Problem 1. Prove or disprove each of the following statements. Namely, either prove that the property always holds, or provide an explicit example where it is false.

a. If \( f : X \to Y \) and \( A, A' \subset X \), then \( f(A \cap A') = f(A) \cap f(A') \).

b. If \( f : X \to Y \) and \( A, A' \subset X \), then \( f(A \cup A') = f(A) \cup f(A') \).

c. If \( f : X \to Y \) and \( B, B' \subset Y \), then \( f^{-1}(B \cap B') = f^{-1}(B) \cap f^{-1}(B') \).

d. If \( f : X \to Y \) and \( B, B' \subset Y \), then \( f^{-1}(B \cup B') = f^{-1}(B) \cup f^{-1}(B') \).

e. If \( A, B, C \) are sets, then \( A - (B - C) = (A - B) \cup (A \cap C) \).

Problem 2. Let \( \{A_i\}_{i \in I} \) be a family of indexed sets, all contained in the set \( X \). Recall from class that \( \bigcup_{i \in I} A_i = \{x; \exists i \in I, x \in A_i\} \) and \( \bigcap_{i \in I} A_i = \{x; \forall i \in I, x \in A_i\} \).

Prove that \( X - \bigcap_{i \in I} A_i = \bigcup_{i \in I} (X - A_i) \).
Problem 1. Show that \( d(x, y) = (x - y)^2 \) does not define a metric on the set \( \mathbb{R} \) of all real numbers.

Problem 2. For every \( x, y \in X \), define
\[
d(x, y) = \begin{cases} 
0 & \text{if } x = y \\
1 & \text{if } x \neq y
\end{cases}
\]
Show that this defines a metric on the set \( X \).

Problem 3. Consider the usual metric \( d(x, y) = |x - y| \) on the set \( \mathbb{R} \) of all real numbers. Give an example of an infinite collection of open sets of \( (\mathbb{R}, d) \), whose intersection is not open.

Problem 4. Prove or disprove the following property: If the metric space \( (X, d) \) has at least two elements, then it admits an open subset which is neither \( X \) nor the empty set \( \emptyset \).
Problem 1. In the metric space \((X, d)\), let the closed ball of radius \(r\) centered at \(x\) be
\[
\bar{B}(x, r) = \{ y \in X; d(x, y) \leq r \}
\]
Show that the closed ball \(\bar{B}(x, r)\) is a closed subset of \(X\).

Problem 2. Let \(d\) be a metric defined on the set \(X\). Show that the function \(d'\) defined by \(d'(x, y) = \min\{d(x, y), 1\}\) is also a metric on \(X\). (We will occasionally use this metric, which has the nice property that \(d'(x, y) \leq 1\) for every \(x, y\)).

Problem 3. Let \(f : X \to Y\) be a map between the topological spaces \(X\) and \(Y\). A neighborhood of \(x \in X\) is a subset \(W\) of \(X\) that contains an open subset \(U\) containing \(x\) (namely \(x \in U \subset W \subset X\)). Show that \(f\) is continuous if and only if, for every \(x \in X\) and every neighborhood \(V\) of \(f(x)\) in \(Y\), there exists a neighborhood \(U\) of \(x\) in \(X\) such that \(f(U) \subset V\). (A property similar to the \(\varepsilon-\delta\) definition of continuity in metric spaces).

Problem 4. Let \(f : X \to \mathbb{R}\) be a continuous map from a topological space \(X\) to the real line \(\mathbb{R}\), endowed with the metric topology defined by the usual metric \(d(x, y)|x - y|\). Show that, for every constant \(c \in \mathbb{R}\), the subset
\[
A = \{ x \in X; f(x) > c \}
\]
is open, and that the subset
\[
B = \{ x \in X; f(x) \geq c \}
\]
is closed. Possible hint: Write these sets as the preimages under \(f\) of suitable open or closed subsets of \(\mathbb{R}\).
Problem 1.  Show that, in a metric space \((X, d)\), the closure of the open ball \(B(x, r)\) is contained in the closed ball \(\bar{B}(x, r)\). Give an example where \(\bar{B}(x, r)\) is different from the closure of \(B(x, r)\).

Problem 2.  In a metric space \((X, d)\), compare the interior of the closed ball \(\bar{B}(x, r)\) to the open ball \(B(x, r)\), namely: Which one is always contained in the other? Are they always equal?

Problem 3.  Let \((X, d)\) be a metric space and let \(A\) and \(B\) be subsets of \(X\). Prove or disprove the following statements (where \(\text{cl}(\ )\) denotes the closure).

\begin{enumerate}
  \item \(\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)\)
  \item \(\text{cl}(A \cap B) = \text{cl}(A) \cap \text{cl}(B)\)
\end{enumerate}

Problem 4.  Let \(A\) be a subset of a metric space \((X, d)\). Show that the boundary of \(A\) is equal to the boundary of the complement \(X - A\).
Problem 1. Let \( f : X \to Y \) be a map between the topological spaces \( X \) and \( Y \). Prove that \( f \) is continuous if and only if \( f(\text{cl}(A)) \subset \text{cl}(f(A)) \) for every \( A \subset X \).
(Hint: Recall that \( f \) is continuous if and only if, for every closed subset \( C \) of \( Y \), \( f^{-1}(C) \) is closed in \( X \)).

Problem 2. Let the topological space \( X \) be the union of two closed subsets \( C_1 \) and \( C_2 \), let \( Y \) be another topological space, and consider two maps \( f_1 : C_1 \to Y \) and \( f_2 : C_2 \to Y \) which are continuous when \( C_1 \) and \( C_2 \) are endowed with the subspace topology. Finally, suppose that \( f_1(x) = f_2(x) \) for every \( x \in C_1 \cap C_2 \), so that we can define a map \( f : X = C_1 \cup C_2 \to Y \) without ambiguity by

\[
f(x) = \begin{cases} 
  f_1(x) & \text{if } x \in C_1 \\
  f_2(x) & \text{if } x \in C_2
\end{cases}
\]

Show that \( f : X \to Y \) is continuous. (Hint: Same as Problem 1).

Problem 3. Show by a counterexample that the property of the previous problem may fail if we do not assume that \( C_1 \) and \( C_2 \) are closed.
Problem 1. Consider the subset \( A = \{ q \in \mathbb{Q} : 0 \leq q \leq 1 \} \) of \( \mathbb{R} \) with the subspace topology. (Recall that \( \mathbb{Q} \) is the set of all rational numbers).

a. Show that \( A \) is not closed in \( \mathbb{R} \). Conclude that it is not compact for the subspace topology.

b. By the previous item, we know that there exists an open covering \( \{ U_i \}_{i \in I} \) which admits no finite subcovering. Provide an explicit example of such a covering.

Problem 2. Let \( S \) be an infinite subset of a compact space \( X \). Prove that there exists a point \( x \in X \) such that for every open set \( U \) containing \( x \), there are infinitely many points in \( U \cap S \).

Problem 3. Let \( A \) be a compact subset of a metric space \( (X, d) \). Let \( b \in X - A \). Define \( d(A, b) = \inf \{ d(a, b) : a \in A \} \). Namely, \( d(a, b) \geq d(A, b) \) for every \( a \in A \) and, for every \( \varepsilon > 0 \), there exists \( a \in A \) such that \( d(A, b) \leq d(a, b) \leq d(A, b) + \varepsilon \).

a. Prove that the function \( \varphi_b : A \rightarrow \mathbb{R} \) defined by \( \varphi_b(a) = d(a, b) \) is continuous.

b. Prove that there exists a point \( a \in A \) such that \( d(A, b) = d(a, b) \); in particular, \( d(A, b) > 0 \).

c. Prove that the function \( d_A : X - A \rightarrow \mathbb{R} \) defined by \( d_A(b) = d(A, b) \) is continuous on \( X - A \).

d. Suppose that \( X = \mathbb{R}^n \) with the usual topology, and \( A \) is closed but not necessarily compact. For \( b \in X - A \), does there still exist a point \( a \in A \) such that \( d(A, b) = d(a, b) \)?
Problem 1. Let the set $\mathbb{R}$ of real numbers be endowed with the usual metric topology, and let $X = [0, 1) \cup \{2\} \cup [3, 4]$ be endowed with the subspace topology. Consider the subsets $A = (0, 1), B = [0, 1), C = \{2\}, D = (3, 4), E = [3, 4), F = [3, 4]$.

a. Which ones of the subsets $A, B, C, D, E, F$ are open in $X$ for the subspace topology?
b. Which ones of the subsets $A, B, C, D, E, F$ are closed in $X$ for the subspace topology?

Problem 2. Let $f: X \to Y$ be a function from a set $X$ to a topological space $Y$.

a. Show that the set $T = \{f^{-1}(V); V$ open in $Y\}$, consisting of all the preimages of open subsets of $Y$, is a topology on $X$.
b. Show that $T$ is the smallest topology on $X$ for which the map $f: X \to Y$ is continuous. Namely, if $T'$ is another topology on $X$ such that $f$ is continuous as a function defined on the topological space $(X, T')$, then $T'$ contains $T$.

Problem 3. Let $A$ and $B$ be two subsets of a topological space $X$. As usual, let $A - B = \{x; x \in A$ and $x \notin B\}$, and let $\text{cl}(A)$ and $\text{int}(A)$ denote the closure and the interior of $A$. Show that $\text{int}(A - B) = \text{int}(A) - \text{cl}(B)$. (Possible hints: Show that each set is contained in the other one; a picture may be helpful).

Problem 4. Let $(X, d)$ be a compact metric space, and let $C(X)$ denote the set of all continuous functions $f: X \to \mathbb{R}$. For every $f, g \in C(X)$, let

$$D(f, g) = \max\{|f(x) - g(x)|; x \in X\}$$

be the maximum value taken by the function $x \mapsto |f(x) - g(x)|$ over $X$. It can be easily shown, and you should assume without proof, that this defines a metric $D$ on $C(X)$. Finally, let $K$ be a compact subset of this metric space $(C(X), D)$.

Show that, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for every $x, y \in X$ with $d(x, y) < \delta$ and for every $f \in K$. (Note that $\delta$ should be independent of $x, y$ and $f$. Possible hint: proof by contradiction).
Problem 1. Let \( f : X \to Y \) be a continuous function between the topological spaces \( X \) and \( Y \). The graph of \( f \) is the subset
\[
G_f = \{(x, y) \in X \times Y ; y = f(x)\}
\]
of the product \( X \times Y \). Endow \( X \times Y \) with the product topology, and \( G_f \) with the subspace topology.

a. Show that the function \( g : X \to G_f \) defined by \( g(x) = (x, f(x)) \) is a homeomorphism.

b. Assume in addition that the topological space \( Y \) is Hausdorff. Show that the graph \( G_f \) is a closed subset of \( X \times Y \).

c. Show that the hypothesis that \( Y \) is Hausdorff is necessary in the previous question. Namely give an example of a continuous function \( f \) whose graph \( G_f \) is not closed in \( X \times Y \). (\( Y \) will necessarily be non-Hausdorff, so you will need to dig outside of our usual source of counterexamples; namely, subspaces of \( \mathbb{R} \) will not suffice).
Let \( \{X_i\}_{i \in I} \) be a family of topological spaces \( X_i \), and let \( X = \prod_{i \in I} X_i \) be their product. Recall that \( X \) consists of all families \( x = (x_i)_{i \in I} \) where \( x_i \in X_i \) for every \( i \in I \). For every \( j \in I \), consider the \( j \)-projection map \( \pi_j: X \to X_j \), which associates to \( x = (x_i)_{i \in I} \) its \( j \)-coordinate \( x_j \).

**Problem 1.** Show that, if the product \( X = \prod_{i \in I} X_i \) is endowed with the product topology, the \( j \)-projection map \( \pi_j: X \to X_j \) is continuous for every \( j \in I \).

**Problem 2.** Show that the product topology is the smallest topology \( T \) on \( X \) for which all projection maps \( \pi_j: (X, T) \to X_j \) are continuous.

**Problem 3.** Let \( Y \) be a topological space, and consider a function \( f: Y \to X \) whose \( i \)-coordinate functions \( f_i: Y \to X_i \) are defined by the property that \( f(y) = (f_i(y))_{i \in I} \); namely \( f_i = f \circ \pi_i \). Show that, if \( X \) is endowed with the product topology, \( f \) is continuous if and only if \( f_i \) is continuous for every \( i \in I \).

**Problem 4.** Show that the product topology is the largest topology \( T \) on \( X \) for which the following property holds: for any topological space \( Y \) and for any function \( f: Y \to (X, T) \) whose coordinate function \( f_i = f \circ \pi_i: Y \to X_i \) are all continuous, then \( f: Y \to (X, T) \) is continuous. (Possible hint: consider \( Y \) to be the set \( X \) endowed with the product topology, and \( f \) the identity map defined by \( f(x) = x \).)

**Problem 5.** Show that the product topology is the only topology \( T \) on \( X \) for which the following property holds: for any topological space \( Y \) and for any function \( f: Y \to (X, T) \), the function \( f: Y \to (X, T) \) is continuous if and only if all its coordinate functions \( f_i = f \circ \pi_i: Y \to X_i \) are continuous.
Problem 1. Let $X$ be a topological space.

a. Show that, if $X$ is connected, every subset $A \subset X$ that is different from $\emptyset$ and $X$ has non-empty boundary.

b. Conversely, suppose that every subset $A \subset X$ that is different from $\emptyset$ and $X$ has non-empty boundary. Does this imply that $X$ is connected?

Problem 2. Let $X$ be the Cantor set. Show that every connected subspace of $X$ consists of only 0 or 1 point.

Problem 3. Let $Y$ be a connected subspace of the topological space $X$. Prove, or disprove by a counterexample:

a. the closure $\text{cl}(A)$ is connected;

b. the boundary $\delta A = \text{cl}(A) \cap \text{cl}(X - A)$ is connected;

c. the interior $\text{int}(A)$ is connected.
Problem 1. Let \( X = \mathbb{R}^2 - \{0\} \) be the plane minus the origin, with the usual topology. Let \( \bar{X} \) be the partition of \( X \) consisting of the sets \( \{(2^n x, 2^n y); n \in \mathbb{Z}\} \). Namely two points are in the same set of the partition if and only if one is obtained from the other one by multiplication by a power of 2. Let \( p: X \to \bar{X} \) be the quotient map, and endow \( \bar{X} \) with the quotient topology.

a. Consider the map \( f: X \to S^1 \times S^1 \) defined by
\[
f(x, y) = \left( \left( \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right), \left( \cos \left( 2\pi \frac{\log \sqrt{x^2+y^2}}{\log 2} \right), \sin \left( 2\pi \frac{\log \sqrt{x^2+y^2}}{\log 2} \right) \right) \right).
\]
Show that \( f \) induces a continuous bijection \( \bar{f}: \bar{X} \to S^1 \times S^1 \) for which \( f = \bar{f} \circ p \).

b. Show that \( \bar{X} \) is compact. Possible hint: Find a compact subset \( K \subset X \) such that \( p(K) = \bar{X} \).

c. Show that \( \bar{f} \) is a homeomorphism.

Problem 2. For every \( \theta_0 \), consider in the plane the curve \( S_{\theta_0} \) whose equation in polar coordinates is
\[
\theta = \theta_0 + \frac{1}{(r-1)(2-r)}, \quad 1 < r < 2.
\]
In particular, one end of \( S_{\theta_0} \) spirals around the circle \( C_1 \) of radius 1 centered at the origin, and the other around the circle \( C_2 \) of radius 2 around the origin.

Let \( X \) be the annulus consisting of all those points of the plane whose polar coordinates \( [r, \theta] \) are such that \( 1 \leq r \leq 2 \). Consider the partition \( \tilde{X} \) of \( X \) whose elements consist of the circle \( C_1 \), the circle \( C_2 \), and all curves \( S_{\theta_0} \) with \( 0 \leq \theta_0 < 2\pi \).

Show that, when \( X \) is endowed with the subspace topology induced by the usual topology of the plane and when \( \tilde{X} \) is induced with the quotient topology, the space \( \tilde{X} \) is not Hausdorff.