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Zero Biasing and Jack Measures

JASON FULMAN[†] and LARRY GOLDSTEIN[‡]

University of Southern California, Los Angeles, CA 90089-2532, USA
(e-mail: fulman@usc.edu, larry@math.usc.edu)

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The tools of zero biasing are adapted to yield a general result suitable for analysing the behaviour of certain growth processes. The main theorem is applied to prove a central limit theorem, with explicit error terms in the L^1 metric, for a natural statistic of the Jack measure on partitions.

1. Introduction

Zero biasing for the normal approximation of a random variable W using Stein's method was introduced in Goldstein and Reinert [16]. One instance in which the zero bias method may be applied is for W for which a Stein pair W, W' may be constructed, that is, for W that may be coupled to a variable W' such that W, W' is exchangeable and satisfies $\mathbb{E}(W'|W) = (1-a)W$ for some $a \in (0, 1]$. After giving a brief review of these methods in Section 2, in Section 3 we provide a general result allowing one to apply zero biasing when the statistic W of interest is formed by certain growth processes and can be coupled in a Stein pair.

Section 4 studies a certain statistic W_α under the Jack$_\alpha$ measure on partitions. We defer precise definitions to Section 4, but for now mention that it is of interest to study statistical properties of the Jack$_\alpha$ measure. The case $\alpha = 1$ corresponds to the actively studied Plancherel measure of the symmetric group. The surveys [1], [5] and [25] and the seminal papers [2], [20] and [24] indicate how the Plancherel measure of the symmetric group is a discrete analogue of random matrix theory, and describe its importance in representation theory and geometry. Okounkov [25] notes that the study of the Jack$_\alpha$ measure is an important open problem, about which relatively little is known. It is a discrete analogue of Dyson's β ensembles from random matrix theory [3].

The particular statistic W_α under the Jack measure which we study is of interest for several reasons. When $\alpha = 1$ it reduces to the character ratio of transpositions under the Plancherel

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measure, or equivalently to the spectrum of the random transposition walk. Also, by Corollary 1 of [6], there is a natural random walk on perfect matchings of the complete graph on n vertices, whose eigenvalues are precisely $\frac{W(\lambda)}{\sqrt{n(n-1)}}$, occurring with multiplicity proportional to the Jack_2 measure of λ. The proofs to date of the central limit theorem for W_α range from combinatorial ones using the method of moments in [21], [17] and [28], and the use of Stein's method, which produces an error term (but with no explicit constant) in the Kolmogorov metric [7, 8, 27]. Our contribution is to prove a central limit theorem in the L^1 metric, with a small explicit constant.

To close the Introduction we mention that the tools of Section 3 are also useful for studying a growth process arising from the Pólya–Eggenberger urn model. More precisely, imagine an urn $\mathcal{U}_{A,B}$ containing A white balls and B black balls. At each time step one ball is drawn, and returned to the urn along with m balls of the same colour. The eprint [11] contains details for obtaining a central limit theorem with explicit error term for the number of white balls drawn after n steps.

2. Stein's method and zero biasing

Stein's lemma [29] states that a random variable Z has the mean zero normal distribution $\mathcal{N}(0, \sigma^2)$ if and only if

$$\sigma^2 \mathbb{E} f'(Z) = \mathbb{E}[Z f(Z)] \tag{2.1}$$

for all absolutely continuous functions f for which these expectations exist. Motivated by this characterization, for a mean zero, variance σ^2 random variable W and a given function h on which to test the difference between $\mathbb{E}h(W)$ and $Nh = \mathbb{E}h(Z)$, Stein [29] considered the differential equation

$$\sigma^2 f'(w) - w f(w) = h(w) - Nh. \tag{2.2}$$

For the unique bounded solution h of (2.2), one can evaluate the required difference by substituting W for w and taking expectation, to yield

$$\mathbb{E}h(W) - Nh = \mathbb{E}[\sigma^2 f'(W) - W f(W)].$$

Though it may not be immediately clear why the right-hand side may be simpler to evaluate than the left, a variety of techniques have been developed to handle various situations. For instance, the exchangeable pair technique, from [30], handles the expectation of the right-hand side when the given random variable W can be coupled to W' so that (W, W') is an a-Stein pair, that is, an exchangeable pair that satisfies

$$\mathbb{E}(W'|W) = (1-a)W \quad \text{for some } a \in (0, 1]. \tag{2.3}$$

Other techniques for handling the Stein equation are discussed in detail in [4] and in the references therein, but of particular relevance here is the zero bias coupling, which we now review.

Though the mean zero normal is the unique distribution satisfying (2.1), one can ask whether a given variable satisfies a like identity of its own. Indeed, it is shown in [16] that for every mean zero, variance σ^2 random variable X, there exists a distribution for a random variable X^*, termed the X-zero biased distribution, such that

$$\sigma^2 \mathbb{E} f'(X^*) = \mathbb{E}[X f(X)] \tag{2.4}$$

for all absolutely continuous functions f for which these expectations exist. The mapping of $\mathcal{L}(X)$, the distribution of X, to $\mathcal{L}(X^*)$, is known as the zero bias transformation. In particular, Stein's lemma (2.1) can be rephrased as the statement that the mean zero normal $\mathcal{N}(0, \sigma^2)$ is the unique fixed point of the zero bias transformation characterized by (2.4).

Heuristically, then, if the transformation has a fixed point at the mean zero normal, then an approximate fixed point should be approximately normal. This heuristic has been made precise for a variety of examples in [16], [12], [13], [14] and [15] (see also [4]) in order to yield bounds in both the Kolmogorov and L^1 metric. For the latter, the following result from [15] is often useful; we use $\|\cdot\|_1$ to denote the L^1 metric.

Theorem 2.1. *If the mean zero, variance* 1 *random variable* W *can be coupled to* W^* *having the* W*-zero bias distribution, then*

$$\|\mathcal{L}(W) - \mathcal{L}(Z)\|_1 \leqslant 2\mathbb{E}|W^* - W|,$$

where Z *is a standard normal variable.*

Hence, to obtain L^1 bounds, the question reduces to finding a way to couple W and W^*. Lemma 2.2 below, from [16], noting here that the result holds also for $a = 1$, shows how the construction of a variable W^* with the W-zero bias distribution can be achieved with the help of the distribution $dF(w, w')$ of a Stein pair. First, it can easily be shown from (2.3) that if W, W' is an a-Stein pair possessing second moments, then

$$\mathbb{E}W = 0 \quad \text{and} \quad \mathbb{E}(W' - W)^2 = 2a\,\mathrm{Var}(W), \tag{2.5}$$

so in particular,

$$dF^\dagger(w, w') = \frac{(w' - w)^2}{2a} dF(w, w') \tag{2.6}$$

is a bivariate distribution.

Lemma 2.2. *If* $W^\dagger, W^\ddagger$ *have distribution* (2.6)*, where* $F(w, w')$ *is the joint distribution of an* a*-Stein pair, and* U *is a uniformly distributed variable, independent of* $W^\dagger, W^\ddagger$*, then*

$$W^* = UW^\dagger + (1 - U)W^\ddagger$$

has the W*-zero bias distribution.*

In particular, if W and $W^\dagger, W^\ddagger$ can be constructed on a common space, then W and W^* can be also.

We remark that a number of results are available when (W, W') is only an approximate Stein pair, that is, an exchangeable pair that satisfies the linearity condition (2.3) with a remainder: see, for instance, [26], and [4]. Correspondingly, here we expect the conclusions of Theorems 2.1 and 3.1 to hold for approximate Stein pairs by including in the bounds the additional terms that arise from such remainders.

In what follows we study processes for which the random variable W of interest can be written as the sum $V + T$, where $V = V_\tau$, a function of a variable τ determined by the process run to a penultimate state, and T a function of running the process for one additional step. In our examples, given a state τ that determines V, a Stein pair $(W, W') = (V + T, V + T')$ can be constructed by running two copies of the last step of a chain, forming T and T' conditionally independent given τ.

In such cases the equality $W' - W = T' - T$ provides a factored representation of the joint distribution (2.6) that implies that the required pair $(W^\dagger, W^\ddagger)$ of random variables can be constructed by forming $(V^\square + T^\dagger_{\tau^\square}, V^\square + T^\ddagger_{\tau^\square})$, where $V^\square$ is a function of a state $\tau^\square$ having the distribution of τ biased in a particular way, and $(T^\dagger_{\tau^\square}, T^\dagger_{\tau^\square})$ is obtained by square biasing the conditional distributions of T and T' given $\tau^\square$ in a way similar to (2.6). By Lemma 2.2 we thereby obtain that

$$W^* = V^\square + T^{\tau^\square}, \quad \text{where} \quad T^{\tau^\square} = UT^\dagger_{\tau^\square} + (1-U)T^\ddagger_{\tau^\square},$$

has the W-zero biased distribution.

Our application of Theorem 3.1 to the Jack measure is particularly simple since the biasing factor to form the $\tau^\square$ distribution from that of τ is unity, and we may therefore take $V = V^\square$. For the Pólya–Eggenberger urn example, discussed in [11], it is shown that biasing draws from the urn $\mathcal{U}_{A,B}$ in our process results in the urn $\mathcal{U}_{A+m,B+m}$.

3. General result

The purpose of this section is to prove the following theorem.

Theorem 3.1. *Consider a bivariate distribution $\mathcal{L}(\tau, T)$ on a random object τ and random variable T, and a τ-measurable random variable $V = V_\tau$ such that sampling τ, and then, given τ, sampling T and T' independently from the conditional distribution $\mathcal{L}(T|\tau)$, the random variables*

$$W = V + T \quad \textit{and} \quad W' = V + T' \tag{3.1}$$

have variance one and are an a-Stein pair. Denoting

$$\mathbb{E}(T|\tau) = \mu_\tau \quad \textit{and} \quad \mathbb{E}((T-\mu_\tau)^2|\tau) = \sigma^2_\tau, \tag{3.2}$$

and the distribution of τ by $dF(\tau)$, the measure $F^\square(\tau)$ specified by

$$dF^\square(\tau) = \frac{\sigma^2_\tau}{a} dF(\tau) \tag{3.3}$$

is a probability measure, and for any coupling of τ to $\tau^\square$ with distribution (3.3), *we have*

$$\|\mathcal{L}(W) - \mathcal{L}(Z)\|_1 \leqslant 2\mathbb{E}|(V_{\tau^\square} - V) + (\mu_{\tau^\square} - \mu_\tau)| + 2\mathbb{E}|T - \mu_\tau| + \frac{\mathbb{E}|T-\mu_\tau|^3}{\mathrm{Var}(T-\mu_\tau)}. \tag{3.4}$$

When μ_τ equals zero and σ^2_τ is constant almost surely, then

$$\|\mathcal{L}(W) - \mathcal{L}(Z)\|_1 \leqslant 2\mathbb{E}|T| + \frac{\mathbb{E}|T^3|}{\mathrm{Var}(T)}. \tag{3.5}$$

Proof. First consider the case where $\mu_\tau = 0$ a.s. Since, conditional on τ, the pair T and T' are independent, we have $\mathbb{E}[T'T|\tau] = \mathbb{E}[T'|\tau]\mathbb{E}[T|\tau] = 0$, and therefore, from (3.1) and (3.2),

$$\mathbb{E}((W'-W)^2|\tau) = \mathbb{E}((T'-T)^2|\tau) = 2\sigma_\tau^2. \tag{3.6}$$

Taking expectation and applying (2.5), we have that

$$\mathbb{E}\sigma_\tau^2 = a, \tag{3.7}$$

verifying that $dF^\square(\tau)$ is a probability measure.

By construction, the joint distribution of (T, T', τ) is, with some abuse of notation, given by

$$dF(t,t',\tau) = dF(t'|\tau)dF(t|\tau)dF(\tau),$$

and therefore the pair (W, W') has distribution

$$dF(w,w') = \int_{\tau,t,t':v+t=w,v+t'=w'} dF(t'|\tau)dF(t|\tau)dF(\tau), \tag{3.8}$$

where $v = V_\tau$. By Lemma 2.2, with U an independent uniform random variable on $[0,1]$,

$$W^* = UW^\dagger + (1-U)W^\ddagger$$

has the W-zero bias distribution when $(W^\dagger, W^\ddagger)$ has distribution given by

$$dF^\dagger(w,w') = \frac{(w'-w)^2}{2a}dF(w,w').$$

For any fixed τ let $F(t|\tau)$ denote the conditional distribution of T given τ. By (3.6), for every τ the measure

$$dF_\tau^\dagger(t,t') = \frac{(t'-t)^2}{2\sigma_\tau^2}dF(t'|\tau)dF(t|\tau) \tag{3.9}$$

is a bivariate probability distribution.

Now, using (3.8), (3.7) and (3.9),

$$\begin{aligned}
dF^\dagger(w,w') &= \frac{(w'-w)^2}{2a}\int_{\tau,t,t':v+t=w,v+t'=w'} dF(t'|\tau)dF(t|\tau)dF(\tau)\\
&= \int_{\tau,t,t':v+t=w,v+t'=w'} \frac{(w-w')^2}{2a}dF(t'|\tau)dF(t|\tau)dF(\tau)\\
&= \int_{\tau,t,t':v+t=w,v+t'=w'} \frac{\sigma_\tau^2}{a}\frac{(t'-t)^2}{2\sigma_\tau^2}dF(t'|\tau)dF(t|\tau)dF(\tau)\\
&= \int_\tau \left(\int_{t,t':v+t=w,v+t'=w'} \frac{(t'-t)^2}{2\sigma_\tau^2}dF(t'|\tau)dF(t|\tau)\right)\frac{\sigma_\tau^2}{a}dF(\tau)\\
&= \int_\tau \left(\int_{t,t':v+t=w,v+t'=w'} dF_\tau^\dagger(t',t)\right)dF^\square(\tau).
\end{aligned} \tag{3.10}$$

The factorization in the integral indicates that, given $\tau^\square$ with distribution $dF^\square(\tau)$, the pair $(W^\dagger, W^\ddagger)$ can be generated by sampling $T_{\tau^\square}^\dagger, T_{\tau^\square}^\ddagger$ from $dF_{\tau^\square}^\dagger(t',t)$, and then setting

$$W^\dagger = V_{\tau^\square} + T_{\tau^\square}^\dagger \quad \text{and} \quad W^\ddagger = V_{\tau^\square} + T_{\tau^\square}^\ddagger,$$

where $V_{\tau^\square}$ is the value of V on $\tau^\square$. In particular, letting

$$T^{\tau^\square} = UT^\dagger_{\tau^\square} + (1-U)T^\ddagger_{\tau^\square}, \tag{3.11}$$

we have that

$$W^* = U(V_{\tau^\square} + T^\dagger_{\tau^\square}) + (1-U)(V_{\tau^\square} + T^\ddagger_{\tau^\square}) = V_{\tau^\square} + T^{\tau^\square}$$

has the W-zero biased distribution.

For a fixed τ, let T_τ and T'_τ denote independent copies of a random variable with distribution $dF(t|\tau)$. Clearly T_τ and T'_τ are exchangeable, and as $\mu_\tau = 0$, we have $\mathbb{E}(T) = \mathbb{E}(\mathbb{E}(T|\tau)) = \mathbb{E}\mu_\tau = 0$ and therefore $\mathbb{E}(T'|T) = \mathbb{E}(T') = 0$. Hence (T, T') is a 1-Stein pair. In view of (3.9), Lemma 2.2 yields that when $T^\dagger_\tau, T^\ddagger_\tau$ have distribution $F^\dagger_\tau(t,t')$ and U is an independent uniform random variable,

$$T^*_\tau = UT^\dagger_\tau + (1-U)T^\ddagger_\tau \tag{3.12}$$

has the T_τ-zero biased distribution.

As $\mathbb{E}(T) = 0$, by (3.7) we obtain

$$a = \mathbb{E}\sigma^2_\tau = \mathbb{E}(\mathbb{E}(T^2|\tau)) = \mathbb{E}(T^2) = \operatorname{Var}(T).$$

Comparing (3.11) and (3.12), we see that the distribution $\mathcal{L}(T^{\tau^\square})$ is the mixture of the distributions $\mathcal{L}(T^*_\tau)$ with mixing measure $\sigma^2_\tau/\operatorname{Var}(T)$, by (3.10). Therefore, by Theorem 2.1 of [14], $T^{\tau^\square}$ has the T-zero bias distribution. Applying the zero bias identity (2.4) with $f(x) = (1/2)x^2\operatorname{sign}(x)$, we have

$$\mathbb{E}|T^{\tau^\square}| = \frac{\mathbb{E}|T^3|}{2\operatorname{Var}(T)}.$$

Now, with τ and $\tau^\square$ the given coupling, letting $V = V_\tau$ and T be sampled from $\mathcal{L}(T|\tau)$, setting $(W, W^*) = (V + T, V^\square + T^{\tau^\square})$ yields a coupling of W and W^* on the same space, satisfying

$$\begin{aligned}\mathbb{E}|W^* - W| &= \mathbb{E}|V_{\tau^\square} - V + T^{\tau^\square} - T| \\ &\leqslant \mathbb{E}|V_{\tau^\square} - V| + \mathbb{E}|T| + \mathbb{E}|T^{\tau^\square}| \\ &= \mathbb{E}|V_{\tau^\square} - V| + \mathbb{E}|T| + \frac{\mathbb{E}|T^3|}{2\operatorname{Var}(T)}.\end{aligned}$$

Theorem 2.1 now yields

$$\|\mathcal{L}(W) - \mathcal{L}(Z)\|_1 \leqslant 2\mathbb{E}|V_{\tau^\square} - V| + 2\mathbb{E}|T| + \frac{\mathbb{E}|T^3|}{\operatorname{Var}(T)}. \tag{3.13}$$

When σ^2_τ is constant we have that $dF^\square(\tau) = dF(\tau)$, and hence may let $\tau^\square = \tau$; taking $V_{\tau^\square} = V$ in (3.13) now yields (3.5).

To obtain the result for general μ_τ, we reduce to the case $\mu_\tau = 0$ by writing

$$(W, W') = (V + T, V + T') = ((V + \mu_\tau) + (T - \mu_\tau), (V + \mu_\tau) + (T' - \mu_\tau)).$$

Replacing V and T in (3.13) by $V + \mu_\tau$ and $T - \mu_\tau$, respectively, yields (3.4). □

4. The Jack measure

In this section we apply Theorem 3.1 to study a property of the Jack_α measure on the set of partitions of size n. For $\alpha > 0$ the Jack_α measure chooses a partition λ of size n with probability

$$\text{Jack}_\alpha(\lambda) = \frac{\alpha^n n!}{\prod_{x\in\lambda}(\alpha a(x) + l(x) + 1)(\alpha a(x) + l(x) + \alpha)}, \tag{4.1}$$

where in the product over all boxes x in the partition λ, $a(x)$ denotes the number of boxes in the same row of x and to the right of x (the 'arm' of x), and $l(x)$ denotes the number of boxes in the same column of x and below x (the 'leg' of x). For example, one calculates that the partition

$$\lambda = \begin{array}{lll}\square & \square & \square \\ \square & \square & \end{array}$$

of 5 has Jack_α measure

$$\text{Jack}_\alpha(\lambda) = \frac{60\alpha^2}{(2\alpha+2)(3\alpha+1)(\alpha+2)(2\alpha+1)(\alpha+1)}.$$

With λ having the Jack_α distribution, we apply the theory of Section 3 to prove an explicit L_1 normal approximation bound for the statistic

$$W_\alpha(\lambda) = \frac{\sum_{x\in\lambda} c_\alpha(x)}{\sqrt{\alpha\binom{n}{2}}},$$

where $c_\alpha(x)$ denotes the 'α-content' of x, defined as

$$c_\alpha(x) = \alpha(\text{column number of } x - 1) - (\text{row number of } x - 1).$$

In the diagram below, representing a partition of 7, each box is filled with its α-content:

0	α	$2\,\alpha$	$3\,\alpha$
-1	$\alpha - 1$		
-2			

.

In the Kolmogorov metric, the paper [7] proved an $O(n^{-1/4})$ error term for the normal approximation of W_α; this rate was sharpened in [10] using martingales to $O(n^{(-1/2)+\epsilon})$ for any $\epsilon > 0$ and in [9] to $O(n^{-1/2})$ using Bolthausen's inductive approach to Stein's method, but without an explicit constant. Hora and Obata [18] prove a central limit theorem, with no error term, for W_α using quantum probability. Here we give an explicit L_1 bound to the normal with small constants.

To obtain our bound we construct an exchangeable pair using Kerov's growth process for generating a random partition distributed according to the Jack_α measure. Given a box x in the diagram of λ, again letting $a(x)$ and $l(x)$ denote the arm and leg of x respectively, set

$$c_\lambda(\alpha) = \prod_{x\in\lambda}(\alpha a(x) + l(x) + 1), \quad c'_\lambda(\alpha) = \prod_{x\in\lambda}(\alpha a(x) + l(x) + \alpha)$$

and, for τ a partition obtained from λ by removing a single corner box,

$$\psi'_{\lambda/\tau}(\alpha) = \prod_{x\in C_{\lambda/\tau} - R_{\lambda/\tau}} \frac{(\alpha a_\lambda(x) + l_\lambda(x) + 1)}{(\alpha a_\lambda(x) + l_\lambda(x) + \alpha)} \frac{(\alpha a_\tau(x) + l_\tau(x) + \alpha)}{(\alpha a_\tau(x) + l_\tau(x) + 1)},$$

where $C_{\lambda/\tau}$ is the union of columns of λ that intersect $\lambda - \tau$ and $R_{\lambda/\tau}$ is the union of rows of λ that intersect $\lambda - \tau$.

The state of Kerov's growth process at times $n = 1, 2, \ldots$ is a partition of size n, starting at time one with the unique partition of 1. If at stage $n-1$ the state of the process is the partition τ, a transition to the partition λ occurs with probability

$$\frac{c_\tau(\alpha)}{c_\lambda(\alpha)}\psi'_{\lambda/\tau}(\alpha).$$

As shown in [22] and [10], if τ is chosen from the Jack$_\alpha$ measure on partitions of size $n-1$, then transitioning according to this rule results in a partition λ of n distributed according to the Jack$_\alpha$ measure.

We now present an L^1 bound for the normal approximation of W_α.

Theorem 4.1. *Let*

$$W_\alpha(\lambda) = \frac{\sum_{x\in\lambda} c_\alpha(x)}{\sqrt{\alpha\binom{n}{2}}} \tag{4.2}$$

and let W_α be the value of $W_\alpha(\lambda)$ when λ has the Jack$_\alpha$ measure distribution for some $\alpha > 0$. Then, for Z a standard normal random variable,

$$\|\mathcal{L}(W_\alpha) - \mathcal{L}(Z)\|_1 \leqslant \sqrt{\frac{2}{n}}\left(2 + \sqrt{2 + \frac{\max(\alpha, 1/\alpha)}{n-1}}\right). \tag{4.3}$$

Proof. First we show that (4.3) holds for all $\alpha \geqslant 1$. Constructing τ from the Jack measure on partitions of size $n-1$ and then taking one step in Kerov's growth process yields λ with the Jack measure on partitions of size n, and we may write

$$W_\alpha = V + T,$$

where

$$V = \frac{\sum_{x\in\tau} c_\alpha(x)}{\sqrt{\alpha\binom{n}{2}}} \quad \text{and} \quad T = \frac{c_\alpha(\lambda/\tau)}{\sqrt{\alpha\binom{n}{2}}},$$

and $c_\alpha(\lambda/\tau)$ denotes the α-content of the box added to τ to form λ.

It is shown in [7] that constructing λ' by taking another step in Kerov's growth process from τ, independently of λ/τ given τ, and then forming W'_α from λ' as W is formed from λ, results in exchangeable variables W_α, W'_α that satisfy (2.3) with $a = 2/n$. Hence, (3.1) of Theorem 3.1 is satisfied. Corollary 5.3 of [7] gives that $\mathrm{Var}(W) = 1$.

From Section 3 of [9], one recalls the following three facts:

(1) $\mathbb{E}[T|\tau] = 0$ for all τ,
(2) $\mathbb{E}[T^2|\tau] = \frac{2}{n}$ for all τ,
(3) $\mathbb{E}[T^4] = \frac{\alpha^2\binom{n}{2} + \alpha(\alpha-1)^2(n-1) + 3\alpha^2\binom{n-1}{2}}{\alpha^2\binom{n}{2}^2}$.

As V is measurable with respect to the σ-algebra generated by τ, condition (3.2) is satisfied. From properties (1) and (2) above we have, respectively, that $\mu_\tau = 0$ and σ_τ^2 is a constant, almost surely. Hence the bound (3.5) of Theorem 3.1 holds.

Applying the Cauchy–Schwarz inequality gives that $\mathbb{E}|T| \leqslant \sqrt{\mathbb{E}T^2} = \sqrt{2/n}$, accounting for the first term in the bound. From property (3), now applying $\alpha \geqslant 1$, we have

$$\mathbb{E}[T^4] \leqslant \left[\frac{\binom{n}{2} + 3\binom{n-1}{2}}{\binom{n}{2}^2}\right] + \frac{\alpha(n-1)}{\binom{n}{2}^2} \leqslant \frac{8}{n^2} + \frac{4\alpha}{n^2(n-1)}.$$

The Cauchy–Schwarz inequality gives that $\mathbb{E}|T^3| \leqslant \sqrt{\mathbb{E}[T^2]\mathbb{E}[T^4]}$, and properties (1) and (2) give $\mathrm{Var}(T) = 2/n$, yielding the final term in the bound (4.3). Thus the result is shown when $\alpha \geqslant 1$.

To obtain a bound for all $\alpha > 0$, note first that when taking the transpose λ^t of a partition λ the roles of the arms $a(x)$ and legs $l(x)$ become interchanged; hence, letting λ_α be a partition with the Jack$_\alpha$ distribution, from (4.1), for all $\alpha > 0$ we have

$$\mathcal{L}(\lambda_\alpha) = \mathcal{L}(\lambda^t_{1/\alpha}).$$

Next, as $W_\alpha(\lambda) = -W_{1/\alpha}(\lambda^t)$ for all λ, and $\mathcal{L}(Z) = \mathcal{L}(-Z)$,

$$\begin{aligned}\|\mathcal{L}(W_\alpha(\lambda_\alpha)) - \mathcal{L}(Z)\|_1 &= \|\mathcal{L}(-W_{1/\alpha}(\lambda^t_\alpha)) - \mathcal{L}(Z)\|_1\\ &= \|\mathcal{L}(-W_{1/\alpha}(\lambda_{1/\alpha})) - \mathcal{L}(-Z)\|_1\\ &= \|\mathcal{L}(W_{1/\alpha}(\lambda_{1/\alpha})) - \mathcal{L}(Z)\|_1.\end{aligned}$$

Hence, as the bound (4.3) holds for all $\alpha \geqslant 1$, it holds for all $\alpha > 0$. □

References

[1] Aldous, D. and Diaconis, P. (1999) Longest increasing subsequences: From patience sorting to the Baik–Deift–Johansson theorem. *Bull. Amer. Math. Soc.* (NS) **36** 413–432.

[2] Borodin, A., Okounkov, A. and Olshanski, G. (2000) Asymptotics of Plancherel measure for symmetric groups. *J. Amer. Math. Soc.* **13** 481–515.

[3] Borodin, A. and Olshanski, G. (2005) Z-measures on partitions and their scaling limits. *Europ. J. Combin.* **26** 795–834.

[4] Chen, L., Goldstein, L. and Shao, Q. (2010) *Normal Approximation by Stein's Method*, Springer.

[5] Deift, P. (2000) Integrable systems and combinatorial theory. *Notices Amer. Math. Soc.* **47** 631–640.

[6] Diaconis, P. and Holmes, S. (2002) Random walks on trees and matchings. *Electron. J. Probab.* **7**.

[7] Fulman, J. (2004) Stein's method, Jack measure, and the Metropolis algorithm. *J. Combin. Theory Ser. A* **108** 275–296.

[8] Fulman, J. (2005) Stein's method and Plancherel measure of the symmetric group. *Trans. Amer. Math. Soc.* **357** 555–570.

[9] Fulman, J. (2006) An inductive proof of the Berry–Esseen theorem for character ratios. *Ann. Combin.* **10** 319–332.

[10] Fulman, J. (2006) Martingales and character ratios. *Trans. Amer. Math. Soc.* **358** 4533–4552.

[11] Fulman, J. and Goldstein, L. (2010) Zero biasing and growth processes. `arXiv:1010.4759`

[12] Goldstein, L. (2005) Berry–Esseen bounds for combinatorial central limit theorems and pattern occurrences, using zero and size biasing. *J. Appl. Probab.* **42** 661–683.

[13] Goldstein, L. (2007) L^1 bounds in normal approximation. *Ann. Probab.* **35** 1888–1930.

[14] Goldstein, L. (2010) Bounds on the constant in the mean central limit theorem. *Ann. Probab.* **38** 1672–1689.

[15] Goldstein, L. (2004) Normal approximation for hierarchical sequences. *Ann. Appl. Probab.* **14** 1950–1969.

[16] Goldstein, L. and Reinert, G. (1997) Stein's method and the zero bias transformation with application to simple random sampling. *Ann. Appl. Probab.* **7** 935–952.

[17] Hora, A. (1998) Central limit theorem for the adjacency operators on the infinite symmetric group. *Comm. Math. Phys.* **195** 405–416.

[18] Hora, A. and Obata, N. (2007) *Quantum Probability and Spectral Analysis of Graphs*, Theoretical and Mathematical Physics, Springer.

[19] Ivanov, V. and Olshanski, G. (2002) Kerov's central limit theorem for the Plancherel measure on Young diagrams. In *Symmetric Functions 2001: Surveys of Developments and Perspectives*, Kluwer.

[20] Johansson, K. (2001) Discrete orthogonal polynomial ensembles and the Plancherel measure. *Ann. of Math.* **153** 259–296.

[21] Kerov, S. V. (1993) Gaussian limit for the Plancherel measure of the symmetric group. *Compt. Rend. Acad. Sci. Paris Ser. I* **316** 303–308.

[22] Kerov, S. V. (2000) Anisotropic Young diagrams and Jack symmetric functions. *Funct. Anal. Appl.* **34** 41–51.

[23] Kerov, S. V. (1996) The boundary of Young lattice and random Young tableaux. In *Formal Power Series and Algebraic Combinatorics: New Brunswick, NJ, 1994*, Vol. 24 of *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, AMS.

[24] Okounkov, A. (2000) Random matrices and random permutations. *Internat. Math. Res. Notices* **20** 1043–1095.

[25] Okounkov, A. (2005) The uses of random partitions. In *XIVth International Congress on Mathematical Physics*, World Scientific, pp. 379–403.

[26] Rinott, Y. and Rotar, V (1997) On coupling constructions and rates in the CLT for dependent summands with applications to the antivoter model and weighted U-statistics. *Ann. Appl. Probab.* **7** 1080–1105.

[27] Shao, Q. and Su, Z. (2006) The Berry–Esseen bound for character ratios. *Proc. Amer. Math. Soc.* **134** 2153–2159.

[28] Sniady, P. (2006) Gaussian fluctuations of characters of symmetric groups and of Young diagrams. *Probab. Theory Rel. Fields* **136** 263–297.

[29] Stein, C. (1972) A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. In *Proc. Sixth Berkeley Symp. Math. Statist. Prob.* Vol. 2, University of California Press, pp. 586–602.

[30] Stein, C. (1986) *Approximate Computation of Expectations*, IMS.