

An Inductive Proof of the Berry-Esseen Theorem for Character Ratios

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Abstract: Bolthausen used a variation of Stein's method to give an inductive proof of the Berry-Esseen theorem for sums of independent, identically distributed random variables. We modify this technique to prove a Berry-Esseen theorem for character ratios of a random representation of the symmetric group on transpositions. An analogous result is proved for Jack measure on partitions.

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1. INTRODUCTION

The Plancherel measure of a finite group G is a probability measure on the set of irreducible representations of G which chooses a representation ρ with probability $\frac{\dim(\rho)^2}{|G|}$, where $\dim(\rho)$ denotes the dimension of ρ . For instance if G is the symmetric group, the irreducible representations are parameterized by partitions λ of n , and the Plancherel measure chooses a partition λ with probability $\frac{n!}{\prod_{x \in \lambda} h(x)^2}$ where the product is over boxes in the partition and $h(x)$ is the hooklength of a box. The hooklength of a box x is defined as 1 + number of boxes in same row as x and to the right of x + number of boxes in same column of x and below x . For example we have filled in each box in the partition of 7 below with its hooklength

6	4	2	1
3	1		
1			

and the Plancherel measure would choose this partition with probability $\frac{7!}{(6*4*3*2)^2}$. Recently there has been interest in the statistical properties of partitions chosen from Plancherel measure and we refer the reader to the surveys [AID], [De] and the seminal papers [J], [O1], [BOO] for a glimpse of the remarkable recent work on Plancherel measure. We recommend [Sa] as an introduction to representation theory of the symmetric group.

Let λ be a partition of n chosen from the Plancherel measure of the symmetric group S_n and let $\chi^\lambda(12)$ be the irreducible character parameterized

by λ evaluated on the transposition (12). The quantity $\frac{\chi^\lambda(12)}{\dim(\lambda)}$ is called a character ratio and is crucial for analyzing the convergence rate of the random walk on the symmetric group generated by transpositions [DSh]. In fact Diaconis and Shahshahani prove that the eigenvalues for this random walk are the character ratios $\frac{\chi^\lambda(12)}{\dim(\lambda)}$ each occurring with multiplicity $\dim(\lambda)^2$. Character ratios on transpositions also play an essential role in work on the moduli space of curves [EO], [OP].

Given these motivations, it is natural to study the distribution of the character ratio $\frac{\chi^\lambda(12)}{\dim(\lambda)}$ and there has been a substantial amount of work in this direction, which we now summarize. Kerov [K1] proved that if λ is chosen from the Plancherel measure of the symmetric group, then for all real x_0 ,

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(\frac{n-1}{\sqrt{2}} \frac{\chi^\lambda(12)}{\dim(\lambda)} \leq x_0 \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_0} e^{-\frac{t^2}{2}} dt.$$

The details of Kerov's argument appeared in [IO], which gave a beautiful development of Kerov's work. Hora [Ho] gave another proof of Kerov's result, exploiting the fact that the k th moment of a Plancherel distributed character ratio is equal to the chance that the random walk generated by random transpositions is at the identity after k steps. Both of these proofs were essentially combinatorial in nature and used the method of moments (and so information about all moments of the character ratio). Recent work of Sniady [Sn1], [Sn2] understands these moments in terms of the genus expansion from random matrix theory.

A more probabilistic approach to Kerov's result appeared in [F1], which proved that for all $n \geq 2$ and real x_0 ,

$$\left| \mathbb{P} \left(\frac{n-1}{\sqrt{2}} \frac{\chi^\lambda(12)}{\dim(\lambda)} \leq x_0 \right) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_0} e^{-\frac{t^2}{2}} dt \right| \leq 40.1n^{-1/4}.$$

The proof used Stein's method (which is fundamentally different from the method of moments as it only uses information about a few lower order moments) and random walk on the set of irreducible representations of the symmetric group. Note that unlike Kerov's original result, this result includes an error term. The paper [F3] used martingale theory to sharpen the error term to $C_s n^{-s}$ for any $s < \frac{1}{2}$ where C_s is a constant depending on s . The paper [ShSu] developed a refinement of Stein's method which led to a proof of the conjecture of [F1] that an error term of $Cn^{-1/2}$ holds where C is a universal constant.

The purpose of the present paper is to use a completely different technique to prove the $Cn^{-1/2}$ bound. The method is based on Bolthausen's [Bol] ingenious inductive proof of the Berry-Esseen theorem for sums of independent identically distributed random variables. As in [F3], we write the character ratio as a sum of martingale differences, but these are neither independent nor identically distributed so some subtle combinatorics

is required to adapt Bolthausen's method. This is not the first example of adapting Bolthausen's method to the non i.i.d. case; Bolthausen [Bol] used the approach to study the distribution of $\sum_{1 \leq i \leq n} A_{i\pi(i)}$ where A is a fixed $n \times n$ matrix and π is a random permutation on n symbols. But the case of character ratios is of considerable interest and quite unlike any other example to which his technique has been applied.

Note that using the method of moments, a central limit theorem is known for character ratios on the conjugacy class of i -cycles in the symmetric group, where i is fixed [K1], [IO], [Ho]. The preprint [F4], written after this paper, uses the "exchangeable pairs" version of Stein's method to obtain an $O(n^{-1/4})$ error term for the class of i -cycles. It also gives analogs for other algebraic structures: Gelfand pairs, twisted Gelfand pairs, and association schemes. It would be interesting to extend the technique of this paper to the case of i -cycles. Significant work would be involved in doing this, since the proof of the central limit theorem uses the fact that in Section 2 when we write $\frac{\binom{n}{2}\chi^\lambda(12)}{\dim(\lambda)}$ as a sum of martingale differences, the expected value of the square of a summand given the previous summands is constant. This is false for general conjugacy classes. Also it is a nontrivial combinatorial problem to give upper bounds on the expected absolute value of the cubes of the summands. Fortunately for the case of transpositions this can be done without much difficulty. And the case of transpositions does seem to have unique practical importance [EO], [OP].

The contents of this paper are as follows. Section 2 develops the combinatorics needed to adapt Bolthausen's method to the case of character ratios, and then proves an upper bound of $Cn^{-1/2}$. Section 3 then recalls the Jack_α measure on partitions (here $\alpha > 0$ is a parameter) and why it is interesting. It then briefly indicates the modifications to the Plancherel case needed to prove a central limit theorem with an error term of $C_\alpha n^{-1/2}$, where C_α is a constant depending on α . This organization is natural since many algebraically inclined readers will want to understand the result for character ratios without needing combinatorics of Jack polynomials; thus a useful lemma is given an algebraic proof in Section 2 and a combinatorial proof in Section 3.

2. CENTRAL LIMIT THEOREM FOR PLANCHEREL MEASURE

The random variable we wish to study is $T_n(\lambda) = \frac{\sqrt{\binom{n}{2}}\chi^\lambda(12)}{\dim(\lambda)}$ where λ is chosen from the Plancherel measure of the symmetric group S_n . To begin we write T_n as a sum of other random variables. For this we need Kerov's growth process on partitions [K2]; this has a natural generalization to arbitrary finite groups [F3], but we only recall it in the case of interest. Given a partition $\lambda(j)$ of size j , one obtains a partition $\lambda(j+1)$ of size $j+1$ by choosing $\lambda(j+1)$ with probability $\frac{\dim(\lambda(j+1))}{(j+1)\dim(\lambda(j))}$ if $\lambda(j+1)$ can be obtained from $\lambda(j)$ by adding a single box, and with probability 0 otherwise. Thus starting from $\lambda(1)$, the

unique partition of size 1, one obtains a random sequence $(\lambda(1), \dots, \lambda(n))$ of partitions. Kerov [K2] proves that each $\lambda(j)$ is distributed according to the Plancherel measure of S_j .

Given Kerov's growth process, one can write $T_n = \frac{1}{\sqrt{\binom{n}{2}}}(X_1 + \dots + X_n)$

where $X_1 = 0$, $\chi^{\lambda(1)}(12)$ is defined as 0, and

$$X_j = \frac{\binom{j}{2}\chi^{\lambda(j)}(12)}{\dim(\lambda(j))} - \frac{\binom{j-1}{2}\chi^{\lambda(j-1)}(12)}{\dim(\lambda(j-1))}$$

for $j \geq 2$.

Lemma 2.1 states that the X_j are martingale differences satisfying special properties. We remark that [F3] extends this lemma to more general conjugacy classes and groups. The notation $\mathbb{E}(A|\cdot)$ means the expected value of A given \cdot .

Lemma 2.1. ([F3])

- (1) $\mathbb{E}(X_j|\lambda(j-1)) = 0$ for $2 \leq j \leq n$ and all partitions $\lambda(j-1)$.
- (2) $\mathbb{E}(X_j|T_n) = \frac{j-1}{\sqrt{\binom{n}{2}}}T_n$ for all $1 \leq j \leq n$.
- (3) $\mathbb{E}(X_j^2) = j - 1$.
- (4) $\mathbb{E}(T_n^2) = 1$.

Frobenius [Fr] found the following explicit formula for the character ratio of the symmetric group on transpositions:

$$\frac{\chi^\lambda(12)}{\dim(\lambda)} = \frac{1}{\binom{n}{2}} \sum_i \left(\binom{\lambda_i}{2} - \binom{\lambda'_i}{2} \right)$$

where λ_i is the length of row i of λ and λ'_i is the length of column i of λ . From his formula it follows that $X_j = c(x)$ where x is the box added to $\lambda(j-1)$ to obtain $\lambda(j)$ and the "content" $c(x)$ of a box is defined as column number of box - row number of box.

Lemma 2.2 gives the conditional second and fourth moments of the X_j 's. We emphasize that these were not derived or even stated in terms of character ratios, but rather were proved in a completely combinatorial way by studying the behavior of the moments of $c(x)$ where x is the box added during Kerov's growth process. We remark that for other conjugacy classes, there is not an analog of the fact that $\mathbb{E}(X_j^2|\lambda(j-1))$ is independent of $\lambda(j-1)$.

Lemma 2.2. *Let $\lambda(j-1)$ be a partition of size $j-1 \geq 1$.*

- (1) ([K3]) $\mathbb{E}(X_j^2|\lambda(j-1)) = j - 1$.
- (2) ([La]) $\mathbb{E}(X_j^4|\lambda(j-1)) = \binom{j}{2} + 3 \sum_{x \in \lambda(j-1)} c(x)^2$.

Lemma 2.3 is a useful identity. Although a combinatorial proof can be given using properties of Schur functions, we defer combinatorial arguments to the more general setting of Jack polynomials in Section 3 and give an algebraic proof.

Lemma 2.3. Let $e_r(z_1, \dots, z_n) = \sum_{1 \leq i_1 < \dots < i_r \leq n} z_{i_1} \cdots z_{i_r}$ be the r th elementary symmetric function of z_1, \dots, z_n . For λ a partition of n , let $e_r(\lambda)$ denote the r th elementary symmetric function of the contents of the boxes of λ . Then $\mathbb{E}(e_r(\lambda)) = 0$ for $1 \leq r \leq n$.

Proof. If $r = n$ the result is clear since the box in the first row and column of λ has content 0, so that $e_n(\lambda) = 0$ for all λ .

For $1 \leq r < n$, we use the theory of Murphy elements [Mu]; a friendly reference giving background on these elements is [DG]. For $2 \leq i \leq n$, the i th Murphy element is defined as the sum of transpositions $R_i = \sum_{1 \leq j < i} (j, i)$. Let z be the element of the group algebra of S_n which is the sum of all permutations with $n - r$ cycles. By Proposition 2.1 of [DG], z is the r th elementary symmetric function of the elements R_2, \dots, R_n .

Since the elements R_2, \dots, R_n are simultaneously diagonalizable in every irreducible representation of the symmetric group, it follows from Murphy's determination of their eigenvalues that in the representation of S_n parameterized by λ , z is a scalar multiple of the $\dim(\lambda) \times \dim(\lambda)$ identity matrix with scalar equal to $e_r(\lambda)$. In the regular representation of S_n the irreducible representation parameterized by λ occurs with multiplicity $\dim(\lambda)$. Hence the trace of z in the regular representation is $n! \mathbb{E}(e_r(\lambda))$. But the coefficient of the identity in z is 0, so the trace of z in the regular representation is 0, implying the result. \square

Lemma 2.4 gives upper bounds for $\mathbb{E}(|X_n|^3)$ and for $\mathbb{E}(|T_{n-1}| |X_n|^3)$. One could prove a similar bound (with slightly worse constants) using the concentration inequality for X_n in the proof of Theorem 2.5.

Lemma 2.4. Suppose that $n \geq 3$.

- (1) $\mathbb{E}(|X_n|^3) \leq (n-1)\sqrt{2n-3}$.
- (2) $\mathbb{E}(|T_{n-1}| |X_n|^3) \leq (n-1)\sqrt{2n-3}$.

Proof. By the Cauchy-Schwarz inequality, $\mathbb{E}(|X_n|^3) \leq \sqrt{\mathbb{E}(X_n^2)\mathbb{E}(X_n^4)}$. By Lemma 2.2, $\mathbb{E}(X_n^2) = n-1$ and

$$\mathbb{E}(X_n^4) = \mathbb{E}(\mathbb{E}(X_n^4 | \lambda(n-1))) = \binom{n}{2} + 3\mathbb{E}\left(\sum_{x \in \lambda(n-1)} c(x)^2\right).$$

By Lemma 2.3 with $r = 2$ and then part 4 of Lemma 2.1,

$$\begin{aligned} \mathbb{E}\left(\sum_{x \in \lambda(n-1)} c(x)^2\right) &= \mathbb{E}\left[\left(\sum_{x \in \lambda(n-1)} c(x)\right)^2 - 2e_2(\lambda(n-1))\right] \\ &= \binom{n-1}{2} \mathbb{E}(T_{n-1}^2) \\ &= \binom{n-1}{2}. \end{aligned}$$

This proves the first assertion.

For the second assertion, note (using part 4 of Lemma 2.1 in the final equality) that

$$\begin{aligned} \mathbb{E}(|T_{n-1}||X_n|^3) &= \mathbb{E}(\mathbb{E}(|T_{n-1}||X_n|^3|\lambda(n-1))) \\ &= \mathbb{E}(|T_{n-1}|\mathbb{E}(|X_n|^3|\lambda(n-1))) \\ &\leq \sqrt{\mathbb{E}(T_{n-1}^2)\mathbb{E}(\mathbb{E}(|X_n|^3|\lambda(n-1))^2)} \\ &= \sqrt{\mathbb{E}(\mathbb{E}(|X_n|^3|\lambda(n-1))^2)}. \end{aligned}$$

The conditional version of the Cauchy-Schwarz inequality and part 1 of Lemma 2.2 give that $\mathbb{E}(|X_n|^3|\lambda(n-1))^2$ is at most

$$\mathbb{E}(X_n^2|\lambda(n-1))\mathbb{E}(X_n^4|\lambda(n-1)) = (n-1)\mathbb{E}(X_n^4|\lambda(n-1)).$$

Thus

$$\sqrt{\mathbb{E}(\mathbb{E}(|X_n|^3|\lambda(n-1))^2)} \leq \sqrt{(n-1)\mathbb{E}(X_n^4)},$$

and the proof of the first assertion showed this to equal $(n-1)\sqrt{2n-3}$, as desired. \square

Now we adapt Bolthausen's [Bol] inductive proof of the Berry-Esseen theorem for i.i.d. random variables to the setting of character ratios. We remark that the unpublished notes of Mann [Man] are a useful exposition of Bolthausen's proof and we refer to them in the proof of Theorem 2.5.

Theorem 2.5. *Let λ be chosen from the Plancherel measure on partitions of size n . Then for all $n \geq 2$ and real x_0 ,*

$$\left| \mathbb{P}(T_n(\lambda) \leq x_0) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_0} e^{-\frac{t^2}{2}} dt \right| \leq Cn^{-1/2},$$

where C is a universal constant.

Proof. It is sufficient to prove the result for $n \geq 3$, so we assume this.

For z real, let $h_{z,0} = \mathbb{I}_{(-\infty, z]}$ be the indicator function of the set $(-\infty, z]$. For z real and $b > 0$, let $h_{z,b}$ be the function which is 1 for $x \leq z$ and then drops linearly to the value 0 at $z+b$ and is 0 for $x \geq z+b$. Let

$$\delta(b, n) = \sup_z \{ |\mathbb{E}(h_{z,b}(T_n)) - \Phi h_{z,b}| \}$$

where Φf is the expected value of a function f under the normal distribution. Note that our ultimate goal is to upper bound $\delta(0, n)$.

As in Stein's method [Stn], let

$$f(x) = f_{z,b}(x) = e^{x^2/2} \int_{-\infty}^x (h_{z,b}(w) - \Phi h_{z,b}) e^{-w^2/2} dw.$$

Then $f'(x) - xf(x) = h_{z,b}(x) - \Phi h_{z,b}$, so that

$$\mathbb{E}(h_{z,b}(T_n)) - \Phi h_{z,b} = \mathbb{E}[f'(T_n) - T_n f(T_n)].$$

Part 2 of Lemma 2.1 with $j = n$ implies that

$$\mathbb{E}(X_n f(T_n)) = \mathbb{E}[f(T_n)\mathbb{E}(X_n|T_n)] = \frac{n-1}{\sqrt{\binom{n}{2}}}\mathbb{E}(T_n f(T_n)),$$

so that

$$\mathbb{E}[f'(T_n) - T_n f(T_n)] = \mathbb{E}\left[f'(T_n) - \frac{\sqrt{\binom{n}{2}}}{n-1}X_n f(T_n)\right].$$

By part 1 of Lemma 2.1 and part 1 of Lemma 2.2, this is equal to

$$\begin{aligned} & \mathbb{E}[f'(T_n)] + \mathbb{E}\left[\frac{X_n^2}{n-1}f'\left(\sqrt{\frac{n-2}{n}}T_{n-1}\right) - f'\left(\sqrt{\frac{n-2}{n}}T_{n-1}\right)\right] \\ & - \mathbb{E}\left[\frac{\sqrt{\binom{n}{2}}}{n-1}X_n f(T_n) - \frac{\sqrt{\binom{n}{2}}}{n-1}X_n f\left(\sqrt{\frac{n-2}{n}}T_{n-1}\right)\right] \\ & = \mathbb{E}\left[f'(T_n) - f'\left(\sqrt{\frac{n-2}{n}}T_{n-1}\right)\right] \\ & - \mathbb{E}\left[\frac{X_n^2}{n-1}\int_0^1 f'\left(\sqrt{\frac{n-2}{n}}T_{n-1} + t\frac{X_n}{\sqrt{\binom{n}{2}}}\right) - f'\left(\sqrt{\frac{n-2}{n}}T_{n-1}\right)dt\right]. \end{aligned}$$

Next we upper bound $\mathbb{E}[f'(T_n) - f'\left(\sqrt{\frac{n-2}{n}}T_{n-1}\right)]$. Recall from [Bol] or [Man] that for any x and Δ ,

$$|f'(x + \Delta) - f'(x)| \leq |\Delta| \left(3 + 2|x| + \frac{1}{b}\int_0^1 \mathbb{I}_{[z, z+b]}(x + s\Delta)ds\right).$$

Thus $\mathbb{E}[f'(T_n) - f'\left(\sqrt{\frac{n-2}{n}}T_{n-1}\right)] \leq A_1 + A_2 + A_3$ where

- $A_1 = \frac{3\mathbb{E}(|X_n|)}{\sqrt{\binom{n}{2}}}$.
- $A_2 = \frac{2\sqrt{\frac{n-2}{n}}}{\sqrt{\binom{n}{2}}}\mathbb{E}(|X_n||T_{n-1}|)$.
- $A_3 = \mathbb{E}\left[\frac{|X_n|}{b\sqrt{\binom{n}{2}}}\int_0^1 \mathbb{I}_{[z, z+b]}(\sqrt{\frac{n-2}{n}}T_{n-1} + \frac{sX_n}{\sqrt{\binom{n}{2}}})ds\right]$.

By part 3 of Lemma 2.1, $\mathbb{E}(|X_n|) \leq \sqrt{\mathbb{E}(X_n^2)} = \sqrt{n-1}$; thus $A_1 \leq \frac{3\sqrt{2}}{\sqrt{n}}$. By parts 3 and 4 of Lemma 2.1,

$$\mathbb{E}(|X_n||T_{n-1}|) \leq \sqrt{\mathbb{E}(X_n^2)\mathbb{E}(T_{n-1}^2)} = \sqrt{n-1}.$$

Thus $A_2 \leq \frac{2\sqrt{2}}{\sqrt{n}}$.

Note that $A_3 = A'_3 + A''_3$ where

$$A'_3 = \mathbb{E} \left[\mathbb{I}(|X_n| \leq 2e\sqrt{n}) \frac{|X_n|}{b\sqrt{\binom{n}{2}}} \int_0^1 \mathbb{I}_{[z, z+b]} \left(\sqrt{\frac{n-2}{n}} T_{n-1} + \frac{sX_n}{\sqrt{\binom{n}{2}}} \right) ds \right]$$

and

$$A''_3 = \mathbb{E} \left[\mathbb{I}(|X_n| > 2e\sqrt{n}) \frac{|X_n|}{b\sqrt{\binom{n}{2}}} \int_0^1 \mathbb{I}_{[z, z+b]} \left(\sqrt{\frac{n-2}{n}} T_{n-1} + \frac{sX_n}{\sqrt{\binom{n}{2}}} \right) ds \right].$$

Clearly

$$A'_3 \leq \frac{2e\sqrt{n}}{b\sqrt{\binom{n}{2}}} \mathbb{E} \left[\mathbb{I}_{\left[z - \frac{2e\sqrt{n}}{\sqrt{\binom{n}{2}}}, z+b + \frac{2e\sqrt{n}}{\sqrt{\binom{n}{2}}} \right]} \left(\sqrt{\frac{n-2}{n}} T_{n-1} \right) \right].$$

Now use the fact (explained in [Man]) that

$$0 \leq \mathbb{E}(\mathbb{I}_B(c_1 T_n + c_2)) \leq \frac{|B|}{c_1 \sqrt{2\pi}} + 2\delta(0, n)$$

for any interval B and constants c_1, c_2 with $c_1 \neq 0$. It follows that $A'_3 \leq \frac{D_1}{\sqrt{n}} + \frac{D_2}{bn} + \frac{D_3\delta(0, n-1)}{b\sqrt{n}}$ where D_1, D_2, D_3 are universal constants. To bound A''_3 , note that since $|X_n| \leq n$, one has that $A''_3 \leq \frac{n}{b\sqrt{\binom{n}{2}}} \mathbb{P}(|X_n| > 2e\sqrt{n})$.

The proof of Proposition 4.6 of [F1] derives the concentration inequality $\mathbb{P}(|X_n| > 2e\sqrt{n}) \leq 2e^{-2e\sqrt{n}}$. Since b will later be chosen to be a constant multiplied by $n^{-1/2}$, it follows that A''_3 is much smaller than A'_3 for large n , and one concludes that

$$A_3 \leq \frac{D_1}{\sqrt{n}} + \frac{D_2}{bn} + \frac{D_3\delta(0, n-1)}{b\sqrt{n}}$$

where D_1, D_2, D_3 are universal constants.

Combining the bounds on A_1, A_2, A_3 , we conclude that

$$\mathbb{E} \left[f'(T_n) - f' \left(\sqrt{\frac{n-2}{2}} T_{n-1} \right) \right] \leq \frac{D_1}{\sqrt{n}} + \frac{D_2}{bn} + \frac{D_3\delta(0, n-1)}{b\sqrt{n}}$$

where D_1, D_2, D_3 are universal constants.

Next, we upper bound

$$\mathbb{E} \left| \frac{X_n^2}{n-1} \int_0^1 \left[f' \left(\sqrt{\frac{n-2}{n}} T_{n-1} + t \frac{X_n}{\sqrt{\binom{n}{2}}} \right) - f' \left(\sqrt{\frac{n-2}{n}} T_{n-1} \right) \right] dt \right|.$$

Arguing as in the previous paragraph this is at most $B_1 + B_2 + B_3$ where

- $B_1 = \frac{1}{n-1} \int_0^1 \frac{3t\mathbb{E}(|X_n|^3)}{\sqrt{\binom{n}{2}}} dt = \frac{3\mathbb{E}(|X_n|^3)}{2(n-1)\sqrt{\binom{n}{2}}}$.
- $B_2 = \frac{1}{n-1} \int_0^1 \frac{2t\sqrt{\frac{n-2}{n}}}{\sqrt{\binom{n}{2}}} \mathbb{E}(|T_{n-1}||X_n|^3) dt = \frac{\sqrt{\frac{n-2}{n}}}{(n-1)\sqrt{\binom{n}{2}}} \mathbb{E}(|T_{n-1}||X_n|^3)$.

$$\bullet B_3 = \frac{1}{n-1} \mathbb{E} \left[\frac{|X_n|^3}{b\sqrt{\binom{n}{2}}} \int_0^1 \int_0^1 t \mathbb{I}_{[z, z+b]} \left(\sqrt{\frac{n-2}{n}} T_{n-1} + st \frac{X_n}{\sqrt{\binom{n}{2}}} \right) ds dt \right].$$

To bound B_1 , use part 1 of Lemma 2.4 to conclude that $B_1 \leq \frac{3}{\sqrt{n}}$. To bound B_2 , use part 2 of Lemma 2.4 to conclude that $B_2 \leq \frac{2}{\sqrt{n}}$. To bound B_3 , one uses an argument almost identical to that for A_3 to conclude that

$$B_3 \leq \frac{E_1}{\sqrt{n}} + \frac{E_2}{bn} + \frac{E_3 \delta(0, n-1)}{b\sqrt{n}}$$

where E_1, E_2, E_3 are universal constants.

Summarizing, it has been proved that

$$\delta(b, n) \leq \frac{C_1}{\sqrt{n}} + \frac{C_2}{bn} + \frac{C_3 \delta(0, n-1)}{b\sqrt{n}}$$

where C_1, C_2, C_3 are universal constants. From [Bol] or [Man], $\delta(0, n) \leq \delta(b, n) + \frac{b}{\sqrt{2\pi}}$ for all b , which implies that

$$\delta(0, n) \leq \frac{C_1}{\sqrt{n}} + \frac{C_2}{bn} + \frac{C_3 \delta(0, n-1)}{b\sqrt{n}} + \frac{b}{\sqrt{2\pi}}.$$

We argue by induction that there is a universal constant C so that $\delta(0, n) \leq \frac{C}{\sqrt{n}}$ for all n . Assuming the result for $n-1$, one obtains that

$$\delta(0, n) \leq \frac{C_1}{\sqrt{n}} + \frac{C_2}{bn} + \sqrt{\frac{3}{2}} \frac{C \cdot C_3}{bn} + \frac{b}{\sqrt{2\pi}}.$$

Choosing $b = \frac{2C_3}{\sqrt{n}}$, it follows that if C is sufficiently large, the induction step will work for all n . This completes the proof. \square

To conclude this section, we note that it would be of interest to prove the following (more general) conjecture. An error term of $O(n^{-1/4})$ has recently been established by the method of exchangeable pairs [F4].

Conjecture: Let $i \geq 2$ be fixed. Then for all $n \geq i$ and real x_0 ,

$$\left| \mathbb{P} \left(\sqrt{\frac{n!}{(n-i)!i}} \frac{\chi^\lambda(12 \cdots i)}{\dim(\lambda)} \leq x_0 \right) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_0} e^{-\frac{t^2}{2}} dt \right| \leq C_i n^{-1/2}$$

where C_i is a constant depending on i .

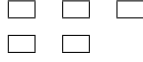
3. CENTRAL LIMIT THEOREM FOR JACK MEASURE

For $\alpha > 0$ the Jack_α measure on partitions of size n chooses a partition λ with probability

$$\frac{\alpha^n n!}{\prod_{x \in \lambda} (\alpha a(x) + l(x) + 1)(\alpha a(x) + l(x) + \alpha)},$$

where the product is over all boxes in the partition. Here $a(x)$ denotes the number of boxes in the same row of x and to the right of x (the ‘‘arm’’ of x)

and $l(x)$ denotes the number of boxes in the same column of x and below x (the “leg” of x). For example the partition of 5 below



would have Jack_α measure

$$\frac{30\alpha^2}{(3\alpha + 1)(\alpha + 2)(2\alpha + 1)(\alpha + 1)^2}.$$

Note that when $\alpha = 1$, Jack measure reduces to Plancherel measure of the symmetric group. The papers [O2], [BO] emphasize that for α fixed the study of Jack_α measure is an important open problem, about which relatively little is known for general values of α . It is a discrete analog of eigenvalue ensembles from random matrix theory and like Jack polynomials [GHJ], should also be relevant to the moduli space of curves.

Given $\alpha > 0$, the quantity to be studied is

$$T_{n,\alpha}(\lambda) = \frac{\sum_i (\alpha \binom{\lambda_i}{2} - \binom{\lambda'_i}{2})}{\sqrt{\alpha \binom{n}{2}}},$$

where as usual λ_i is the length of the i th row of λ and λ'_i is the length of the i th column of λ . It is of interest to study the quantity $T_{n,\alpha}(\lambda)$ under Jack measure for several reasons. When $\alpha = 1$ it reduces to the study of the character ratio of transpositions under Plancherel measure. When $\alpha = 2$ it is a spherical function of the Gelfand pair (S_{2n}, H_{2n}) where H_{2n} is the hyperoctahedral group of size $2^n n!$. Also by Corollary 1 of [DHol], there is a natural random walk on perfect matchings of the complete graph on n vertices, whose eigenvalues are precisely $\frac{T_{n,2}(\lambda)}{\sqrt{n(n-1)}}$, occurring with multiplicity proportional to the Jack_2 measure of λ .

The paper [F2] used the “exchangeable pairs” version of Stein’s method to prove a central limit theorem for $T_{n,\alpha}$ with error term $C_\alpha n^{-1/4}$ where C_α is a constant depending on α . This was sharpened in [F3] using martingales to $C_{\alpha,s} n^{-s}$ for any $s < \frac{1}{2}$.

The main result of this section is Theorem 3.1.

Theorem 3.1. *Suppose that $\alpha \geq 1$ and let λ be chosen from the Jack_α measure on partitions of size n . Then there is a constant C_α depending on α so that for all $n \geq 2$ and real x_0 ,*

$$\left| \mathbb{P}(T_{n,\alpha}(\lambda) \leq x_0) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_0} e^{-\frac{t^2}{2}} dt \right| \leq C_\alpha n^{-1/2}.$$

Note that in Theorem 3.1 we suppose that $\alpha \geq 1$ since the Jack_α probability of λ is equal to the $\text{Jack}_{\frac{1}{\alpha}}$ probability of the transpose of λ , implying that for any x , the Jack_α probability that $T_{n,\alpha} = x$ is equal to the $\text{Jack}_{\frac{1}{\alpha}}$ probability that $T_{n,\frac{1}{\alpha}} = -x$. Also note that C_α must depend on α , since by

Corollary 5.3 of [F2], the random variable $T_{n,\alpha}$ has mean 0, variance 1, and third moment $\frac{\alpha-1}{\sqrt{\alpha\binom{n}{2}}}$.

There is no need to write out a proof of Theorem 3.1, which uses exactly the same logic as that of Theorem 2.5. But it is necessary to give analogs of Lemmas 2.1, 2.2, 2.3, and 2.4, and we do that. The concentration inequality needed for $X_{n,\alpha}$ is Lemma 6.6 of [F2] and can be used to give another proof of some results in this section.

There is an α -analog of Kerov's growth process (due to Kerov [K4]) giving a sequence of partitions $(\lambda(1), \dots, \lambda(n))$ with $\lambda(j)$ distributed according to the Jack_α measure on partitions of size j ; see [F3] for details. Moreover from the definition of $T_{n,\alpha}$, it follows that

$$T_{n,\alpha} = \frac{1}{\sqrt{\alpha\binom{n}{2}}}(X_{1,\alpha} + \dots + X_{n,\alpha}).$$

Here $X_{1,\alpha} = 0$ and if $j \geq 2$ then $X_{j,\alpha} = c_\alpha(x)$ where x is the box added to $\lambda(j-1)$ to obtain $\lambda(j)$ and the " α -content" $c_\alpha(x)$ of a box x is defined to be α (column number of x) - (row number of x).

Lemma 3.2 is an analog of Lemma 2.1 and is generalized in [F3] to arbitrary spherical functions of the Gelfand pair (S_{2n}, H_{2n}) .

Lemma 3.2. ([F3])

- (1) $\mathbb{E}(X_{j,\alpha}|\lambda(j-1)) = 0$ for $2 \leq j \leq n$ and all partitions $\lambda(j-1)$.
- (2) $\mathbb{E}(X_{j,\alpha}|T_{n,\alpha}) = \frac{(j-1)\sqrt{\alpha}}{\sqrt{\binom{n}{2}}}T_{n,\alpha}$ for all $1 \leq j \leq n$.
- (3) $\mathbb{E}(X_{j,\alpha}^2) = \alpha(j-1)$.
- (4) $\mathbb{E}(T_{n,\alpha}^2) = 1$.

Lemma 3.3 is the α version of Lemma 2.2.

Lemma 3.3. Let $\lambda(j-1)$ be a partition of size $j-1 \geq 1$.

- (1) ([K4]) $\mathbb{E}(X_{j,\alpha}^2|\lambda(j-1)) = \alpha(j-1)$.
- (2) ([La])

$$\begin{aligned} \mathbb{E}(X_{j,\alpha}^4|\lambda(j-1)) &= \alpha^2 \binom{j}{2} + \alpha(\alpha-1)^2(j-1) + 3\alpha \sum_{x \in \lambda(j-1)} c_\alpha(x)^2 \\ &\quad + 3\alpha(\alpha-1) \sum_{x \in \lambda(j-1)} c_\alpha(x). \end{aligned}$$

Lemma 3.4 is the α version of Lemma 2.3. The proof is combinatorial, as opposed to the algebraic argument given for Lemma 2.3.

Lemma 3.4. Consider the Jack_α measure on partitions of size n .

- (1) If $m \geq 1$ is an integer then

$$\mathbb{E} \left(\prod_{x \in \lambda} (m + c_\alpha(x)) \right) = m^n.$$

- (2) Let $e_{r,\alpha}(\lambda)$ denote the r th elementary symmetric function of the α -contents of the boxes of λ . Then $\mathbb{E}(e_{r,\alpha}(\lambda)) = 0$ for $1 \leq r \leq n$.

Proof. It suffices to prove the first assertion since the second assertion follows from the first by taking the coefficient of m^{n-r} on both sides. Page 324 of [Mac] proves the identity

$$\sum_{\lambda} b_{\lambda}(q, t) P_{\lambda}(y; q, t) P_{\lambda}(z; q, t) = e^{\sum_{n \geq 1} (\frac{1}{n} \frac{1-t^n}{1-q^n} p_n(y) p_n(z))}$$

where the sum is over all λ of all sizes, $P_{\lambda}(y; q, t)$ denotes a Macdonald symmetric function, $p_n(y) = \sum_i y_i^n$ denotes the n th power sum symmetric function, and $b_{\lambda}(q, t)$ is a number to be discussed more below. We apply the homomorphism of the ring of symmetric functions determined by $p_r(y) \mapsto mu^r$, $p_r(z) \mapsto l^{1-r}$ for all $r \geq 1$ where m, l are positive integers; this is possible since the p_r 's are algebraically independent. Then we take the limit $q = t^{\alpha}$, $t \mapsto 1$ in which Macdonald polynomials become Jack polynomials.

With these substitutions, consider the left hand side of the identity. By pages 380 and 381 of [Mac],

$$\begin{aligned} b_{\lambda}(q, t) &\mapsto \prod_{x \in \lambda} \frac{\alpha a(x) + l(x) + 1}{\alpha a(x) + l(x) + \alpha} \\ P_{\lambda}(y; q, t) &\mapsto u^{|\lambda|} \prod_{x \in \lambda} \frac{m + c_{\alpha}(x)}{\alpha a(x) + l(x) + 1} \\ P_{\lambda}(z; q, t) &\mapsto \frac{1}{l^{|\lambda|}} \prod_{x \in \lambda} \frac{l + c_{\alpha}(x)}{\alpha a(x) + l(x) + 1}. \end{aligned}$$

Letting $l \rightarrow \infty$, one sees that the coefficient of u^n in the left-hand side of the identity is $\frac{1}{\alpha^n n!} \mathbb{E}(\prod_{x \in \lambda} (m + c_{\alpha}(x)))$.

Consider the right hand side of the identity with these substitutions. One obtains $e^{\sum_{n \geq 1} (\frac{mu^n}{n\alpha^{n-1}})}$. Letting $l \rightarrow \infty$, one obtains $e^{\frac{mu}{\alpha}}$, and taking the coefficient of u^n gives $\frac{m^n}{\alpha^n n!}$. Comparing with the previous paragraph proves the first assertion of the lemma. \square

Finally, we give the analog of Lemma 2.4.

Lemma 3.5. *Suppose that $n \geq 3$. There is a constant D_{α} such that*

- (1) $\mathbb{E}(|X_{n,\alpha}|^3) \leq D_{\alpha} n^{3/2}$.
- (2) $\mathbb{E}(|T_{n-1,\alpha}| |X_{n,\alpha}|^3) \leq D_{\alpha} n^{3/2}$.

Proof. The proof method is the same as that of Lemma 2.4, using the Cauchy-Schwarz inequality in the first part and the conditional Cauchy-Schwarz inequality in the second part. One uses that $\mathbb{E}(X_{n,\alpha}^2 | \lambda(n-1)) =$

$\alpha(n-1)$ for all $\lambda(n-1)$ (part 1 of Lemma 3.3). Also one needs that

$$\begin{aligned}\mathbb{E}(X_{n,\alpha}^4) &= \alpha^2 \binom{n}{2} + \alpha(\alpha-1)^2(n-1) + 3\alpha \mathbb{E} \left(\sum_{x \in \lambda(n-1)} c_\alpha(x)^2 \right) \\ &= \alpha^2 \binom{n}{2} + 3\alpha^2 \binom{n-1}{2} + \alpha(\alpha-1)^2(n-1).\end{aligned}$$

The first equality used part 2 of Lemma 3.3 and the fact that $\mathbb{E}(T_{n-1,\alpha}) = 0$. The second equality used part 4 of Lemma 3.2 and Lemma 3.4 with $r = 2$. \square

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