Math 440, Topology
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Fall 2006

Course outline
CHAPTER 1

Basic Set Theory

Sets

“Definition”. A set is a collection of objects. If $x$ is an object in the set $A$, we write $x \in A$ and we say that $x$ belongs to $A$, or that $x$ is an element of $A$.

The quotes around the word “definition” are here because there are a few touchy issues of mathematical logic about which collections of objects can be called sets and which ones cannot. However, we will not concern ourselves with these subtleties.

It is often convenient to describe the content of a set with curly brackets { }.

Examples.

$Z = \{ \text{all integers} \} = \{ \ldots, -1, 0, 1, 2, 3, \ldots \}$

$2Z = \{ \text{all even integers} \} = \{ \ldots, -2, 0, 2, 4, 6, \ldots \}$

$= \{ x \in Z; \exists y \in Z, x = 2y \}$

Notation. $\exists$ means “there exists”, and $\forall$ means “for every”

Definitions. If $A$ and $B$ are sets, $A$ is contained in $B$, or $A$ is a subset of $B$, if every element of $A$ is also an element of $B$. We then write $A \subset B$.

The empty set $\emptyset$ is the set consisting of no element.

The union of the sets $A$ and $B$ is the set $A \cup B = \{ x; x \in A \text{ or } x \in B \}$.

The intersection of the sets $A$ and $B$ is the set $A \cap B = \{ x; x \in A \text{ and } x \in B \}$.

The complement of $B$ in $A$ is $A - B = \{ x; x \in A \text{ and } x \notin B \}$.

Example. If $A = \{1, 2\}$ and $B = \{1, 2, 3\}$, then $1 \in A, 3 \notin A, A \subset B, \{1, 3\} \not\subset A, A \cup B = B, A \cap B = \{1\}, A - B = \emptyset \text{ and } B - A = \{3\}$. Here a bar \slash across a symbol means the negation of that symbol.

A set can be an element of a set. For instance, $\{1, 3\} \in \{ \text{subsets of } \{1, 2, 3\} \} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

Functions

Definition. If $X$ and $Y$ are sets, a function or a map $f: X \to Y$ is a rule which to every $x \in X$ associates an element $f(x) \in Y$. 


The function \( f: X \to Y \) is \textit{surjective} or \textit{onto} if \( \forall y \in Y, \exists x \in X, f(x) = y \).

The function \( f: X \to Y \) is \textit{injective} or \textit{one-to-one} if \( \forall x, x' \in X \) with \( x \neq x' \), \( f(x) \neq f(x') \).

A function is \textit{bijective} or is a \textit{one-to-one correspondence} if it is both surjective and injective.

**Definition.** When the function \( f: X \to Y \) is bijective, its \textit{inverse function} is the function \( f^{-1}: Y \to X \) defined by the property that, for every \( y \in Y \), \( f^{-1}(y) \) is equal to the unique \( x \in X \) such that \( f(x) = y \).

**Definition.** Given a function \( f: X \to Y \) and two subsets \( A \subseteq X \) and \( B \subseteq Y \), the \textit{image} of \( A \) is \( f(A) = \{ y \in Y; \exists x \in X, y = f(x) \} \). The \textit{preimage} of \( B \) is \( f^{-1}(B) = \{ x \in X; f(x) \in B \} \).

### Indexed families of sets

If \( A_1, A_2, \ldots, A_n \) is a finite family of sets,

\[
A_1 \cup A_2 \cup \cdots \cup A_n = \{ x; x \in A_1 \text{ or } x \in A_2 \text{ or } \ldots \text{ or } x \in A_n \}
= \{ x; \exists i \in \{1, 2, \ldots, n\}, x \in A_i \}
\]

and

\[
A_1 \cap A_2 \cap \cdots \cap A_n = \{ x; x \in A_1 \text{ and } x \in A_2 \text{ and } \ldots \text{ and } x \in A_n \}
= \{ x; \forall i \in \{1, 2, \ldots, n\}, x \in A_i \}
\]

More generally:

**Definition.** An \textit{indexed family of sets} \( \mathcal{A} \) consists of a set \( \mathcal{A} \) of sets and of a surjective map \( f: I \to \mathcal{A} \). The set \( I \) is the \textit{index set} of the family.

**Example.** If \( \mathcal{A} = \{A_1, A_2, \ldots, A_n\} \), choose \( I = \{1, 2, \ldots, n\} \) and set \( f(i) = A_i \).

Similarly, or a general family of indexed sets, we usually write the set \( f(i) \in \mathcal{A} \) with \( i \) as a subscript, for instance \( f(i) = A_i \), so that we can write

\[ \mathcal{A} = \{ A_i; i \in I \} = \{ A_i \}_{i \in I}. \]

**Definition.** For an indexed family of sets \( \mathcal{A} = \{A_i\}_{i \in I} \), their \textit{union} is

\[
\bigcup_{i \in I} A_i = \{ x; \exists i \in I, x \in A_i \}
\]

and their \textit{intersection} is

\[
\bigcap_{i \in I} A_i = \{ x; \forall i \in I, x \in A_i \}
\]

**Proposition.** (Distributivity of \( \cup \) and \( \cap \)) Given sets \( X \) and \( \{A_i\}_{i \in I} \),

\[
X \cap \left( \bigcup_{i \in I} A_i \right) = \bigcup_{i \in I} (X \cap A_i)
\]
and
\[ X \cup \left( \bigcap_{i \in I} A_i \right) = \bigcap_{i \in I} (X \cup A_i). \]

**Proposition.** (De Morgan’s laws) Given sets \( X \) and \( \{A_i\}_{i \in I} \),
\[ X - \bigcup_{i \in I} A_i = \bigcap_{i \in I} (X - A_i) \]
and
\[ X - \bigcap_{i \in I} A_i = \bigcup_{i \in I} (X - A_i). \]

**Products**

**Definition.** The product \( A \times B \) of the sets \( A \) and \( B \) consists of all couples \((a, b)\) where \( a \in A \) and \( b \in B \). Namely
\[ A \times B = \{(a, b); a \in A, b \in B\}. \]

More generally:

**Definition.** The product \( A_1 \times A_2 \times \cdots \times A_n \) of the sets \( A_1, A_2, \ldots, A_n \) consists of all \( n \)-uples \((a_1, a_2, \ldots, a_n)\) where each \( a_i \) is an element of \( A_i \). Namely
\[ A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n); \forall i, a_i \in A_i\}. \]

Even more generally,

**Definition.** The product \( \prod_{i \in I} A_i \) of an indexed family of sets \( \{A_i\}_{i \in I} \) is the set of all functions \( a: I \to \bigcup_{i \in I} A_i \) such that \( a(i) \in A_i \) for every \( i \in I \). The element \( a(i) \in A_i \) is the \( i \)-th coordinate of \( a \). We often write \( a(i) = a_i \), and \( a = (a_i)_{i \in I} \).
CHAPTER 2

Metric Spaces

Distance functions

Definition. A metric or distance function on the set $X$ is a function $d : X \times X \to \mathbb{R}$ such that:

1. $d(x, y) \geq 0$, $\forall x, y \in X$;
2. $d(x, y) = 0$ if and only if $x = y$;
3. $d(y, x) = d(x, y)$, $\forall x, y \in X$ (symmetry);
4. $d(x, z) \leq d(x, y) + d(y, z)$, $\forall x, y, z \in X$ (triangle inequality).

Example 1. $X = \mathbb{R}$, $d(x, y) = |x - y|$.

Example 2. $X = \mathbb{R}^n = \{(x_1, x_2, \ldots, x_n); \forall i, x_i \in \mathbb{R}, \}$, with the standard (euclidean) distance

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2} = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

Example 3. $X = \mathbb{R}^n$, with the distance

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n| = \sum_{i=1}^{n} |x_i - y_i|.$$

Example 4. $X = \mathbb{R}^n$, with the distance

$$d(x, y) = \max_{i=1, \ldots, n} |x_i - y_i|.$$

Example 5. (The $L^2$–distance in analysis)

$X = \{\text{all continuous functions } f: [0, 1] \to \mathbb{R}\}$

and

$$d(f, g) = \left( \int_{0}^{1} (f(x) - g(x))^2 dx \right)^{\frac{1}{2}}.$$

(The triangle inequality is not that easy to prove in this case.)

Example 6. (The diadic distance in number theory) $X$ is the set $\mathbb{Q}$ of all rational numbers $x = \frac{p}{q}$ with $p, q \in \mathbb{Z}$ with $q \neq 0$. Every rational number $x \neq 0 \in \mathbb{Q}$ can be written as $x = 2^n \frac{p'}{q'}$ for odd integers $p', q' \in \mathbb{Z}$ and for $n \in \mathbb{Z}$, possibly negative. Define the diadic norm $|x|_2$ to be equal to $2^{-n}$ in this case and, when $x = 0$, decide that $|0|_2 = 0$. 

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The **diadic distance** on \( X = \mathbb{Q} \) is defined by the property that
\[
d(x, y) = |x - y|_2.
\]
It is a fundamental example of an ultrametric, as defined below.

**Definition.** An **ultrametric** on the set \( X \) is a distance function \( X \) which satisfies the following stronger version of the triangle inequality:
\[
d(x, z) \leq \max\{d(x, y), d(y, z)\}, \quad \forall x, y, z \in X.
\]

**Definition.** A **metric space** is a pair \((X, d)\) consisting of a set \( X \) and of a metric \( d \) on \( X \).

### Balls, and open subsets

**Definition.** In a metric space \((X, d)\), the (open) **ball of radius** \( r > 0 \) **centered at** \( x \in X \) is the subset
\[
B_d(x, r) = B(x, r) = \{ y \in X ; d(x, y) < r \}.
\]

**Example 1.** In \( X = \mathbb{R} \) with the distance \( d(x, y) = |x - y| \), the ball \( B(x, r) \) is the interval \((x - r, x + r)\).

**Example 2.** In \( X = \mathbb{R}^3 \) with the euclidean distance
\[
d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2} = \sqrt{\sum_{i=1}^{3} (x_i - y_i)^2},
\]
\( B(x, r) \) is a standard open ball, namely the inside of the sphere of radius \( r \) centered at \( x \).

**Example 3.** In \( X = \mathbb{R}^3 \) with the distance
\[
d(x, y) = \max_{i=1,2,3} |x_i - y_i|,
\]
\( B(x, r) \) is the inside of a cube.

**Example 4.** In \( X = \mathbb{R}^2 \) with the distance
\[
d(x, y) = |x_1 - y_1| + |x_2 - y_2| = \sum_{i=1}^{2} |x_i - y_i|,
\]
\( B(x, r) \) is the inside of a diamond.

**Example 4’.** In \( X = \mathbb{R}^3 \) with the distance
\[
d(x, y) = |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3| = \sum_{i=1}^{3} |x_i - y_i|,
\]
\( B(x, r) \) is the inside of an octahedron.

**Definition.** Let \((X, d)\) be a metric space. The subset \( U \) of \( X \) is **open** if, for every \( x \in U \), there exists a ball \( B(x, r) \) centered at \( x \) such that \( B(x, r) \) is contained in \( U \).

**Silly examples.** The whole space \( X \), and the empty set \( \emptyset \) are open subsets of \( X \).
More serious examples. For $x = \mathbb{R}$ with the usual metric $d(x, y) = |x - y|$, the open interval $(a, b)$ is an open subset, and the closed interval $[a, b]$ is not an open subset.

Proposition. In a metric space $(X, d)$, every ball $B(x, r)$ is open.

Proposition.
1. If $\{U_i\}_{i \in I}$ is a family of open subsets of $X$, then the union $\bigcup_{i \in I} U_i$ is also open.
2. If $\{U_1, U_2, \ldots, U_n\}$ is a finite family of open subsets of $X$, then the intersection $\bigcap_{i=1}^n U_i$ is also open.

Definition. A subset $C$ of $X$ is closed if its complement $X - C$ is open.

Silly example. The whole space $X$, and the empty set $\emptyset$ are always closed subsets of $X$.

More serious examples. For $X = \mathbb{R}$ with the usual metric $d(x, y) = |x - y|$, the closed interval $[a, b]$ is an closed subset, and the open interval $(a, b)$ is not an closed subset.

Example. The closed ball $\overline{B}(x, r) = \{y \in X; d(x, y) \leq r\}$ is a closed subset of $X$.

The following will be important later on to explain the definition of topological spaces.

Proposition. Let $(X, d)$ be a metric space.

1. If $\{C_i\}_{i \in I}$ is a family of closed subsets of $X$, then the intersection $\bigcap_{i \in I} C_i$ is also closed.
2. If $\{C_1, C_2, \ldots, C_n\}$ is a finite family of closed subsets of $X$, then the union $\bigcup_{i=1}^n C_i$ is also closed.

Remark. In a metric space, a subset can be both open and closed, and a subset can be neither open nor closed. For instance, in any metric space $(X, d)$, the whole space and the empty set $\emptyset$ are both open and closed. A less trivial example occurs when $X = \mathbb{R} - \{0\}$ with the metric $d(x, y) = |x - y|$, in which case the subset consisting of the interval $(0, \infty)$ is both open and closed.

Example. In $X = \mathbb{R}$ with the metric $d(x, y) = |x - y|$, the semi-open interval $[a, b)$ is neither open nor closed, for $-\infty < a < b < +\infty$.

Continuous functions and converging sequences

Definition. A function $f: X \to Y$ from a metric space $(X, d)$ to the metric space $(Y, d')$ is continuous at the point $x_0 \in X$ if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d'(f(x), f(x_0)) < \varepsilon$ for every $x \in X$ with $d(x, x_0) < \delta$.

The function $f$ is continuous, for short, if it is continuous at every $x \in X$.

Definition. A sequence in $X$ is a function $\mathbb{N} \to X$. If the sequence associates the element $x_n \in X$ to the number $n$, it is usually denoted by $(x_n)_{n \in \mathbb{N}}$. 
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DEFINITION. The sequence \((x_n)_{n \in \mathbb{N}}\) in the metric space \((X, d)\) converges to \(x \in X\) if, for every \(\varepsilon > 0\), there exists a number \(n_0 \geq 0\) such that \(d(x_n, x) < \varepsilon\) for every \(n \geq n_0\). We then write

\[
\lim_{n \to \infty} x_n = x.
\]

THEOREM. Let \(f : X \to Y\) be a function from a metric space \((X, d)\) to the metric space \((Y, d')\). The function \(f\) is continuous at \(x\) if and only if, for every sequence \((x_n)_{n \in \mathbb{N}}\) converging to \(x\) in \((X, d)\), the sequence \((f(x_n))_{n \in \mathbb{N}}\) converges to \(f(x)\) in \((Y, d)\).

The following result is crucial to motivate the introduction of topological spaces.

THEOREM. Let \(f : X \to Y\) be a function from a metric space \((X, d)\) to the metric space \((Y, d')\). The function \(f\) is continuous if and only if, for every open subset \(U\) in \(Y\), the preimage \(f^{-1}(U)\) is open in \(X\).

Similarly:

THEOREM. The sequence \((x_n)_{n \in \mathbb{N}}\) in the metric space \((X, d)\) converges to \(x \in X\) if and only if, for every open subset \(U\) of \(X\) which contains \(x\), there exists a number \(n_0\) such that \(x \in U\) for every \(n \geq n_0\).
CHAPTER 3

Topological Spaces

Definition. A topological space consists of a set $X$ and of a family $T$ of subsets of $X$ such that the following three conditions hold.

1. The empty set $\emptyset$ and the whole space $X$ belong to $T$;
2. If $\{U_i\}_{i \in I}$ is a (possibly infinite) family of elements of $T$, then the union $\bigcup_{i \in I} U_i$ also belongs to $T$;
3. If $\{U_1, U_2, \ldots, U_n\}$ is a finite family of elements of $T$, the intersection $\bigcap_{i=1}^n U_i$ is also an element of $T$.

Such a family $T$ is called a topology on the set $X$. The elements of $T$ are referred to as open subsets of the topological space $X$.

Fundamental example. If $(X, d)$ is a metric space, then the family of all open subsets of $X$ forms a topology on $X$.

Definition. In this case, $T$ is the metric topology defined on $X$ by the metric $d$.

Silly examples.

1. The trivial topology, or indiscrete topology, on the set $X$ is the topology $T = \{\emptyset, X\}$.
2. The discrete topology on the set $X$ consists of the family $T$ of all subsets of $X$.

Useless example. On a set $X$, the finite complement topology is

$$T = \{U \subset X; X - U \text{ finite}\} \cup \{\emptyset\}.$$

Definition. A topological space $X$ is metrizable if its topology $T$ is the metric topology defined by some metric $d$ on $X$.

Examples.

1. Most interesting examples of topological spaces are metrizable.
2. The discrete topology is metrizable, as it is the metric topology associated to the metric $d$ defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

3. The finite complement is metrizable if and only if $X$ is finite (in which case it coincides with the discrete topology).
4. The trivial topology is not metrizable when $X$ has more than two elements.
Proposition. Consider two metrics $d$ and $d'$ on the same set $X$. Let $B_d(x, r)$ denote the ball of radius $r$ centered at $x$ for the metric $d$, and let $B_{d'}(x, r')$ similarly denote the ball for the metric $d'$. Then, $d$ and $d'$ define the same metric topology (namely a subset of $X$ is open for $d$ if and only if it is open for $d'$) if and only if the following two conditions hold:

1. $\forall x \in X, \forall r > 0, \exists r' > 0, B_{d'}(x, r') \subset B_d(x, r)$;
2. $\forall x \in X, \forall r' > 0, \exists r > 0, B_d(x, r) \subset B_{d'}(x, r')$.

Example. On $X = \mathbb{R}^n$, the three metrics

$$d_1(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2} = \sum_{i=1}^{n} (x_i - y_i)^2,$$

$$d_2(x, y) = |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n| = \sum_{i=1}^{n} |x_i - y_i|$$

and

$$d_3(x, y) = \max_{i=1, 2, \ldots, n} |x_i - y_i|$$

define the same metric topology $\mathcal{T}$ on $\mathbb{R}^n$. Namely, a subset of $\mathbb{R}^n$ is open for one of these metrics $d_i$ if and only if it is open for any of the other two $d_j$.

In particular, a function $f: \mathbb{R}^n \to Y$ or $g: X \to \mathbb{R}^n$ is continuous for $d_i$ if and only if it is continuous for any of the other two $d_j$.

Continuous functions

Definition. A function $f: X \to Y$ from the topological space $X$ to the topological space $Y$ is continuous if, for every open subset $U$ of $Y$, the preimage $f^{-1}(U)$ is open in $X$.

Example. A map between metric spaces is continuous in the $\varepsilon$-$\delta$ sense if and only if it is continuous in this sense.

Silly example. A map $f: X \to Y$ is always continuous if $X$ is endowed with the discrete topology, or if $Y$ is endowed with the trivial topology.

Definition. Given two functions $f: X \to Y$ and $g: Y \to Z$, their composition is the map $g \circ f: X \to Z$ defined by $g \circ f(x) = g(f(x))$ for every $x \in X$.

Proposition. If $X$, $Y$, and $Z$ are topological spaces and if the maps $f: X \to Y$ and $g: Y \to Z$ are continuous, then the composition $g \circ f: X \to Z$ is continuous.

Definition. A homeomorphism between the topological spaces $X$ and $Y$ is a map $f: X \to Y$ such that:

1. $f$ is bijective;
2. $f$ is continuous;
3. the inverse function $f^{-1}: Y \to X$ is continuous.
The topological spaces $X$ and $Y$ are homeomorphic if there exists a homeomorphism $f: X \to Y$.

**Proposition.** If $f: X \to Y$ is bijective, it is a homeomorphism if and only if the map $U \mapsto f(U)$ establishes a one-to-one correspondence between open subsets of $X$ and open subsets of $Y$.

In particular, a homeomorphism is a “dictionary” translating any property of $X$ involving its topology to a property of the topology of $Y$. As a consequence, homeomorphic topological spaces have exactly the same properties.

**Example.** In $\mathbb{R}^n$ endowed with the usual euclidean metric $d$, let $X = B(0, 1) - 0$ be an open ball minus its center, and let $Y = \mathbb{R}^n - \bar{B}(0, 1)$ be the complement of a closed ball. Endow $X$ and $Y$ with the metric topology defined by $d$. Then, the map $f: X \to Y$ defined by $f(x) = x/d(x, 0)^2$ is a homeomorphism. Indeed, $f$ is continuous by the standard $\varepsilon$–$\delta$ arguments of calculus, and so is its inverse $f^{-1}: Y \to X$ since $f^{-1}(y) = y/d(0, y)^2$. (The fact that we get the same formula is a coincidence)

**Closed subsets**

**Definition.** A subset $C$ of a topological space $X$ is closed if and only if its complement $X - C$ is open, namely if and only if $X - C$ belongs to the topology $T$ of $X$.

**Proposition.** Let $X$ be a topological space.

1. If $\{C_i\}_{i \in I}$ is a family of closed subsets of $X$, then the intersection $\bigcap_{i \in I} C_i$ is also closed.
2. If $\{C_1, C_2, \ldots, C_n\}$ is a finite family of closed subsets of $X$, then the union $\bigcup_{i=1}^n C_i$ is also closed.

**Proposition.** A map $f: X \to Y$ between topological spaces is continuous if and only, for every closed subset $C$ of $Y$, its preimage $f^{-1}(C)$ is open in $X$. 
CHAPTER 4

Subsets and subspaces

Closure, interior and boundary

Definition. Let $A$ be a subset of the topological space $X$.

1. The closure $\text{cl}(A) = \overline{A}$ of $A$ is the intersection of all the closed subsets of $X$ that contain $A$.
2. The interior $\text{int}(A) = \overset{\circ}{A}$ of $A$ is the union of all the open subsets of $X$ that are contained in $A$.

Definition. In a topological space $X$, a neighborhood of the point $x \in X$ is a subset $W$ which contains an open subset $U$ such that $x \in U \subset W$.

Warning. There are competing definitions in the literature. Some textbooks require a neighborhood to be open.

Proposition. Let $A$ be a subset of the topological space $X$.

1. The closure of $A$ is the smallest of all the closed subsets of $X$ that contain $A$.
2. A point $x \in X$ is in the closure of $A$ if and only if every neighborhood of $x$ has a non-trivial intersection with $A$.

Proposition. Let $A$ be a subset of the topological space $X$.

1. The interior of $A$ is the largest of all the open subsets of $X$ that are contained $A$.
2. A point $x \in X$ is in the interior of $A$ if and only if there exists a neighborhood of $x$ which is contained in $A$.

Proposition. If $A$ is a subset of the topological space $X$,

1. $\text{cl}(X - A) = X - \text{int}(A)$
2. $\text{int}(X - A) = X - \text{cl}(A)$

Proposition. If $A$ is a subset of the topological space $X$,

1. $A$ closed $\iff A = \text{cl}(A)$
2. $A$ open $\iff A = \text{int}(A)$

Definition. If $A$ is a subset of the topological space $X$, the boundary of $A$ is the subset $\delta A = \text{cl}(A) - \text{int}(A)$. 

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Proposition. The boundary $\delta A$ is equal to $\text{cl}(A) \cap \text{cl}(X - A)$. As a consequence, a point $x \in X$ is in the boundary of $A$ if and only if every neighborhood $W$ of $x$ meets both $A$ and its complement $X - A$.

Proposition. On the set $\mathcal{P}(X) = \{A; A \subset X\}$ of all subsets of a topological space $X$, consider the operations of closure $A \mapsto \text{cl}(A)$ and complementation $X \mapsto X - A$.

1. For every subset $A$ of $X$, $\text{cl}(X - \text{cl}(X - \text{cl}(X - \text{cl}(A)))) = \text{cl}(X - \text{cl}(A))$.

2. If we start from any subset $A \in \mathcal{P}(X)$ and successively apply these two operations, one can reach at most 14 elements of $\mathcal{P}(X)$, namely:
   (i) $A$,
   (ii) $\text{cl}(A)$,
   (iii) $X - \text{cl}(A)$,
   (iv) $\text{cl}(X - \text{cl}(A))$,
   (v) $X - \text{cl}(X - \text{cl}(A))$,
   (vi) $\text{cl}(X - \text{cl}(X - \text{cl}(A)))$,
   (vii) $X - \text{cl}(X - \text{cl}(X - \text{cl}(A)))$,
   (viii) $X - A$,
   (ix) $X - \text{cl}(X - A)$,
   (x) $X - \text{cl}(X - A)$,
   (xi) $\text{cl}(X - \text{cl}(X - A))$,
   (xii) $X - \text{cl}(X - \text{cl}(X - A))$,
   (xiii) $\text{cl}(X - \text{cl}(X - \text{cl}(X - A)))$,
   (xiv) $X - \text{cl}(X - \text{cl}(X - \text{cl}(X - A)))$.

3. For the subset $A = [0, 1] \cup [2, 3] \cup (3, 4] \cup \{5\} \cup [6, 7] \cap \mathbb{Q}$ of $\mathbb{R}$, the above 14 subsets are distinct.

The subspace topology

Definition. Let $Y$ be a subset of the topological spaces $X$. The subspace topology on $Y$ is the topology

$$\mathcal{T} = \{U \subset Y; \exists V \text{ open in } X, U = Y \cap V\}.$$ 

Namely, a subset $U$ of $Y$ is open for the subspace topology if and only if there is an open subset $V$ of $X$ such that $U = Y \cap V$.

When $Y$ is endowed with the subspace topology, we say that $Y$ is a subspace of the topological space $X$.

Fundamental example. If $(X, d)$ is a metric space with metric $d: X \times X \to \mathbb{R}$, and if $X$ is endowed with the corresponding metric topology, the subspace topology on $Y \subset X$ coincides with the metric topology defined by the metric $d|_{Y \times Y}$ on $Y$, where $d|_{Y \times Y}: Y \times Y \to \mathbb{R}$ is the restriction of $d$ (defined by $d|_{Y \times Y}(y, y) = d(y, y)$).

Proposition. If $Y$ is a subset of $X$, a subset $C$ of $Y$ is closed for the subspace topology if and only if there is an closed subset $D$ of $X$ such that $C = Y \cap D$.

Warning. When $Y$ is a subspace of $X$, a subset $A \subset Y \subset X$ can be open (or closed) as a subset of $Y$ but not necessarily as a subset of $X$. Similarly, one has to be careful in distinguishing the closure $\text{cl}_Y(A)$ and interior $\text{int}_Y(A)$ of $A$ in the topological space $Y$ from the closure $\text{cl}_X(A)$ and interior $\text{int}_X(A)$ of $A$ in the
topological space $X$. As a consequence, you have to be careful with statements like “$A$ is open” or “consider the closure of $A$”. It is a good idea to always specify the topology which you are referring to, unless there is absolutely no ambiguity.

**Example.** Let $X$ be $\mathbb{R}$ endowed with the usual metric topology, and let $Y = \mathbb{Q}$ (the set of rational numbers) be endowed with the subspace topology. The subset $A = \mathbb{Q} \cap [\sqrt{2}, \sqrt{3}] = \mathbb{Q} \cap (\sqrt{2}, \sqrt{3})$ is both open and closed for the topology of $Y$, but is neither open nor closed for the topology of $X$. More precisely, $\text{cl}_Y(A) = \text{int}_Y(A) = A$, but $\text{cl}_X(A) = [\sqrt{2}, \sqrt{3}]$ and $\text{int}_X(A) = \emptyset$.

**Definition.** Let $f: X \to Y$ be a map between two sets $X$ and $Y$. If $A$ is a subset of $X$, the restriction of $f$ to the domain $A$ is the map $f|_A: A \to Y$ defined by $f|_A(a) = f(a)$ for every $a \in A$. If $B$ is a subset of $Y$ which contains the image $f(X)$ of $f$, the restriction of $f$ to the range $B$ is the map $f|_B: X \to B$ defined by $f|_B(x) = f(x)$ for every $x \in X$.

**Proposition.** Let $f: X \to Y$ be a continuous map between the topological spaces $X$ and $Y$. If $X'$ is a subspace of $X$, the restriction $f|_{X'}: X' \to Y$ to the domain $X'$ is also continuous. If $Y'$ is a subspace of $Y$ which contains $f(X)$, the restriction $f|_{Y'}: X \to Y'$ to the range $Y'$ is also continuous.
CHAPTER 5

The Hausdorff Property

Definition. A topological space $X$ has the Hausdorff property, or is a Hausdorff space if, for every $x, y \in X$ with $x \neq y$, there exists open subsets $U$ and $V$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Proposition. If $X$ is a Hausdorff topological space, then the subset $\{x\}$ is closed for every $x \in X$.

Example 1. Every metric space is Hausdorff. Consequently, every metrizable spaces.

Example 2. If $X$ is infinite and is endowed with the finite complement topology, then $X$ is not Hausdorff. However, every one-element subset $\{x\}$ is closed in $X$.

Example 3. If $X$ is endowed with the trivial topology $\{\emptyset, X\}$ and has at least 2 elements, then no one-element subset $\{x\}$ is closed in $X$. As a consequence (or by a direct argument), $X$ is not Hausdorff.
CHAPTER 6

Compact spaces

Definition. An open covering of the topological space $X$ is a family $\{U_i\}_{i \in I}$ where each $U_i$ is open in $X$ and where $\bigcup_{i \in I} U_i = X$.

A subcovering of $\{U_i\}_{i \in I}$ is a subfamily $\{U_j\}_{j \in J}$ with $J \subset I$ which is still a covering, namely such that $\bigcup_{j \in J} U_j = X$.

Definition. The topological space $X$ is compact if every open covering of $X$ admits a finite subcovering, namely if for every open covering $\{U_i\}_{i \in I}$ of $X$ there exists $i_1, i_2, \ldots, i_n \in I$ such that $X = U_{i_1} \cup U_{i_2} \cup \ldots U_{i_n}$.

Compact metric spaces

Definition. A subsequence of the sequence $(x_n)_{n \in \mathbb{N}}$ is a sequence of the form $(x_{n_k})_{k \in \mathbb{N}}$ with $n_1 < n_2 < \cdots < n_k < n_{k+1} < \ldots$.

Recall that the sequence $(x_n)_{n \in \mathbb{N}}$ is convergent if it admits a limit $x = \lim_{n \to \infty} x_n$.

Theorem. A metric space $(X, d)$ is compact if and only if every sequence $(x_n)_{n \in \mathbb{N}}$ admits a converging subsequence.

A key step in the proof is the following result, which is quite useful for applications.

Proposition (Lebesgue Number Lemma). Let $\{U_i\}_{i \in I}$ be an open covering of a compact metric space $(X, d)$. Then, there exists a number $\varepsilon$ such that, for every $x \in X$, there exists a $U_i$ such that the ball $B(x, \varepsilon)$ is completely contained in $U_i$.

Note that $\varepsilon$ is independent of $x$. It is called the Lebesgue number of the covering.

Compact subsets of $\mathbb{R}^n$

Theorem. The closed interval $[0, 1]$, with the usual topology, is compact.

Definition. A metric space $(X, d)$ is bounded if, for some (or every) point $x_0 \in X$, there exists a number $K > 0$ such that $d(x, x_0) \leq K$ for every $x \in X$.

The triangle inequality shows that this property does not depend on the point $x_0$.

Proposition. Every compact metric space is bounded.

Theorem. If $\mathbb{R}^n$ is endowed with the usual euclidean metric, a subset $A \subset \mathbb{R}^n$ is compact for the subspace topology if and only if it is closed and bounded.
EXAMPLE. Consider the subset $X$ of $[0, 1]$ consisting of all those points whose decimal expansion uses only the digits 3, 4 and 7. The space $X$ is compact as a subspace of $\mathbb{R}$.

EXAMPLE. Any Cantor set $X = \bigcap_{n=1}^{\infty} C_n$ where each $C_n$ consists of $2^n$ disjoint intervals in $\mathbb{R}$, and is obtained from $C_{n-1}$ by removing an interval from the interior of each interval of $C_{n-1}$.

Applications

Theorem (Maximum/Minimum Value Theorem). Let $f: X \to \mathbb{R}$ be a continuous map from a compact space $X$ to $\mathbb{R}$, with the usual topology. Then there exists $x_{\min}, x_{\max} \in X$ such that $f(x_{\min}) \leq f(x) \leq f(x_{\max})$ for every $x \in X$.

Definition. A function $f: X \to Y$ from a metric space $(X, d)$ to a metric space $(X', d')$ is uniformly continuous if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d'(f(x), f(y)) < \varepsilon$ for every $x, y \in X$ such that $d(x, y) < \delta$.

The key point here is that $\delta$ is independent of $x$.

Theorem (Uniform Continuity). Let $f: X \to \mathbb{R}$ be a continuous map from a compact metric space $X$ to $\mathbb{R}$. Then $f$ is uniformly continuous.

Remark. This is false if $X$ is not compact. For instance, the function $f: (0, \infty) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is not uniformly continuous.

Compact spaces, continuous maps, and closed subsets

Proposition. Let $X$ be a compact topological space. Then every closed subset $C$ of $X$ is compact for the subspace topology.

Proposition. Let $X$ be a Hausdorff topological space. Then every subset $A \subset X$ which is compact for the subspace topology is also a closed subset of $X$.

Proposition. If $f: X \to Y$ is continuous and if $X$ is compact, then the image $f(X)$ is compact for the subspace topology induced by the topology of $Y$.

Theorem (The Fundamental Theorem of Topology!). Let $f: X \to Y$ be a continuous map between topological spaces. Suppose that $f$ is bijective, that $X$ is compact, and that $Y$ is Hausdorff. Then $f$ is a homeomorphism (namely the inverse map $f^{-1}: Y \to X$ is also continuous).
CHAPTER 7

The product topology

Basis for a topology

Definition. A basis for a topology on the set $X$ is a family $\mathcal{B}$ of subsets of $X$ which satisfies the following two conditions:

1. every $x \in X$ belongs to at least some $B \in \mathcal{B}$, namely $\bigcup_{B \in \mathcal{B}} B = X$;
2. for every $B_1, B_2 \in \mathcal{B}$ and every $x \in B_1 \cap B_2$, there exists $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$.

Fundamental example. Let $(X, d)$ be a metric space. Then the family $\mathcal{B} = \{B(x, r)\}_{x \in X, r > 0}$ of all balls in $X$ is a basis for a topology.

Proposition. If $\mathcal{B}$ is a basis for a topology on $X$, then $T = \{U \subset X; \forall x \in U, \exists B \in \mathcal{B}, x \in B \subset U\}$ is a topology for $X$.

Definition. The topology $T$ is the topology defined by the basis $\mathcal{B}$.

Example. In the metric space $(X, d)$, the topology defined by the basis $\mathcal{B}$ of all balls of $X$ is exactly the metric topology.

Example. In a topological space $X$ with topology $T$, then $\mathcal{B} = T$ is a basis for a topology, and the topology defined by this $\mathcal{B}$ is exactly $T$.

Proposition. Let the topology of $X$ be defined by a basis $\mathcal{B}$, and let the topology of $X'$ be defined by a basis $\mathcal{B}'$. A function $f: X \to X'$ is continuous if and only if, for every $x \in X$ and every basis element $B' \in \mathcal{B}'$ containing $f(x) \in X'$, there exists a basis element $B \in \mathcal{B}$ containing $x$ such that $f(B) \subset B'$. (Note the analogy with the $\varepsilon$-$\delta$ definition of continuity in metric spaces).

Finite products

Definition. If $X_1$ and $X_2$ are two sets, the product of $X_1$ and $X_2$ is the set $X_1 \times X_2$ consisting of all ordered pairs $(x_1, x_2)$ with $x_1 \in X_1$ and $x_2 \in X_2$.

More generally, if $X_1, X_2, \ldots, X_n$ is a family of $n$ sets, their product is the set $X_1 \times X_2 \times \cdots \times X_n$ consisting of all ordered $n$-uples $(x_1, x_2, \ldots, x_n)$ where each $x_i$
is an element of $X_i$, namely
\[ \prod_{i=1}^{n} X_i = X_1 \times X_2 \times \cdots \times X_n = \{(x_1, x_2, \ldots, x_n) : \forall i, x_i \in X_i\}. \]

**Proposition.** Let $X_1, \ldots, X_n$ be a finite family of topological spaces, and consider their product $X = X_1 \times \cdots \times X_n$. Then the family
\[ \mathcal{B} = \{U_1 \times \cdots \times U_n : \forall i, U_i \text{ open in } X_i\} \]
of subsets of $X$ is a basis for a topology.

**Definition.** The topology defined by $\mathcal{B}$ is the *product topology* on the product of the topological spaces $X_1, \ldots, X_n$.

For $i = 1, \ldots, n$, consider the projection map $\pi_i : X_1 \times \cdots \times X_n \to X_i$ defined by $\pi_i(x_1, \ldots, x_n) = x_i$.

**Proposition.** The product topology is the smallest topology on the product $X_1 \times \cdots \times X_n$ for which the maps $\pi_i : X_1 \times \cdots \times X_n \to X_i$ are all continuous.

**Proposition.** Let the product $\prod_{i=1}^{n} X_i = X_1 \times \cdots \times X_n$ of the topological spaces $X_1, \ldots, X_n$ be endowed with the product topology, and let $Y$ be another topological space. For every $i = 1, \ldots, n$, let a function $f_i : Y \to X_i$ be given, and define $f : Y \to \prod_{i=1}^{n} X_i$ by the property that $f(y) = (f_1(y), \ldots, f_n(y))$ for every $y \in Y$. Then $f$ is continuous if and only if $f_1, \ldots, f_n$ are continuous.

**General products**

**Definition.** If $\{X_i\}_{i \in I}$ is an indexed family of sets, possibly infinite, their *product* $\prod_{i \in I} X_i$ is the set of all maps $x : I \to \bigcup_{i \in I} X_i$ such that $x(i) \in X_i$ for every $i \in I$.

In practice, knowing $x$ is equivalent to knowing the family of the elements $x(i) \in X_i$. Most of the time, we write $x(i) = x_i$ and $x = (x_i)_{i \in I}$.

**Definition.** If $x = (x_i)_{i \in I}$ is an element of $\prod_{i \in I} X_i$, then $x_i \in X_i$ is the $i$-th component, or $i$-th coordinate of $x$.

The map $\pi_i : \prod_{j \in I} X_j \to X_i$ defined by $\pi_i((x_j)_{j \in I}) = x_i$ is the $i$-th canonical projection, or the $i$-th coordinate map of $\prod_{j \in I} X_j$.

**Special case.** If $X_i$ is equal to a fixed set $X$ for every $i$, the product $\prod_{i \in I} X_i$ is just the set of all maps $f : I \to X$, and is also denoted by $X^I$. In particular, when $I = \mathbb{N}$, $X^\mathbb{N}$ is the set of all sequences in $X$.

**Proposition.** Let $X = \prod_{i \in I} X_i$ be the product of an indexed family $\{X_i\}_{i \in I}$ of topological spaces $X_i$. Let $\mathcal{B}$ be the family of subsets of $X$ of the form $\prod_{i \in I} U_i$ where each $U_i$ is open in $X_i$ and where, in addition, all but finitely many $U_i$ are equal to $X_i$ in the sense that the set $\{i \in I : U_i \neq X_i\}$ is finite. Then $\mathcal{B}$ is a basis for a topology on $X$.

**Definition.** The topology defined on $X = \prod_{i \in I} X_i$ by the basis $\mathcal{B}$ is the *product topology* on $X$. 
Remark. The family $B'$ consisting of all subsets of the form $\prod_{i \in I} U_i$ where each $U_i$ is open in $X_i$ (without assumption of finiteness for the set $\{i \in I; U_i \neq X_i\}$) is also a basis for a topology. The topology it defines is called the box topology. It of course coincides with the product topology when $I$ is finite. However, when $I$ is infinite, the box topology is less convenient than the product topology. See the next two propositions and the example afterwards.

**Proposition.** The product topology is the smallest topology on the product $X = \prod_{i \in I} X_i$ for which the coordinate maps $\pi_i: \prod_{j \in I} X_j \to X_i$ are all continuous.

**Proposition.** Let $f: Y \to \prod_{i \in I} X_i$ be a map from a topological space $Y$ to the product of a family $\{X_i\}_{i \in I}$ of topological spaces. Let $\prod_{i \in I} X_i$ be endowed with the product topology. Then $f$ is continuous if and only if each coordinate map $\pi_i \circ f: Y \to X_i$ is continuous, where $\pi_i: \prod_{j \in I} X_j \to X_i$ is the canonical projection.

**Example.** Let $I = \mathbb{N}$ and $Y = X_n = \mathbb{R}$, with the usual topology, for every $n \in \mathbb{N}$. Consider the map $f: \mathbb{R} \to \prod_{n \in \mathbb{N}} X_n = \mathbb{R}^\mathbb{N}$ which to $x \in \mathbb{R}$ associates the constant sequence $(x_n)_{n \in \mathbb{N}}$ with $x_n = x$ for every $n \in \mathbb{N}$. Then $f$ is continuous if $\mathbb{R}^\mathbb{N}$ is endowed with the product topology, but not if it is endowed with the box topology.

**Products of metric spaces**

Consider a countable family of metric spaces $(X_i, d_i)$, with $i \in \mathbb{N}$. On the product $\prod_{i \in \mathbb{N}} X_i$, consider the metric $d$ defined by

$$d(x, y) = \sum_{i=1}^{\infty} \min\{d_i(x_i, y_i), \frac{1}{2^n}\}$$

for every $x = (x_i)_{i \in \mathbb{I}}$ and $x = (x_i)_{i \in \mathbb{I}}$ in $\prod_{i \in \mathbb{N}} X_i$.

**Theorem.** The metric topology defined by $d$ is the same as the product topology on $\prod_{i \in \mathbb{N}} X_i$.

**Example: the Cantor set**

Consider a Cantor set $C = \bigcap_{n=1}^{\infty} C_n$, where each $C_n$ consists of $2^n$ disjoint intervals in $\mathbb{R}$ and is obtained from $C_{n-1}$ by removing an interval from the interior of each interval of $C_{n-1}$. For every $n = 1, 2, \ldots, n, \ldots$, let $X_n = \{0, 1\}$ be endowed with the discrete topology. If the product $\prod_{n=1}^{\infty} X_n = \{0, 1\}^\infty$ is endowed with the product topology, we will construct a homeomorphism

$$f: C \to \prod_{n=1}^{\infty} X_n = \{0, 1\}^\infty.$$

For every $x \in C$, let $I_n(x)$ be the interval of $C_n$ that contains $x$. By construction, $I_n(x)$ contains two intervals of $C_{n+1}$, one of which is $I_{n+1}(x)$; let $f_{n}(x) \in \{0, 1\}$ be defined by the property that $f_n(x) = 0$ if, among these two intervals, $I_{n+1}(x)$ is the interval on the left, and $f_n(x) = 1$ if $I_{n+1}(x)$ is the interval on the right. Then $f: C \to \prod_{n=1}^{\infty} X_n$ is defined by the property that $f(x) = (f_n(x))_{n=1,2,\ldots,n,\ldots}$. 
Proposition. $f: C \to \prod_{n=1}^\infty X_n = \{0, 1\}^\infty$ is a homeomorphism.
CHAPTER 8

Connected spaces

Recall that, in an arbitrary topological space $X$, the subsets $\emptyset$ and $X$ are always open and closed.

Definition. A topological space $X$ is connected if $\emptyset$ and $X$ are the only subsets of $X$ that are both open and closed.

This is better understood in the following way. Let a separation of $X$ be a pair $\{A, B\}$ of two non-empty closed subsets $A$, $B$ such that $A \cup B = X$ and $A \cap B = \emptyset$.

Lemma. The topological space $X$ is connected if and only if it admits no separation.

In other words, $X$ is connected if and only if it cannot be separated into two disjoint closed pieces.

Example. The subspace $X = \mathbb{R} - \{0\}$ of $\mathbb{R}$, endowed with the subspace topology induced by the usual topology of $\mathbb{R}$, is not connected because it can be separated into the union of the two disjoint closed subsets $A = (-\infty, 0)$ and $B = (0, +\infty)$.

Theorem. A subspace $X$ of the real line $\mathbb{R}$ is connected (for the subspace topology) if and only if $X$ is an interval.

Proposition. Suppose that $X$ can be written as a union $X = \bigcup_{i \in I} X_i$ of subspaces $X_i$. If each $X_i$ is connected for the subspace topology and if the intersection $\bigcap_{i \in I} X_i$ is non-empty, then $X$ is connected.

Proposition. In the topological space $X$, consider subspaces $Y$ and $Z$ such that $Y \subset Z \subset \text{cl}(Y)$. If $Y$ is connected (for the subspace topology), then $Z$ is connected.

Example. In the plane $\mathbb{R}^2$, let $Z$ be the union of the graph $Y = \{(x, \sin \frac{1}{x}); x > 0\}$ of the function $\sin \frac{1}{x}$ and of the point $(0, 0)$. Then $Z$ is connected.

Proposition. If $f : X \to Y$ is continuous and if $X$ is connected, then $f(X)$ is connected for the subspace topology induced by the topology of $Y$.

Proposition. A product $X = \prod_{i \in I} X_i$ of connected topological spaces $X_i$ is connected.

Applications

Theorem. (Intermediate Value Theorem) Let $f : X \to \mathbb{R}$ be a continuous function from a connected topological space to the real line $\mathbb{R}$. For every $x, y \in X$ and every $c \in \mathbb{R}$ that is between $f(x)$ and $f(y)$, there exists a point $z \in X$ such that $f(z) = c$. 

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Theorem. (Border Crossing Theorem) Let $A$ be a subset of the topological space $X$. If $Z$ is a connected subset of $X$ that meets both $A$ and $X - A$, then $Z$ must meet the boundary of $A$.

Path connected spaces

Definition. A path in a topological space is a continuous map $\alpha : [0, 1] \to X$, where $[0, 1]$ is the interval in $\mathbb{R}$ with the usual topology. The topological space $X$ is path connected if, for every $x, y \in X$, there exists a path $\alpha$ with $\alpha(0) = x$ and $\alpha(1) = y$.

Proposition. Every path connected topological space is connected.

Example. For every $n \geq 2$, $\mathbb{R}^n - \{0\}$ is path connected and therefore connected. On the other hand, $\mathbb{R} - \{0\}$ is disconnected.

Corollary. If $n \geq 2$, there exists no homeomorphism between $\mathbb{R}^n$ and $\mathbb{R}$.

Example. In the plane $\mathbb{R}^2$, let $Z$ be the union of the graph $Y = \{(x, \sin \frac{1}{x}); x > 0\}$ of the function $\sin \frac{1}{x}$ and of the point $(0, 0)$. Then $Z$ is connected, but not path-connected.

Connected components

Definition. In a topological space $X$, the connected component of $x \in X$ is the set of all $y \in X$ for which there exists a connected subspace $Z$ of $X$ which contains both $x$ and $y$.

Proposition. Every connected component of $X$ is a connected subspace of $X$.

Proposition. The connected components of $X$ form a partition of $X$. Namely, every point of $X$ belongs to one and only one connected component. In particular, two connected components are either disjoint or equal.
 CHAPTER 9

The quotient topology

Definition. A quotient map is a surjective map \( p: X \to \bar{X} \) between two sets \( X \) and \( \bar{X} \).

Example. Let \( \bar{X} \) be a partition of \( X \). Namely, \( \bar{X} \) is a set of subsets of \( X \) such that every \( x \in X \) belongs to one and only one subset \( A \in \bar{X} \). Define \( p: X \to \bar{X} \) by the property that \( p(x) \) is this subset \( p(x) \in \bar{X} \) with \( x \in p(x) \in \text{bar}X \).

Example. If \( \sim \) is an equivalence relation on \( X \) (if you know what that is) and if \( \bar{X} \) is the set of equivalence classes of \( \sim \), the map \( p: X \to \bar{X} \) which to \( x \in X \) associates its equivalence class \( \bar{x} \in \bar{X} \) is a quotient map.

Lemma. If \( X \) is a topological space and if \( p: X \to \bar{X} \) is a quotient map, then
\[ T = \{ U \subset \bar{X}; p^{-1}(U) \text{ is open in } X \} \]
is a topology on \( \bar{X} \). It is the largest topology on \( \bar{X} \) for which \( p \) is continuous.

Definition. The above topology \( T \) on \( \bar{X} \) is the quotient of the topology of \( X \) by the quotient map \( p: X \to \bar{X} \).

The intuitive idea is that \( \bar{X} \) is obtained from \( X \) by gluing together those points of \( X \) which have the same image under \( p \).

Proposition. Let \( f: X \to Y \) be a continuous map between topological spaces, and let \( p: X \to \bar{X} \) be a quotient map. Suppose in addition that \( f(x) = f(y) \) for every \( x, y \in X \) such that \( p(x) \). Then, if \( \bar{X} \) is endowed with the quotient topology, there is a unique continuous map \( \bar{f}: \bar{X} \to Y \) such that \( \bar{f} \circ p = f \). In fact, the quotient topology is the smallest topology on \( \bar{X} \) for which this property holds.

Examples.

1. Let \( X \) be the interval \([0, 1]\), and consider the partition \( \bar{X} \) consisting of the subset \( \{0, 1\} \) and of all the subsets \( \{x\} \) with \( 0 < x < 1 \). Endow \( \bar{X} \) with the quotient topology. Then \( \bar{X} \) is homeomorphic to the unit circle \( S^1 \) in the plane.

2. Let \( X \) be the square \([0, 1] \times [0, 1] \) in the plane, and let \( \bar{X} \) be the partition consisting of the subsets
   - \( \{(x, y)\} \) with \( 0 < x < 1 \) and \( 0 \leq y \leq 1 \);
   - \( \{(0, y), (1, y)\} \) with \( 0 \leq y \leq 1 \).

   Then \( \bar{X} \) is homeomorphic to the cylinder \( S^1 \times [0, 1] \).

3. Let \( X \) be the square \([0, 1] \times [0, 1] \) in the plane, and let \( \bar{X} \) be the partition consisting of the subsets
   - \( \{(x, y)\} \) with \( 0 < x < 1 \) and \( 0 \leq y \leq 1 \);
   - \( \{(0, y), (1, 1 - y)\} \) with \( 0 \leq y \leq 1 \).
Then $\tilde{X}$ is homeomorphic to the *Möbius strip*, namely the surface in $\mathbb{R}^3$ parametrized by

$$(u, v) \mapsto \left( (2 + u \cos(\pi v)) \cos(2\pi v), (2 + u \cos(\pi v)) \sin(2\pi v), u \sin(\pi v) \right)$$

for $-1 \leq u \leq 1$, $0 \leq v \leq 1$.

4. Let $X$ be the square $[0, 1] \times [0, 1]$ in the plane, and let $\tilde{X}$ be the partition consisting of the subsets

- $\{(x, y)\}$ with $0 < x < 1$ and $0 < y < 1$;
- $\{(x, 0), (x, 1)\}$ with $0 < x < 1$;
- $\{(0, y), (1, y)\}$ with $0 < y < 1$;
- $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$.

Then $\tilde{X}$ is homeomorphic to the torus $S^1 \times S^1$. The torus is also homeomorphic to the surface in $\mathbb{R}^3$ parametrized by

$$(u, v) \mapsto ((2 + \cos(2\pi u)) \cos(2\pi v), (2 + \cos(2\pi u)) \sin(2\pi v), \sin(2\pi u))$$

for $0 \leq u \leq 1$, $0 \leq v \leq 1$.

5. Let $X$ be the square $[0, 1] \times [0, 1]$ in the plane, and let $\tilde{X}$ be the partition consisting of the subsets

- $\{(x, y)\}$ with $0 < x < 1$ and $0 < y < 1$;
- $\{(x, 0), (x, 1)\}$ with $0 < x < 1$;
- $\{(0, y), (1, 1 - y)\}$ with $0 < y < 1$;
- $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$.

Then $\tilde{X}$ is the *Klein bottle*. It is the union of two Möbius strip meeting along their boundary.
6. Let $X$ be the unit sphere $S^2$ in $\mathbb{R}^3$, and let the partition $\tilde{X}$ consist of all pairs of antipodal points $\{x, -x\}$ with $x \in S^2$. Then $\tilde{X}$ is the projective plane. It is the union of a disk and of a Möbius strip meeting along their boundaries.

7. Let $\tilde{X}$ be the space of half-lines emanating from the origin in $\mathbb{R}^3$. Namely, let $X$ be $\mathbb{R}^3 - \{0\}$, and let $\tilde{X}$ consist of all the subsets $\{ax; 0 < a < +\infty\}$ with $x \in X$. Then the quotient space $\tilde{X}$ is homeomorphic to the projective plane of the previous example.