it from the other bidder. If the winner of the auction makes a take-it-or-leave-it offer to the loser, it is called the monopoly resale market. If the loser of the auction makes offers, it is called the monopsony resale market. We assume that no bid information is disclosed at the end of the auction and prior to the resale stage.[12] However, at the end of the auction each learns about whether he won or lost in the auction and will use this information to update his belief about the other bidder's private value. We are interested in determining perfect Bayesian equilibrium of the two-stage game and focus on the equilibrium with strictly increasing bidding strategies in the auction stage.

We now proceed by presenting a set of notations to describe how bidders update their beliefs and how the equilibrium price is determined in the resale stage.[13] Given a pair of strictly increasing bidding strategies in the first stage, $\{b_i(\cdot)\}$, and its inverse bidding functions, $\{\phi_i(\cdot)\}$, let bidder i be the winner of the auction with private value

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[12] This no disclosure policy is adopted for reasons similar to that of Hafalir and Krishna (2008a). In the monopoly or monopsony resale markets, the disclosure of the winning bid does not change the result. If both bids are announced, there may not be any strictly monotone equilibrium. In this case, it is natural to consider mixed strategy equilibria. Lebrun (2007) showed that the no-disclosure equilibrium is observationally equivalent to a mixed strategy equilibrium with full disclosure of bids. Our ranking results then hold for such equilibrium with full disclosure of bids.

[13] In the resale stage, there is a hierarchy of beliefs in the sense of Bergemann and Morris (2005) and Borgers and McQuade (2007). In more general resale mechanisms than the monopoly and monopsony resale markets in this paper, the hierarchy of beliefs has interesting implications (Cheng and Tan 2009).

x_i. From bidder i's viewpoint, bidder j's private value x_j is distributed according to a conditional distribution $F_j(x_j)/F_j(y)$ on support $[0, y]$, where $y = \phi_j(b_i(x_i))$ depends on x_i through the bidding strategies. Resale does not occur when $x_i > y$, and we focus on the case $x_i \leq y$. In the monopoly resale market, the seller chooses a price p to maximize the following monopoly profit function

$$(p - x_i)\frac{F_j(y) - F_j(p)}{F_j(y)}. \tag{9}$$

The solution to the above profit-maximization problem, denoted by $p(x_i, y)$, satisfies the first-order condition

$$p - \frac{F_j(y) - F_j(p)}{f_j(p)} = x \tag{10}$$

and depends on the distribution of the buyer. Note that $p(x, x) = x$ for all $x \leq \min(a_1, a_2)$. We then extend the definition of $p(x, y)$ symmetrically to the region with $x > y$ and obtain a symmetric pricing function p.

Similarly, in the monopsony resale market, let x_i be the valuation of the buyer. From the buyer's perspective, the seller's private value x_j is distributed according to $[F_j(x_j) - F_j(y)]/[1 - F_j(y)]$ on support $[y, a_j]$ with $y < x_i$. The buyer chooses p to maximize the monopsony profit function

$$(x_i - p)\frac{F_j(p) - F_j(y)}{1 - F_j(y)}. \tag{11}$$

Remarkably, when the optimal price is an interior solution, it satisfies the same first-order condition as in (10). It is a corner solution at a_j if

$$p - \frac{F_j(y) - F_j(p)}{f_j(p)} \leq x_i$$

for all $p \leq a_j$.

To the extent possible, we shall use i for the offer-maker and j for the offer-receiver. The following lemma summarizes the properties of the extended optimal price functions.

Lemma 2 *The monopoly pricing function $p(x, y)$ defined in (10) with its extension is symmetric, increasing and continuously differentiable. The same properties hold for the monopsony pricing function except that there may be a kink when $p(x, y) = a_j$.*

When the optimal price is unique, Proposition 4 in the appendix of Hafalir and Krishna (2008a) offers existence of perfect Bayesian equilibrium of the two-stage game. They also show that the equilibrium bidding functions satisfy an ordinary differential equation system (the first-order condition). We note that this differential equation system is identical to (7) for the equilibrium of the common-value auction constructed as follows: let $s_i = x_i$ and the common value function be defined by $w(s_1, s_2) = p(s_1, s_2)$, or equivalently, $w(t_1, t_2) = p(v_1(t_1), v_2(t_2))$. Since the boundary conditions are identical in the two cases, the two sets of ordinary differential equations yield the identical equilibrium bidding strategies. We can then apply the revenue formula for the common-value auction. We summarize the results in the following proposition.

Proposition 4 *The equilibrium bidding strategy in the first-stage of the first-price private-value auction with resale and the monopoly offer mechanism* (*the monopsony offer mechanism*) *is the same as that of the first-price common-value auction with a common-value function defined by*

$$w(s_1, s_2) = p(s_1, s_2), \textit{ or } w(t_1, t_2) = p(v_1(t_1), v_2(t_2))$$

where $p(x, y)$ is the resale price function from the monopoly offer mechanism (*the monopsony offer mechanism*).[14] *Moreover, the expected revenue of the auctioneer from the first-price auction with resale is given by*

$$R^{FR} = 2 \int_0^1 (1 - t) p(v_1(t), v_2(t)) \mathrm{d}t.$$

Regularity conditions on distribution functions are often imposed to insure uniqueness of the optimal price in the resale market. It should be noted that in the literature on mechanism design and auction theory, there are two kinds of regularity conditions (see Williams 1987 for instance). In a monopoly market, the virtual value of a buyer is defined as

$$J_B(x; F_j) = x - \frac{1 - F_j(x)}{f_j(x)}. \tag{12}$$

The distribution F_j is called buyer-regular if $J_B(x; F_j)$ is increasing in x. Note that if F_j is buyer-regular then its conditional distribution $F_j(x_j)/F_j(y)$ on support $[0, y]$ for any $y > 0$ is also buyer-regular. Buyer-regularity thus implies that the monopoly profit function (9) is single-peaked and hence has a unique maximum determined by the first-order condition (10). It can be easily verified that a sufficient condition for a distribution to be buyer-regular is that it is convex.

In the monopsony market, the virtual value of a seller is defined as

$$J_S(x; F_j) = x + \frac{F_j(x)}{f_j(x)}. \tag{13}$$

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[14] We note that the bid equivalence between the first-price private-value auction with resale and the first-price common-value auction holds generally for other resale mechanism when the common value function is properly defined and is based on the transaction price function in the resale stage. This issue is further explored in Cheng and Tan (2009).

The distribution F_j is called seller-regular if $J_S(x; F_j)$ is increasing in x. Note that if F_j is seller-regular then its conditional distribution $[F_j(x_j) - F_j(y)]/[1 - F_j(y)]$ on support $[y, a_j]$ for any $y < a_j$ is seller-regular, which implies that the monopsony profit function (11) is single-peaked and hence has a unique maximum. Concavity is sufficient for a distribution to be seller-regular.

Note that the buyer-regularity differs from the seller-regularity.[15] There are two separate implications of the regularity. As we mentioned above, one implication of the two regularity conditions is that the optimal monopoly and monopsony offer prices are unique, respectively. In the two-stage game of the auction with resale, when bids are off equilibrium path, both kinds of regularity may be needed for a unique optimal offer, as a bidder might be a seller or a buyer in the resale stage. However, requiring a bidder's distribution to satisfy both regularity conditions are clearly too strong. We may often have a unique optimal solution without buyer-regularity. For instance, if the buyer distribution is $F(x) = x^k, k \in (0, 1)$, the monopoly profit function (9) is strictly concave and hence has a unique optimal monopoly price, but F is not buyer-regular. An alternative is to assume that the monopoly or monopsony profit function has a unique maximum. Even when optimal prices in the resale stage are not unique, it is still possible to select a monotonic optimal pricing function and obtain existence of equilibrium in the two-stage game. In this sense, the uniqueness of the optimal monopoly or monopsony price is not necessary for equilibrium existence but rather for expositional convenience. A second implication of regularity is that the buyer-regularity will be closely related to Condition (R) which plays an important role for revenue ranking, as we will illustrate in Lemma 4. In particular, we show that if the strong bidder is buyer-regular, then Condition (R) holds for the monopoly resale mechanism and if the weak bidder is buyer-regular then (R) holds for the monopsony resale mechanism. The seller-regularity does not have this implication.

We now turn to discuss the second-price auction with resale. The two-stage game differs only in the first stage in which the first-price auction is replaced by a second-price auction. There is a continuum of equilibria in the two-stage game (Blume and Heidhues 2004; Lebrun 2007). Proposition 2 in Hafalir and Krishna (2008a) shows that there is one equilibrium in which both bidders bid their private values and that this is an efficient equilibrium in the sense that there is no need for resale after the auction, implying that the revenue is the same with or without resale. When there is no resale, the "bid-your-value" strategies constitute a weakly dominant equilibrium. When resale is permitted, this pair of strategies is no longer weakly dominant. Nevertheless, it is robust in the sense of Borgers and McQuade (2007) information-invariant criterion and is the only such a robust equilibrium (Hafalir and Krishna 2008b).[16] This

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[15] For example distribution function $\sqrt{x}$ on $[0, 1]$ is seller-regular, but not buyer-regular. The function $1 - \sqrt{1-x}$ on $[0, 1]$ is buyer-regular, but not seller-regular.

[16] In Borgers and McQuade (2007), an "information invariant" equilibrium is robust with respect to a broad set of possible hierarchies of beliefs. For static games such as a common-value auction, this concept is equivalent to the ex post equilibrium. For multi-stage games such as an auction with resale, they use a concept of weakly information invariant equilibrium. In a multi-stage game with finite types, Borgers and McQuade (2007) assume that there is uncertainty of player types but no uncertainty on the history of plays. This is not satisfied for our model of auctions with resale in which there are continuous types and unobserved bids. However their concept can be easily adapted to this case.

is the equilibrium that we use in ranking revenues of the auctions with resale. Since there is no transaction in the resale stage, the expected revenue for the auctioneer from the second-price auction does not depend on which resale mechanism is allowed in the second stage, and is given by

$$R^{SR} = \int_0^{\bar{a}} (1 - F_1(x))(1 - F_2(x))\mathrm{d}x,$$

which is identical to the expected revenue of the second-price auction equilibrium (8) of the common-value model in Proposition 2.

We now utilize our results in the last section to rank revenues between the two auction formats in the AIPV auction with resale. Note that the common-value function in Proposition 4 is derived from the distributions of the bidders' private values. This means that Conditions (R) and (S) can be seen as assumptions on these distributions. In particular, we will show that if the strong bidder is buyer-regular, then (R) holds for the monopoly resale mechanism. Similarly, if the weak bidder is buyer-regular, (R) holds for the monopsony resale mechanism. Moreover, the necessary conditions on revenue ranking relate to whether the derivative of the buyer-virtual value is above or less than -2. A concave distribution is seller-regular, but need not be buyer-regular. There are examples in which a weak bidder is seller-regular, but the derivative of the buyer-virtual value is below -2. This can lead to higher revenue for the second-price auction, which will be illustrated in Example E.

The following theorem provides sufficient conditions for revenue ranking, which follows immediately from Theorem 3 and Corollary 2.

Theorem 5 *Suppose $F_1(x) \geq F_2(x)$ with strict inequality for a subset of $[0, a_1]$ of non-zero measure. Then*

(i) *$R^{FR} > R^{SR}$ if the optimal (monopoly or monopsony) price in the resale market satisfies $p(x, y) \geq rx + (1 - r)y$ for whenever $F_1(x) = F_2(y), x \leq y \leq a_1$ for some $r \in (0, 0.5]$.*

(ii) *$R^{FR} < R^{SR}$ if there is a constant $r \in (0.5, 1)$ such that $p(x, y) \leq rx+(1-r)y$ whenever $F_1(x) = F_2(y)$ and $v_2(1-2r+2rt) \leq v_1(t) \leq v_2(t), 0 \leq t \leq 1$ for the case of optimal monopoly price, and*

(iii) *$R^{FR} < R^{SR}$ if there is a constant $r \in (0.5, 1)$ such that $p(x, y) \leq rx+(1-r)y$ whenever $F_1(x) = F_2(y)$ and $v_1(t) \leq v_2(t) \leq v_1(2r-1+(2-2r)t), 0 \leq t \leq 1$ for the case of optimal monopsony price.*

The following lemma shows that the sufficient conditions in Theorem 5 can be reduced to primitive conditions. For the monopoly resale mechanism, if the strong bidder has a convex value distribution, then the optimal monopoly price satisfies conditions (C) and (C′) in Sect. 2 and hence part (i) of Theorem 5 applies. For the monopsony resale mechanism, if the weak bidder has a concave value distribution, then the optimal monopsony price satisfies the opposite condition $p(x, y) \leq (x + y)/2$, and part (iii) of Theorem 5 may apply.

Lemma 3 *In the monopoly resale market, if the strong bidder has a convex value distribution F_2, then the optimal monopoly price by the weak bidder satisfies Condition* (C)*. In the monopsony resale market, if the weak bidder has a concave value distribution F_1, then the monopsony price by the strong bidder satisfies $p(x, y) \leq (x + y)/2$.*

Theorem 5 combined with Lemma 3 offers simple sufficient conditions for ranking revenues of auctions with resale. The intuition for the ranking results is roughly as follows. When the optimal offer price is relatively high, the first-price auction generates higher revenue than the second-price does. This happens when the resale market is a monopoly and when the strong bidder has a convex distribution of private values.[17] When the optimal offer price is relatively low, the above ranking may be reversed. This occurs if the resale market is a monopsony and the weak bidder has a concave value distribution.

While the concavity and convexity of the distributions are strong, we next use Conditions (R) and (S) developed in Sects. 2 and 3 to rank revenues. Since $p(x, x) = x$, we have

$$H(x_i, x_j) = 2p_i(x_i, x_j)[1 - F_j(x_j)] - 1 + F_j(x_i).$$

Furthermore, given our focus on a pair of weak–strong bidders, we can decompose Condition (R) into two parts, each of which is weaker than (R) and together with one of the two resale mechanisms will be sufficient for the first-price auction to dominate the second-price auction in revenues.

Condition R1 $H(x_1, x_2) < 0$ if $v_1(F_2(x_2)) < x_1 < x_2$.

Condition R2 $H(x_1, x_2) > 0$ if $\min(v_1(F_2(x_2)), a_2) > x_1 > x_2$.

For a similar reason, Condition (S) can be decomposed into the following two parts.

Condition S1 $H(x_1, x_2) > 0$ if $v_1(F_2(x_2)) < x_1 < x_2$.

Condition S2 $H(x_1, x_2) < 0$ if $\min(v_1(F_2(x_2)), a_2) > x_1 > x_2$.

The following lemma illustrates the relationship between buyer-regularity and Conditions (R1) and (R2). Buyer-regularity is weaker than the convexity of the distribution.

Lemma 4 *In the monopoly resale market, if F_2 is buyer-regular, then* (R1) *holds for the monopoly price function and for $j = 2$. In the monopsony resale market, if F_1 is buyer-regular, then* (R2) *holds for the monopsony price function and for $j = 1$.*

The following theorem is a consequence of Theorem 2.

Theorem 6 *Suppose $F_1(x) \geq F_2(x)$ with strict inequality for a subset of $[0, a_1]$ of non-zero measure. Then*

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[17] For a monopolist who cannot commit to a take-it-or-leave-it offer, and cuts prices subsequently after the offer is rejected in a dynamic bargaining model, the monopoly power can be weakened and part (ii) of Theorem 5 then becomes applicable and the ranking is reversed. For such examples, see Cheng (2008).

(i) $R^{FR} > R^{SR}$ *if either* (R1) *holds for the case of the monopoly resale mechanism, or* (R2) *holds for the case of the monopsony resale mechanism.*

(ii) $R^{FR} < R^{SR}$ *if either* (S1) *holds for the case of the monopoly resale mechanism, or* (S2) *holds for the case of the monopsony resale mechanism.*

Hafalir and Krishna (2008a) have shown that the buyer-regularity condition is sufficient for $R^{FR} > R^{SR}$ to hold. This ranking result is a consequence of Lemma 4 and part (i) of Theorem 6. That is, we have provided an alternative proof of the main revenue ranking in Hafalir and Krishna (2008a). Here, we further note that condition (R1) is weaker than buyer-regularity, as Example D illustrates, where by choosing a distribution with the derivative of the buyer's virtual value between -2 and 0, Condition (R) holds but the buyer-regularity fails. Therefore, our study provides an extension of their analysis on revenue ranking in AIPV auctions with resale. In addition, Theorem 6 offers sufficient conditions for a reversed ranking of revenues.

Example D Let $F_2(x) = \sqrt{x}$ for $x \in [0.04, 1]$, $F_2(x) = 5x$ for $x \in [0, 0.04]$. The buyer virtual value of F_2 is $J_B(x; F_2) = 3x - 2x^{0.5}$. Since $\mathrm{d}J_B/\mathrm{d}x = 3 - x^{-0.5}$ which is close to -2 when x is close to 0.04 from above, F_2 is not buyer-regular. To show condition (R1) holds, note that when $p(x, y) \leq 0.04$, we have either $p_1(x, y) = 0$, or 0.5, and hence (R1) holds trivially. Consider next the case of $p(x, y) > 0.04$. The monopoly price offer by bidder 1 maximizes

$$(F_2(y) - F_2(p))(p - x) = (y^{0.5} - p^{0.5})(p - x),$$

which is strictly concave in p. Hence the first-order condition

$$y^{0.5} - 1.5p^{0.5} + 0.5p^{-0.5}x = 0$$

yields a unique solution given by $p(x, y) = \frac{1}{9}\left(\sqrt{3x + y} + \sqrt{y}\right)^2$. Setting $\frac{1}{9}\left(\sqrt{3x + y} + \sqrt{y}\right)^2 = 0.04$ yields

$$y = (0.3 - 2.5x)^2.$$

Further note that

$$p_1(x, y) = \frac{1}{3} + \frac{\sqrt{y}}{3\sqrt{3x + y}}.$$

Since $F_2(x) \geq 5x$ for $x \leq 0.04$, it follows that for $x \leq 0.04 \leq y$,

$$H(x, y) = \frac{2}{3}[1 - \sqrt{y}] + \frac{2(\sqrt{y} - y)}{3\sqrt{3x + y}} - 1 + \sqrt{x}.$$

Figure 1 plots the equation $H(x, y) = 0$, where the thin curve and the 45 degree line divide the type space into four regions with $H(x, y) > 0$ in the East and West and $H(x, y) < 0$ in the South and North. To show (R1), it suffices to show that the thick

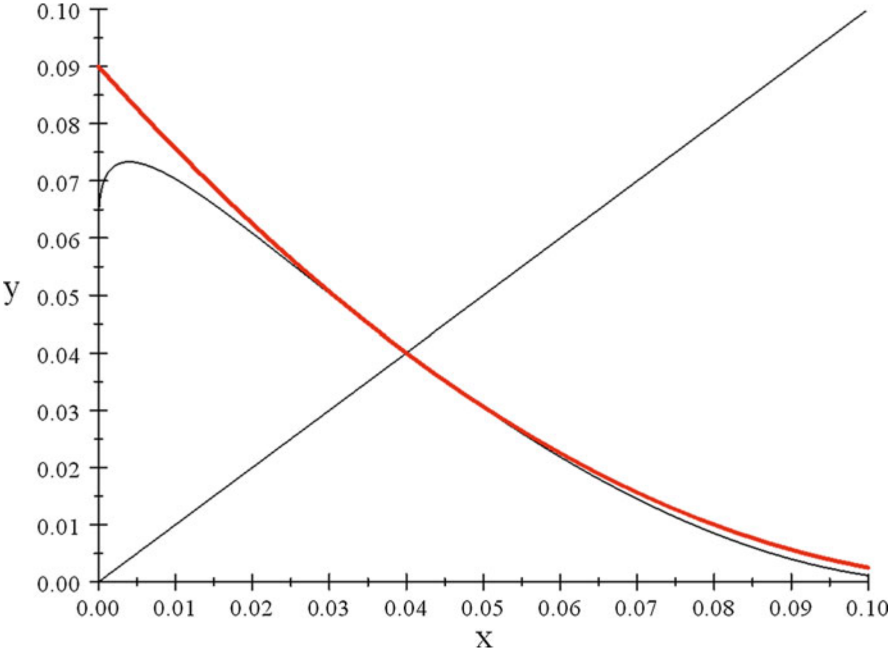

Fig. 1 (Example D): The *thick curve* represents $y = (0.3 - 2.5x)^2$ and the *thin curve* and the *diagonal line* represent $H(x, y) = 0$

curve $\sqrt{3x + y} + \sqrt{y} = 0.6$ is to the right of the thin curve when $x \leq 0.04 \leq y$. This is true when $H(x, y) < 0$ for (x, y), $x < 0.04$ on the thick curve, or

$$0.4\frac{1 - \sqrt{y}}{0.6 - \sqrt{y}} < 1 - \sqrt{x}.$$

We have

$$0.4\frac{1 - \sqrt{y}}{0.6 - \sqrt{y}} < 0.4\frac{1 - \sqrt{0.04}}{0.6 - \sqrt{0.04}} = 0.8 < 1 - \sqrt{x}.$$

Therefore, (R1) holds.

One may wonder whether the seller-regularity of F_1 is sufficient for the ranking $R^{FR} > R^{SR}$ in auctions with the monopsony resale mechanism. The following example illustrates that this is not the case. Thus, in terms of revenue ranking, there is an asymmetric implication of the buyer-regularity and seller-regularity corresponding to the monopoly and monopsony resale mechanisms, respectively.

Example E Consider the monopsony resale market and let the value distribution of the weak bidder be $F_1(x) = x^{\frac{1}{2}}$ with support $[0, 1]$ and $F_2(x) = 10x$ for $x \leq$

0.01, $F_2(x) = F_1(x)$ for $x \geq 0.01$.[18] It follows that $v_2(t) = 0.1t$ for $t \leq 0.1$ and $v_1(t) = t^2$. Since F_1 is concave, it is seller-regular. The strong bidder is also seller-regular (with a jump in virtual values). Given $v_1 = x$, and the maximum valuation $v_2 = y > x$ of the strong bidder, the optimal monopsony price is derived in the same way as in Example D and is given by

$$p(x, y) = \left(\frac{\sqrt{x} + \sqrt{x + 3y}}{3}\right)^2$$

for $x \leq y$. Hence

$$H(x, y) = 2p_2(x, y)[1 - F_1(x)] - 1 + F_1(y)$$

for $x \leq y$. This is just like Example D with x, y switched. Figure 1 illustrates that (S2) holds. Note also from Example D that $\mathrm{d}J_B(x; F_1)/\mathrm{d}x = 3 - x^{-0.5}$ and is less than -4 when $x < 0.01$, and hence the buyer-regularity fails for F_1.

The revenues can easily be computed as follows:

$$\begin{aligned} R^{FR} &= 2\int_0^{0.1} (1 - t)p(t^2, 0.1t)\mathrm{d}t + 2\int_{0.1}^{1} (1 - t)t^2\mathrm{d}t \\ &= 2\int_0^{0.1} (1 - t)\left(\frac{t + \sqrt{t^2 + 0.3t}}{3}\right)^2 \mathrm{d}t + 2\int_{0.1}^{1} (1 - t)t^2\mathrm{d}t = 0.16681595, \end{aligned}$$

and

$$R^{SR} = \int_0^{0.01} (1 - 10x)(1 - x^{0.5})\mathrm{d}x + \int_{0.01}^{1} (1 - x^{0.5})^2\mathrm{d}x = 0.16682333,$$

leading to $R^{FR} < R^{SR}$.

We have the following necessary conditions for revenue ranking in AIPV auctions with resale, which are simple consequences of the necessary conditions for ranking in common-value auctions in Theorem 4. In the following result, j refers to the offer-receiver (the buyer in the case of monopoly and the seller in the case of monopsony).

Theorem 7 *Suppose that F_j is continuously differentiable up to the third derivatives. In the monopoly (monopsony) resale mechanism, if $R^{FR} \geq R^{SR}$ for all F_i in some C^1 neighborhood of F_j satisfying $F_i \geq F_j$ and other conditions in the model, then the following holds*

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[18] Although the density function of F_2 has a kink at $x = 0.01$, the example can be slightly modified to produce an example satisfying all the smooth conditions, and the ranking is still reversed.

$$\frac{\mathrm{d}J_B(x;F_j)}{\mathrm{d}x} \geq -2$$

for all $x \in [0, a_j]$*. If there is some* $d < a_j$ *such that* $R^{FR} \leq R^{SR}$ *for all* F_i *in some* C^1 *neighborhood of* F_j *satisfying* $F_i \geq F_j$, $F_1 = F_2$ *on* $[d, a_j]$ *and other conditions in the model, then the following holds*

$$\frac{\mathrm{d}J_B(x;F_j)}{\mathrm{d}x} \leq -2 \quad \textit{for all} \;\; x \in [0, d].$$

Note that $\frac{\mathrm{d}J_B(x;F_j)}{\mathrm{d}x} > 0$ at $x = a_j$. Therefore, the second inequality in Theorem 7 can only hold outside a neighborhood of a_j. While a sufficient condition for the first-price auction to dominate the second-price auction in revenues is the buyer-regularity of the offer-receiver, i.e., the virtual-value function of the buyer is monotonically increasing, the above necessary condition for the same ranking requires the virtual-value function not decrease too fast and imposes a lower bound of the derivative of the virtual-value function. On the other hand, the necessary condition for the reverse ranking of revenues imposes an upper bound of the derivative of the virtual-value function over an interval excluding the upper bound of the support.

The following example illustrates that for the monopoly resale mechanism, the buyer-regularity fails for the strong bidder and $R^{FR} < R^{SR}$ holds. The idea of this example is similar to E, except that the resale is a monopoly market. The strong bidder is not regular, but the derivative of the buyer-virtual value is below -2. This will lead to higher revenue for the second-price auction with resale.

Example F Consider the monopoly resale mechanism and the value distribution of the strong bidder as $F_2(x) = x^{\frac{1}{2}}$ with support [0, 1]. For $k > 2$, let the weak bidder be defined by

$$\begin{aligned} F_1(x) &= 0.02^{\frac{1}{2}-\frac{1}{k}} x^{\frac{1}{k}}, x \leq 0.02; \\ &= x^{0.5}, 0.02 \leq x \leq 1. \end{aligned}$$

Graphs are shown in Fig. 2. Note that $v_2(t) = t^2$ and from Example E the buyer-regularity fails. The optimal monopoly price is given by $p(x, y) = (\frac{\sqrt{y}+\sqrt{y+3x}}{3})^2$. Figure 1 illustrates that (S1) holds. The revenue from the first-price auction is given by

$$\begin{aligned} R^{FR} &= 2\int_0^1 (1-t)p(v_1(t), v_2(t))\mathrm{d}t \\ &= 2\int_0^{\sqrt{0.02}} (1-t)p(0.02^{1-\frac{k}{2}}t^k, t^2)\mathrm{d}t + 2\int_{\sqrt{0.02}}^1 (1-t)t^2\mathrm{d}t. \end{aligned}$$

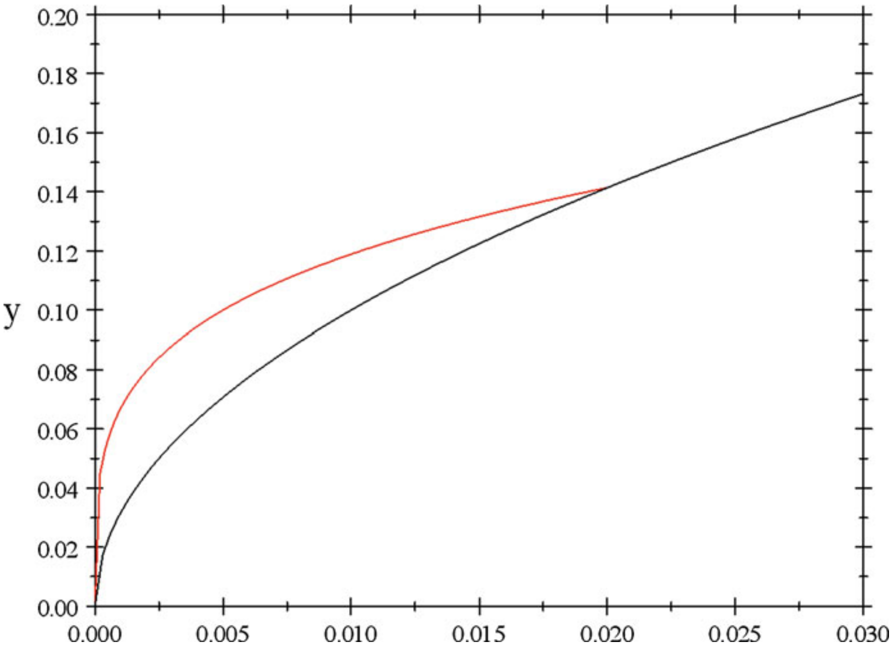

Fig. 2 (Example F): The *upper and lower curves* represent $F_w(x)$ and $F_s(x)$ and coincide for $x \geq 0.02$

When $k = 4$, we have

$$R^{FR} = 2\int_0^{\sqrt{0.02}} (1-t)\left(\frac{t+\sqrt{t^2+150t^4}}{3}\right)^2 \mathrm{d}t + 2\int_{\sqrt{0.02}}^1 (1-t)t^2\mathrm{d}t = 0.1663054,$$

and

$$\begin{aligned} R^{SR} &= \int_0^1 (1-F_1(x))(1-F_2(x))\mathrm{d}x \\ &= \int_0^{0.02} (1-0.02^{0.25}x^{0.25})(1-x^{0.5})\mathrm{d}x + \int_{0.02}^1 (1-x^{0.5})^2\mathrm{d}x \\ &= 0.16631811 > R^{FR}. \end{aligned}$$

Hence the ranking is reversed. When k is larger, the difference in revenues is greater.

5 Discussion

In this paper, we have analyzed two auction models, namely a pure common-value auction and a private-value auction with resale, both of which involving two bidders with asymmetrically and independently distributed private signals. By establishing a close link between the two auction models, we are able to provide sufficient and necessary conditions for ranking revenues between first-price and second-price auctions. Our analysis contributes to the literature on common-value auctions and extend Hafalir and Krishna (2008a) analysis of private-value auctions with resale.

In our analysis, we have made a number of assumptions. In this final section, we discuss how these assumptions may affect our findings. We have focused on two simple resale mechanisms, namely the monopoly and monopsony take-it-or-leave-it price mechanisms. In many applications, it may not be clear at the beginning of the auction who will be the offer-maker in the resale stage. In such situations, one approach is to assume that there is an exogenous random mechanism that determines who makes offers. Let $p(x, y)$ and $r(x, y)$ be the monopoly and monopsony pricing functions, respectively, and π and $1-\pi$ be the probability that the winner and loser of the auction makes offers, respectively. Define a new common-value function

$$w(s_1, s_2) = \pi p(s_1, s_2) + (1 - \pi)r(s_1, s_2).$$

Then the equilibrium bidding strategy of the auction with resale is the same as that of the common-value auction defined by the common-value function $w(s_1, s_2)$. The same first-order condition for both models has also been noted in Hafalir and Krishna (2008a). Thus, our approach of analyzing private-value auctions with resale through the study of the common-value auctions is still applicable. Note that Conditions (C) and (R) are satisfied by $w(s_1, s_2)$ if they are satisfied by both $p(s_1, s_2)$ and $r(s_1, s_2)$. If the first-price auction generates higher revenue than the second-price auction for both the monopoly and monopsony resale mechanisms, it follows immediately that the same is true for the above random offer-maker model for any π. If the ranking is reversed for the monopsony market, on the other hand, then there is some π_0 above which the first-price auction generates higher revenue and below which the second-price auction yields higher revenue.

A broader question is how the ranking result is affected by the type of trading mechanisms in the resale stage. A general pattern seems to be that the direction of ranking is affected by the bargaining power between the buyer and the seller in the resale stage. From this perspective, under the buyer-regularity the weaker bidder has more bargaining power and this favors the first-price auction, since it makes the weak bidder more aggressive in the bidding and raises the auctioneer's revenue in the first-price auction, while the revenue in the second-price auction is not affected by the type of resale mechanisms or bargaining power. The seller-regularity instead may weaken the bargaining power of the weak bidder, and make the strong bidder less aggressive in the bidding as she can let the weak bidder win the auction and then buy it from him. Furthermore, bargaining power can be affected by who is making an offer in the resale stage, as is the case in this paper. A more general approach however deals with a bilateral trade in which both sides may make offers simultaneously as in a double-auction resale, or sequentially as in dynamic bargaining models. These issues are further explored in Cheng (2008), which offers a general message that more bargaining power to the winner increases the first-price auction revenue of the auctioneer, while the revenue of the second-price auction revenue remains the same.

In the case of more than two bidders, ranking revenues across auctions with resale turns out to be a challenging task. Cheng and Tan (2009) attempt to use the same approach to study auctions with resale. There we have shown bid-equivalence between two auction models under some restrictions on the resale mechanism. We believe that bid-equivalence holds more generally and, however, its formulation is still in progress. In addition, the symmetry of the equilibrium bidding strategy is no longer true with more than two bidders. This complicates the revenue formulas used in this paper.

We have made the assumption that the signals are independent in our model, which is shared by Hafalir and Krishna (2008a). When signals are affiliated and asymmetrically distributed, Parreiras (2006) ranks revenues of auctions without resale, where he selects a second-price auction equilibrium which does not correspond to the truth-telling equilibrium for the auction with resale. Consequently, his result will not directly apply to the study of auctions with resale. Ranking auctions with resale in the context of affiliated signals remains an open question. We expect that the bid equivalence result still holds in this case and hence our approach in this paper is applicable. Given the complexity of the revenue formula, ranking revenues requires further investigation.

Appendix: Proofs

Proof of Lemma 1 Note that

$$\frac{\partial H(s_i, s_j)}{\partial s_j} = 2w_{ij}(s_i, s_j)(1 - F_j(s_j)) - 2w_i(s_i, s_j) f_j(s_j).$$

Submodularity of w implies $w_{ij} < 0$, which in turn implies $\frac{\partial H(s_i, s_j)}{\partial s_j} < 0$. It follows from $H(s_i, s_i) = 0$ that $H(s_i, s_j) > 0$ for $s_j < s_i$ and $H(s_i, s_j) < 0$ for $s_j > s_i$. Thus, (R) holds.

To prove part (ii), we take the derivative of both sides of $w(s, s) = s$ with respect to s twice and get

$$w_{11}(s, s) + w_{12}(s, s) + w_{21}(s, s) + w_{22}(s, s) = 0.$$

Symmetry implies

$$w_{ii}(s, s) = -w_{ij}(s, s).$$

Since

$$H(s_i, s_j) = 2w_i(s_i, s_j)(1 - F_j(s_j)) - 1 + F_j(s_i),$$

it follows that

$$\begin{aligned}\frac{\partial H}{\partial s_i}(s_i, s_j) &= 2w_{ii}(s_i, s_j)(1 - F_j(s_j)) + f_j(s_i)\\ &= -2w_{ij}(s_i, s_j)(1 - F_j(s_j)) + f_j(s_i)\end{aligned}$$

By (6), and continuity of $\frac{\partial H}{\partial s_i}$, we have $H(s_i, s_j) < 0$ for (s_i, s_j) in a neighborhood of (s, s). Hence (S) holds in a neighborhood of (s, s).

Proof of Corollary 1 Proposition 1 implies

$$R^F = 2\int_0^1 (1 - t)w(v_1(t), v_2(t))\mathrm{d}t = \int_0^1 (1 - t)(v_1(t) + v_2(t))\mathrm{d}t.$$

Integrating by parts yields

$$\int_0^1 (1 - t)v_1(t)\mathrm{d}t = \frac{1}{2}\int_0^1 (1 - t)^2\mathrm{d}v_1(t) = \frac{1}{2}\int_0^1 (1 - t)^2\mathrm{d}v_1(t)$$

and

$$\int_0^1 (1 - t)v_2(t)\mathrm{d}t = \frac{1}{2}\int_0^1 (1 - t)^2\mathrm{d}v_2(t).$$

The claim follows.

Proof of Proposition 2 Given the selected equilibrium in the second-price auction,

$$b_i(s) = w(s, s) \text{ for } i = 1, 2,$$

the expected revenue is given by

$$\begin{aligned}R^S &= \int_0^a w(x, x)d[1 - (1 - F_1(x))(1 - F_2(x))]\\ &= -\int_0^a w(s, s)d[(1 - F_1(x))(1 - F_2(x))].\end{aligned}$$

The claim then follows from integration by parts,

$$R^S = \int_0^a (1 - F_1(x))(1 - F_2(x)) \mathrm{d}w(x, x).$$

Proof of Proposition 3 This can be proved easily by our revenue formulas. We have

$$\begin{aligned} R^F &= \int_0^1 2(1-t)w(v(t), v(t))\mathrm{d}t = \int_0^1 (1-t)^2 \mathrm{d}w(v(t), v(t)) \\ &= \int_0^a (1 - F(x))^2 \mathrm{d}w(x, x) = R^S. \end{aligned}$$

Proof of Theorem 1 From Proposition 2 and the assumption that $F_1 \neq F_2$ for a subset of non-zero measure, we have

$$\begin{aligned} R^S &= \int_0^{\bar{a}} (1 - F_1(x))(1 - F_2(x))\mathrm{d}w(x, x) \\ &< \frac{1}{2}\int_0^{\bar{a}} [(1 - F_1(x))^2 + (1 - F_2(x))^2]\mathrm{d}w(x, x) \\ &= \frac{1}{2}\int_0^{a_1} (1 - F_1(x))^2 \mathrm{d}w(x, x) + \frac{1}{2}\int_0^{a_2} (1 - F_2(x))^2 \mathrm{d}w(x, x). \end{aligned}$$

It follows from integration by parts and $w(0, 0) = 0$ that

$$\begin{aligned} R^S &< \int_0^{a_1} (1 - F_1(x))w(x, x)dF_1(x) + \int_0^{a_2} (1 - F_2(x))w(x, x)dF_2(x) \\ &= \int_0^1 (1-t)w(v_1(t), v_1(t))\mathrm{d}t + \int_0^1 (1-t)w(v_2(t), v_2(t))\mathrm{d}t \\ &= \int_0^1 (1-t)\left[w(v_1(t), v_1(t)) + w(v_2(t), v_2(t))\right]\mathrm{d}t. \end{aligned}$$

Either (C) or (C′) then implies

$$R^S < 2\int_0^1 (1-t)w(v_1(t), v_2(t))\mathrm{d}t = R^F,$$

and the claim follows.

Proof of Theorem 2 Let $t = F_i(x)$, $h(x) = w(x, x)$, and rewrite the revenue from the second-price auction as follows,

$$\begin{aligned} R^S &= \int_0^1 (1-F_i(x))(1-F_j(x))dh(x) \\ &= \int_0^1 (1-t)(1-F_j(v_i(t)))h'(v_i)v_i'(t)\mathrm{d}t. \end{aligned}$$

Integrating by parts yields

$$\begin{aligned} R^S &= \int_0^1 (1-t)d\left[\int_0^{v_i(t)} (1-F_j(v))h'(v)\mathrm{d}v\right] \\ &= \int_0^1 \left[\int_0^{v_i(t)} (1-F_j(v))h'(v)\mathrm{d}v\right]\mathrm{d}t. \end{aligned}$$

It follows that

$$R^F - R^S = \int_0^1 2(1-t)w(v_i(t), v_j(t))\mathrm{d}t - \int_0^1 \left[\int_0^{v_i(t)} (1-F_j(v))h'(v)\mathrm{d}v\right]\mathrm{d}t.$$

Let $u\,(k,t) = v_j(t) + k(v_i(t) - v_j(t))$, $0 \le k \le 1$, and

$$D(k) = \int_0^1 2(1-t)w(u(k,t), v_j(t))\mathrm{d}t - \int_0^1 \left[\int_0^{u(k,t)} (1-F_j(v))h'(v)\mathrm{d}v\right]\mathrm{d}t.$$

Note that by Proposition 3, $D(0) = 0$. We want to show that if (R) holds then $D'(k) > 0$, which in turn implies $D(1) = R^F - R^S > 0$. Indeed, note that

$$\begin{aligned} D'(k) &= \int_0^1 2(1-t)w_i(u(k,t), v_j(t))(v_i(t) - v_j(t))\mathrm{d}t \\ &\quad - \int_0^1 ((1 - F_j(u(k,t)))h'(u(k,t))(v_i(t) - v_j(t))\mathrm{d}t \\ &= \int_0^1 (v_i(t) - v_j(t))H(u(k,t), v_j(t))\mathrm{d}t. \end{aligned}$$

Since $v_i(t) > v_j(t)$ holds if and only if $u(k,t) > v_j(t)$ for $k > 0$. It follows from (R) and $v_i(t) \neq v_j(t)$ for a subset of non-zero measure that $D'(k) > 0$ for $k > 0$. The claim follows. The proof for the case of (S) is similar.

Proof Theorem 3 If $r \le 0.5$, (4) holds for $s_1 \le s_2$, and the same proof for Theorem 1 applied to the half space implies that $R^F > R^S$. For the case $r > 0.5$, we apply Theorem 2. For Condition (S) to hold for $i = 1$, we need

$$H(s_1, s_2) = 2r[1 - F_2(s_2)] - 1 + F_2(s_1) > 0, \quad \text{for } s_1 < s_2$$

or

$$s_1 > v_2(1 - 2r + 2rF_2(s_2)).$$

Since $H(s_2, s_2) = 2r - 1 > 0$, this inequality holds in a neighborhood of s_2 as long as $s_2 < a_2$. Thus, if

$$v_2(1 - 2r + 2rt) \le v_1(t) \le v_2(t), \quad 0 \le t \le 1,$$

then (S) applies to w, $F_1 = v_1^{-1}$, $F_2 = v_2^{-1}$, and $R^F < R^S$ holds. This proves (ii).

To show part (iii), we apply (S) for $i = 2$ and have

$$H(s_2, s_1) = 2(1 - r)[1 - F_1(s_1)] - 1 + F_1(s_2).$$

When $s_2 < v_1(2r - 1 + (2 - 2r)F_1(s_1))$, we have

$$F_1(s_2) < 2r - 1 + (2 - 2r)F_1(s_1).$$

Thus, if

$$v_1(t) \le v_2(t) \le v_1(2r - 1 + (2 - 2r)t), \quad 0 \le t \le 1,$$

then (S) holds for w, F_1, $F_2 = v_2^{-1}$, and $R^F < R^S$ follows. This proves (iii).

Proof Theorem 4 For simplicity of notations, let $i = 1$. The proof is by contradiction. Suppose the first claim in Theorem 4 does not hold. That is, there exists $s_0 \in (0, a_2)$ such that $H_1(s_0, s_0) < 0$. The continuity of $H_1(s_1, s_2)$ implies that $H_1(s_1, s_2) < 0$ for $s_1 > s_2$ near (s_0, s_0). Since $H(s_0, s_0) = 0$, it follows that $H(s_1, s_2) > 0$ for $s_1 < s_2$ near (s_0, s_0). That is, there exists a neighborhood N around (s_0, s_0) such that $H(s_1, s_2) < 0$ for $(s_1, s_2) \in N$ and $s_1 > s_2$.

Let $v_j(t_0) = s_0$. There exists a smooth function $\delta(t)$ and a neighborhood T of t_0 such that $\delta(t) = 1$ for $t \notin T$ and $1 - \varepsilon < \delta(t) < 1$ for $t \in T$, satisfying $(\delta(t)v_j(t), v_j(t)) \in N$ for $t \in T$. Now, define $v_i(t) = v_j(t)$ for $t \notin T$ and $v_i(t) = \delta(t)v_j(t)$ for $t \in T$. And define $u(k, t) = v_j(t) + k(v_j(t) - v_i(t))$, $k \in [0, 1]$ as in the proof of Theorem 2. It follows from the argument in that proof that $D(k)$ is decreasing in k. Since $D(0) = 0$, we have $D(1) < 0$. That is, for the pair of bidders (v_i, v_j) so defined, we obtain $R^F < R^S$, which contradicts the assumption of the theorem. The claim follows. The proof for the second claim is similar.

Proof of Lemma 2 First note that by the monotone selection theorem, the monopoly price function $p(x, y)$ is increasing within the region $x \leq y$. Fixing y, as x increases beyond y, its increasing property is a consequence of the increasing property in the vertical direction and symmetry. We want to show that $p(x, y)$ is differentiable on the diagonal. Let

$$J(p, y) = p - \frac{F_2(y) - F_2(p)}{f_2(p)},$$

then it follows from the first-order condition that

$$J(p(x, y), y) = x. \tag{14}$$

Note that p is differentiable at (x, x). To see this, taking the partial derivatives of (14) with respect to x and y, respectively, when $x \leq y$, we obtain

$$\frac{\partial p}{\partial x} = \frac{1}{\frac{\partial J}{\partial p}}, \quad \frac{\partial p}{\partial y} = -\frac{\frac{\partial J}{\partial y}}{\frac{\partial J}{\partial p}}.$$

When $x = y = p$, we have $\frac{\partial J}{\partial y} = -1$ and hence $\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y}$. This proves the differentiability of p at the diagonal. Hence p is continuously differentiable everywhere. For the monopsony price function, the first-order condition is similar, with the exception of the trivial corner solution, and hence the proof is identical.

Proof of Lemma 3 In the monopoly case, assume that bidder 1 wins the object and makes an offer to sell the object to bidder 2. The monopoly price $p(x, y)$ satisfies (C) if

$$p(x, y) \geq \frac{x + y}{2}.$$

By definition $z = p(x, y)$ maximizes the following function in z

$$K(z) = [F_2(y) - F_2(z)](z - x) = F_2(y)(z - x) - F_2(z)(z - x),$$

which is concave, as $F_2(z)(z - x)$ is a product of two increasing positive convex functions. It is sufficient to show that

$$K'\left(\frac{x+y}{2}\right) > 0,$$

or

$$F_2(y) - F_2\left(\frac{x+y}{2}\right) - F_2'\left(\frac{x+y}{2}\right)\left(\frac{x+y}{2} - x\right) > 0.$$

Equivalently, it suffices to show that

$$\frac{F_2(y) - F_2\left(\frac{x+y}{2}\right)}{\frac{y-x}{2}} > F_2'\left(\frac{x+y}{2}\right). \tag{15}$$

Note that the left-hand side (15) is the slope of the line through the two points $(\frac{x+y}{2}, F_2(\frac{x+y}{2}))$, $(y, F_2(y))$, while the right-hand side is the slope of F_2 at $\frac{x+y}{2}$. The convexity of F_2 is sufficient for (15) to hold.

In the monopsony case, if bidder 2 loses the auction and makes buying offers to bidder 1, the arguments are similar. Since $z = r(x, y)$ maximizes the following concave objective function in z

$$K(z) = (F_1(z) - F_1(x))(y - z),$$

it suffices to show that

$$K'\left(\frac{x+y}{2}\right) < 0,$$

or

$$F_1'\left(\frac{x+y}{2}\right)\left(y - \frac{x+y}{2}\right) - F_1\left(\frac{x+y}{2}\right) + F_1(x) < 0.$$

Equivalently, we need to show that

$$F_1'\left(\frac{x+y}{2}\right) < \frac{F_1\left(\frac{x+y}{2}\right) - F_1(x)}{\frac{y-x}{2}}. \tag{16}$$

Note that the right-hand side (16) is the slope of the line through the two points $(x, F_1(x))$, $(\frac{x+y}{2}, F_1(\frac{x+y}{2}))$, while the left-hand side is the slope of F_1 at $\frac{x+y}{2}$. The concavity of F_1 is sufficient for (16) to hold. The proof is complete.

Proof of Lemma 4 Let bidder j be the offer-receiver, and assume that F_j is buyer-regular. Fix $x_j = y$ and suppress the variable x_j in the function p by letting $p(x) = p(x, y)$. The first-order condition for the optimal $p(x)$ yields

$$p(x) - x = \frac{F_j(y) - F_j(p(x))}{f_j(p(x))}, \tag{17}$$

or

$$(p(x) - x)f_j(p(x)) + F_j(p(x)) = F_j(y). \tag{18}$$

Taking the derivative of (18) with respect to x yields

$$p'(x) = \frac{1}{2 + (p(x) - x)\frac{f_j'(p(x))}{f_j(p(x))}} > 0. \tag{19}$$

We need to show

$$p'(x) < (>)\frac{1}{2}\frac{1 - F_j(x)}{1 - F_j(y)} \quad \text{when } j = 2(\text{or } 1),$$

or

$$\frac{1}{2 + (p(x) - x)\frac{f_j'(p(x))}{f_j(p(x))}} < (>)\frac{1}{2}\frac{1 - F_j(x_i)}{1 - F_j(x_j)} \quad \text{when } j = 2(\text{or } 1).$$

Since $F_j(x) < (>)F_j(p(x))$ when $j = 2(\text{or } 1)$, it suffices to show

$$\frac{1}{2 + (p(x) - x)\frac{f_j'(p(x))}{f_j(p(x))}} \leq (\geq)\frac{1}{2}\frac{1 - F_j(p(x))}{1 - F_j(y)} \quad \text{when } j = 2(\text{or } 1),$$

which is equivalent to

$$2\frac{F_j(y) - F_j(p(x))}{1 - F_j(p(x))} + (p(x) - x)\frac{f_j'(p(x))}{f_j(p(x))} \geq (\leq)0.$$

Dividing both sides by $F_j(y) - F_j(p(x)) > (<)0$, we need to show

$$\frac{2}{1 - F_j(p(x))} + \frac{p(x) - x}{F_j(y) - F_j(p(x))}\frac{f_j'(p(x))}{f_j(p(x))} \geq 0. \tag{20}$$

Using (17), we know (20) is equivalent to

$$\frac{2}{1 - F_j(p(x))} + \frac{f_j'(p(x))}{f_j(p(x))^2} \geq 0,$$

which is implied by the buyer-regularity of F_j.

The above argument shows that (R1) holds with strict inequality for the monopoly offer by bidder 1 and $j = 2$. In the monopsony case, it also shows (R2) with strict inequality for the monopsony offer by bidder 2, and $j = 1$, $p(x, y) < a_1$. When $p(x, y) = a_1$, we have $p_1(x, y) = 0$. We also have $y > a_1$ and $F_1(y) = 1$. Hence $H(x, y) = 0$ in this case.

Proof of Theorem 6 By Proposition 4, we can apply Theorem 2. In the proof for the monopoly case, we only need (R1) for the monopoly price function and $j = 2$. For the monopsony case, we only need (R2) for the monopsony price function and $j = 1$. We obtain strict inequality if strict inequality in (R1) or (R2) holds for a set of positive measure.

Proof of Theorem 7 By Lemma 2, $p(x, y)$ is symmetric. From (19), we know that p is twice continuously differentiable. Taking the second-order derivative of (19) and evaluating at (y, y), $y < a_j$, we obtain

$$p''(y) = \frac{1}{8}\frac{f_j'(y)}{f_j(y)}.$$

In the proof of Theorem 4, the necessary condition is obtained when restricting F_i to be in a C^1 neighborhood of F_j, and$F_i \geq F_j$. Hence,

$$\frac{1}{8}\frac{f_j'(y)}{f_j(y)} + \frac{1}{2}\frac{f_j(y)}{1 - F_j(y)} \geq 0,$$

or

$$\frac{(1 - F_j(y))f_j'(y)}{f_j^2(y)} + 4 \geq 0.$$

Therefore,

$$\frac{\mathrm{d}J_B(y; F_j)}{\mathrm{d}y} = 2 + \frac{(1 - F_j(y))f_j'(y)}{f_j^2(y)} \geq -2,$$

and the proof is complete. The proof for the monopsony case is similar.

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