Additional exercises for the book

Low-dimensional geometry:
from euclidean surfaces to hyperbolic knots

by Francis Bonahon

The exercises in the book are often challenging. Here is a list of additional exercises which are a little easier and student-friendly, for the first six chapters of the book.

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Exercises on complex numbers.

Additional Exercise 1. Let $z = 2 + \mathrm{i}$ and $z' = 3 - 2\mathrm{i}$. Write the product zz' and the quotient $\frac{z}{z'}$ in the form $a + \mathrm{i}b$, with a, $b \in \mathbb{R}$.

Additional Exercise 2. Let $z \in \mathbb{C}$ be a complex number, and let $\bar{z}$ be its conjugate. Show that z is a real number if and only if $z = \bar{z}$. Namely:

- **a.** First show that, if z is a real number, then $z = \bar{z}$.
- **b.** Then show that, if $z = \bar{z}$, then z is a real number.

Additional Exercise 3. Find r and θ so that $\mathrm{i} - 1 = r\mathrm{e}^{\mathrm{i}\theta}$. Hint: First plot $\mathrm{i} - 1$ in the complex plane, and use polar coordinates.

Exercises on Chapter 1.

Additional Exercise 4. Let $\varphi\colon \mathbb{C} \to \mathbb{C}$ be the rotation of angle θ around the point $z_0 \in \mathbb{C}$. Express $\varphi(z)$ in terms of z, z_0 and $\mathrm{e}^{\mathrm{i}\theta}$.

Additional Exercise 5. The map $\psi\colon \mathbb{C} \to \mathbb{C}$ defined by $\psi(z) = -\bar{z}$ is a relatively simple transformation of the plane. What is it? (Namely describe it with words, such as "the rotation of angle $\frac{\pi}{7}$ around the point $2 - \mathrm{i}$"; of course, this one is not the answer.)

Additional Exercise 6. Let (X, d) be a metric space. Show that

$$-d(Q, R) \leqslant d(P, Q) - d(P, R) \leqslant d(Q, R)$$

for every P, Q, $R \in X$.

Additional Exercise 7. Let (X, d) and (X', d') be two metric spaces. We saw in class that a map $\varphi\colon X \to X'$ is *continuous* at the point $P_0 \in X$ if, for every $\varepsilon > 0$, there exists a number $\delta > 0$ such $d'\big(\varphi(P), \varphi(P_0)\big) < \varepsilon$ whenever $d(P, P_0) < \delta$. Also, φ is an isometry if it is bijective and if $d'\big(\varphi(P), \varphi(Q)\big) = d(P, Q)$ for every P, $Q \in X$.

Show that, if $\varphi\colon X \to X'$ is an isometry, then it is continuous at every point $P_0 \in X$.

Additional Exercise 8. Consider the metric space $(\mathbb{R}, d)$ where $\mathbb{R} = (-\infty, \infty)$ is the real line and where d is the usual distance, defined by the property that $d(x, y) = |y - x|$ for every $x, y \in \mathbb{R}$. Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be an isometry from $(\mathbb{R}, d)$ to itself. Consider the image $x_0 = \varphi(0)$.

- **a.** Show that, for every $x \in \mathbb{R}$, the image $\varphi(x)$ is equal to $x_0 + x$ or to $x_0 - x$.
- **b.** By Problem 1, the map φ is continuous. Use this property to show that, either $\varphi(x) = x_0 + x$ for every $x \in \mathbb{R}$, or $\varphi(x) = x_0 - x$ for every $x \in \mathbb{R}$. (Make sure that you understand the difference with the statement of Part a.)

Additional Exercise 9. In the euclidean plane $(\mathbb{R}^2, d_{\mathrm{euc}})$, consider four points P, Q, P', $Q' \in \mathbb{R}^2$ such that $d_{\mathrm{euc}}(P, Q) = d_{\mathrm{euc}}(P', Q') \neq 0$.

- **a.** Show that there is a euclidean translation $\psi\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\psi(P) = P'$.
- **b.** Composing this translation with a rotation around P', show that there is an isometry $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\varphi(P) = P'$ and $\varphi(Q) = Q'$.
- **c.** Let $\varphi_1\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $\varphi_2\colon \mathbb{R}^2 \to \mathbb{R}^2$ be two isometries such that $\varphi_1(P) = P'$, $\varphi_1(Q) = Q'$, $\varphi_2(P) = P'$ and $\varphi_2(Q) = Q'$. Show that the composition $\varphi_1^{-1} \circ \varphi_2$ is an isometry of $(\mathbb{R}^2, d_{\mathrm{euc}})$ such that $\varphi_1^{-1} \circ \varphi_2(P) = P$ and $\varphi_1^{-1} \circ \varphi_2(Q) = Q$. Conclude that $\varphi_1^{-1} \circ \varphi_2$ is, either the identity map, or the reflection across the line PQ.
- **d.** Show that there are exactly two isometries $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\varphi(P) = P'$ and $\varphi(Q) = Q'$.

Additional Exercise 10. In the euclidean plane $\mathbb{R}^2 = \mathbb{C}$, consider the line L that passes through the point z_0 and makes an angle of θ with the horizontal. If φ denotes the euclidean reflection across this line L, find a formula expressing $\varphi(z)$ as a function of z, z_0 and θ.

Additional Exercise 11. The metric space (X, d) is *locally isometric* to the metric space (X', d') if, for every $P \in X$, there exists an isometry $\varphi\colon B_d(P, r) \to B_{d'}(P', r)$ from a small ball $B_d(P, r)$ centered at P in X and a small ball $B_{d'}(P', r)$ in X'. Also, (X, d) is *locally homogeneous* if, for every P, $Q \in X$, there exists an isometry $\varphi\colon B_d(P, r) \to B_d(Q, r)$ from a small ball $B_d(P, r)$ centered at P in X to a small ball $B_d(Q, r)$ centered at Q.

Show that, if (X, d) is locally isometric to (X', d') and if (X', d') is locally homogeneous, then (X, d) is locally homogeneous.

Additional Exercise 12. For two points P and Q in the plane $\mathbb{R}^2$, define

$$d(P, Q) = \begin{cases} \ell_{\mathrm{euc}}([O, P]) + \ell_{\mathrm{euc}}([O, Q]) & \text{if } P \neq Q \\ 0 & \text{if } P = Q \end{cases}$$

where $[O, P]$ denotes the line segment joining P to the origin O, and where ℓ_{euc} is the euclidean length.

- **a.** Show that $(\mathbb{R}^2, d)$ is a metric space, namely that the function d so defined satisfies all the properties required to be a metric.
- **b.** Suppose in addition that $d_{\mathrm{euc}}(O, P) = 1$. Describe the balls $B_d(P, \frac{1}{2})$ and $B_d(P, \frac{3}{2})$. (I remind you that $B_d(P, r) = \{Q \in \mathbb{R}^2; d(P, Q) < r\}$.)

Additional Exercise 13. We want to endow the real line $\mathbb{R}$ with a new metric d, defined by the property that

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ \max\{\frac{1}{q};\ p,q \text{ integers},\ q>0,\ x<\frac{p}{q}<y\} & \text{if } x<y \\ \max\{\frac{1}{q};\ p,q \text{ integers},\ q>0,\ y<\frac{p}{q}<x\} & \text{if } x>y. \end{cases}$$

(Namely, $d(x,y)$ is 1 over the smallest denominator of a rational number siting between x and y.) Show that d is indeed a metric, and that $(\mathbb{R}, d)$ is a metric space.

Additional Exercise 14. Let (X, d) be a metric space where the metric d satisfies the property that

$$d(P,R) \leqslant \max\bigl\{d(P,Q), d(Q,R)\bigr\}$$

for every P, Q, $R \in X$.

- **a.** Show that, for every Q in the ball $B_d(P,r)$, the ball $B_d(Q,r)$ is contained in $B_d(P,r)$ (namely that every point $R \in B_d(Q,r)$ is contained in $B_d(P,r)$).
- **b.** Conclude that, if $Q \in B_d(P,r)$, the balls $B_d(Q,r)$ and $B_d(P,r)$ are equal.
- **c.** Finally conclude that, if there is a point Q that is in both $B_d(P,r)$ and $B_d(P',r)$, then the two balls $B_d(P,r)$ and $B_d(P',r)$ are equal.

Exercises on Chapter 2.

Additional Exercise 15. In the hyperbolic plane $\mathbb{H}^2$, consider the two points $P = \mathrm{i}$ and $Q = 4+\mathrm{i}$. For $t>0$, let $P_t = t\mathrm{i}$, let $Q_t = 4 + t\mathrm{i}$, and let γ_t be the curve going from P to Q that is made up of the vertical line segment $[P, P_t]$, followed by the horizontal line segment $[P_t, Q_t]$, and finally followed by the vertical segment $[Q_t, Q]$.

- **a.** Draw a picture of γ_t.
- **b.** Compute the hyperbolic length $\ell_{\text{hyp}}(\gamma_t)$.
- **c.** For which value of t is $\ell_{\text{hyp}}(\gamma_t)$ minimum? (Remember calculus?)
- **d.** Use Part c to show that $d_{\text{hyp}}(P,Q) \leqslant 2\ln 2 + 2$.

Additional Exercise 16. Let $\varphi\colon \mathbb{H}^2 \to \mathbb{H}^2$ be the map defined by the property that $\varphi(x,y) = (-x,y)$. (Namely, φ is the euclidean reflection across the y–axis.)

- **a.** Show that, if γ is a curve in $\mathbb{H}^2$ and if γ_1 is the image of γ under φ, then $\ell_{\text{hyp}}(\gamma_1) = \ell_{\text{hyp}}(\gamma)$.
- **b.** Use Part a to show that φ is an isometry from $(\mathbb{H}^2, d_{\text{hyp}})$ to itself.

Additional Exercise 17. Let g be a complete geodesic of $\mathbb{H}^2$, and consider a point $P \in \mathbb{H}^2$.

- **a.** First consider the case where g is a vertical half-line $g = \{(x_0,y) \in \mathbb{R}^2; y > 0\}$. Show that there exists a unique complete geodesic h containing P and orthogonally cutting g at some point Q (namely h and g meet in Q and form an angle of $\frac{\pi}{2}$ there).

b. In the case of a general complete geodesic g, show that there exists a unique complete geodesic h containing P and orthogonally cutting g at some point Q. (It may be useful to use the following two results. The first one, already seen in class or in Lemma 2.6 of the textbook, is that there is an isometry φ of $(\mathbb{H}^2, d_{\text{hyp}})$ that sends g to a vertical half-line. The second result, which we have not yet seen in class but can be found in the textbook as Corollary 2.17, is that an isometry φ of $(\mathbb{H}^2, d_{\text{hyp}})$ respects angles in the sense that, if two curves γ_1 and γ_2 meet at a point P_1 and form an angle θ there, then the angle between their images $\varphi(\gamma_1)$ and $\varphi(\gamma_2)$ at $\varphi(P_1)$ is also equal to θ.)

c. For g, h, P and Q as above, suppose now that h is a vertical half-line and that the point P lies above Q on this half-line. Show that Q is the point of g that is closest to P, in the sense that $d_{\text{hyp}}(P, Q') > d_{\text{hyp}}(P, Q)$ for every $Q' \in g$ different from Q.

d. Now consider the general case. Let Q be associated to g and P as in Part b. Show that Q is the point of g that is closest to P. (Same hint as for Part b.)

(Note that this construction is the same as the construction, in euclidean geometry, of the orthogonal projection Q of a point $P \in \mathbb{R}^2$ to a straight line g.)

Additional Exercise 18. Let g be a complete geodesic of $\mathbb{H}^2$, which is a semi-circle of radius R centered at $x_0 \in \mathbb{R}$. Consider the map

$$\varphi(z) = x_0 + R^2 \frac{z - x_0}{|z - x_0|^2}.$$

a. Find a, b, c, $d \in \mathbb{R}$ with $ad - bc = 1$ such that

$$\varphi(z) = \frac{c\bar{z} + d}{a\bar{z} + b}$$

for every z.

b. Combine Part a with a result from class to conclude that φ defines an isometry $\varphi\colon \mathbb{H}^2 \to \mathbb{H}^2$ of $(\mathbb{H}^2, d_{\text{hyp}})$.

c. Show that $\varphi(z) = z$ for every z in the geodesic g, and that φ exchanges the two halves of $\mathbb{H}^2$ delimited by g.

d. Consider a point $P \in \mathbb{H}^2$. As in Problem 1, let h be the complete geodesic that contains P and orthogonally meets g at a point Q. Show that φ sends h to h. (As in the hint for Part b for Problem 1, use the fact that φ respects angles). Conclude that $\varphi(P)$ is the point of the geodesic h that is at the same distance from Q as P, but is on the other side of g.

(Again, this construction is the hyperbolic analogue of the euclidean reflection across a straight line.)

Additional Exercise 19. In the hyperbolic plane $\mathbb{H}^2$, consider the usual vertical half-line $L = \{\mathrm{i}y; y > 0\}$. Given two points $P_1 = \mathrm{i}y_1$ and $P_2 = \mathrm{i}y_2$ on L, find the point $M = \mathrm{i}y$ of L such that

$$d_{\text{hyp}}(P_1, M) = d_{\text{hyp}}(P_2, M).$$

Additional Exercise 20. Show that all isometries φ of $(\mathbb{H}^2, d_{\text{hyp}})$ such that $\varphi(\mathrm{i}) = \mathrm{i}$ are of the form

$$\varphi(z) = \frac{az+b}{-bz+a} \quad \text{with } a^2+b^2=1$$

or

$$\varphi(z) = \frac{-b\bar{z}+a}{a\bar{z}+b} \quad \text{with } a^2+b^2=1.$$

Additional Exercise 21. Let $\varphi\colon \mathbb{H}^2 \to \mathbb{H}^2$ be the reflection across the y–axis defined by $\varphi(x,y) = (-x,y)$, and consider the vertical half-line $L = \{(0,y); y>0\}$ in $\mathbb{H}^2$. Consider a point $P_0 = (x_0, y_0) \in \mathbb{H}^2$ with $x_0 > 0$ in $\mathbb{H}^2$, and its image $P_0' = \varphi(P_0) = (-x_0, y_0)$.

- **a.** Let P be a point on L. Show that $d_{\text{hyp}}(P, P_0) = d_{\text{hyp}}(P, P_0')$.
- **b.** Let $P = (x,y) \in \mathbb{H}^2$ be a point with $x > 0$, namely a point located in the half of $\mathbb{H}^2$ delimited by L that contains P_0.
  - (i) Let g be the geodesic arc going from P to P_0', and decompose it in two pieces: a geodesic arc g_1 going from P to a point $Q \in L$, and a geodesic arc g_2 going from Q to P_0' (draw a picture). Use some of the arcs g_1, g_2, $\varphi(g_1)$, $\varphi(g_2)$ to construct a curve going from P to P_0 whose hyperbolic length is exactly equal to the hyperbolic length of g.
  - (ii) Use Part (ii) to show that $d_{\text{hyp}}(P, P_0) < d_{\text{hyp}}(P, P_0')$.

Additional Exercise 22. Let $\varphi\colon \mathbb{H}^2 \to \mathbb{H}^2$ be the isometry of $(\mathbb{H}^2, d_{\text{hyp}})$ defined by

$$\varphi(z) = \frac{az+b}{cz+d} \quad \text{with } a,b,c,d \in \mathbb{R} \text{ and } ad-bc=1.$$

Suppose in addition that $|a+d| > 2$ and $c \neq 0$.

- **a.** Show that there exists exactly two points $x \in \mathbb{R}$ such that $\varphi(x) = x$. Hint: quadratic formula.
- **b.** Use Part a to show that there is a unique complete geodesic g in $\mathbb{H}^2$ such that $\varphi(g) = g$.

Additional Exercise 23. Let $\varphi\colon \mathbb{H}^2 \to \mathbb{H}^2$ be the isometry of $(\mathbb{H}^2, d_{\text{hyp}})$ defined by

$$\varphi(z) = \frac{az+b}{cz+d} \quad \text{with } a,b,c,d \in \mathbb{R} \text{ and } ad-bc=1.$$

Suppose in addition that $a+d = \pm 2$.

- **a.** Show that if, in addition, $c = 0$ then φ is a horizontal translation $\varphi(z) = z+u$ for some $u \in \mathbb{R}$.
- **b.** Show that, if $c \neq 0$, there exists exactly one point $x_0 \in \mathbb{R}$ such that $\varphi(x_0) = x_0$. Hint: quadratic formula.
- **c.** Let $\psi\colon \mathbb{H}^2 \to \mathbb{H}^2$ be a hyperbolic isometry sending the point x_0 of Part b to the point ∞. (We saw in class that such a ψ always exists.) Show that $\psi \circ \varphi \circ \psi^{-1}$ sends ∞ to ∞, and fixes no point $x \in \mathbb{R}$.
- **d.** Conclude that $\psi \circ \varphi \circ \psi^{-1}$ is a horizontal translation.

Additional Exercise 24. Consider the linear fractional map

$$\varphi(z) = \frac{3\bar{z} - 2}{\bar{z} - 1}.$$

Express φ as the composition $\varphi = \alpha \circ \beta \circ \gamma$ of a horizontal translation α, a standard inversion β and another horizontal translation γ.

Additional Exercise 25. Consider a hyperbolic isometry

$$\varphi(z) = \frac{az + b}{cz + d} \quad \text{with } a, b, c, d \in \mathbb{R},\ ad - bc = 1 \text{ and } c \neq 0$$

and the horizontal line

$$L = \{x + \mathrm{i}; x \in \mathbb{R}\}$$

defined by the equation $y = 1$.

- **a.** Compute $\varphi(\infty)$.
- **b.** Remember that we saw in class that a linear fractional map sends circle to circle (if we consider a line plus the point ∞ as a circle of infinite radius). Use this property to show that φ sends L to a $C - \{\frac{a}{c}\}$, where C is a circle in $\mathbb{C}$ that is tangent to the real line $\mathbb{R}$ at the point $\frac{a}{c}$, and where $C - \{\frac{a}{c}\}$ denotes the circle C from which the point $\frac{a}{c}$ has been removed.
- **c.** Compute the imaginary part of $\varphi(x + i)$, and find the maximum of this imaginary part as x ranges over all points of $\mathbb{R}$.
- **d.** Use Part b to find the radius of the circle C.

Additional Exercise 26. Let $\varphi\colon \mathbb{H}^2 \to \mathbb{H}^2$ be a hyperbolic isometry sending the point ∞ to itself. Show that, either φ is a horizontal translation $\varphi(z) = z + x_0$ with $x_0 \in \mathbb{R}$, or there exists a complete geodesic g such that $\varphi(g) = g$. (Possible hint: Write φ as $\varphi(z) = \frac{az+b}{cz+d}$ or $\varphi(z) = \frac{c\bar{z}+d}{a\bar{z}+b}$ and look for the end points of g.)

Exercises on Chapters 4-5.

Additional Exercise 27. In the plane $X = \mathbb{R}^2$, consider for each $c \in \mathbb{R}$ the hyperbola

$$H_c = \{(x, y) \in \mathbb{R}^2; xy = c\}.$$

(When $c = 0$, the "hyperbola" H_0 is somewhat degenerate.)

- **a.** Draw a picture of H_1, H_{-1}, H_0 and $H_{\frac{1}{2}}$.
- **b.** Show that the hyperbolas H_c form a partition $\bar{X}$ of $X = \mathbb{R}^2$, in the sense that every point $P \in \mathbb{R}^2$ belongs to one and only one hyperbola H_c.
- **c.** Consider the hyperbolas H_{c_1} and H_{c_2} associated to positive numbers c_1, $c_2 > 0$. Show that, for every $\varepsilon > 0$, there exist two points $P_1 \in H_{c_1}$ and $P_2 \in H_{c_2}$ such that $d_{\mathrm{euc}}(P_1, P_2) < \varepsilon$.
- **d.** More generally, consider the hyperbolas H_{c_1} and H_{c_2} associated to arbitrary numbers c_1, $c_2 \in \mathbb{R}$. Show that, for every $\varepsilon > 0$, there exist two points $P_1 \in H_{c_1}$ and $P_2 \in H_{c_2}$ such that $d_{\mathrm{euc}}(P_1, P_2) < \varepsilon$.
- **e.** Let $\bar{d}_{\mathrm{euc}}$ be the quotient semi-metric on the partition $\bar{X}$ defined (using discrete walks as seen in class) by the euclidean metric d_{euc} of $X = \mathbb{R}^2$. In particular, for $P \in \mathbb{R}^2$, let $\bar{P} \in \bar{X}$ denote the hyperbola H_c that contains it.

(i) Show that $\bar{d}_{\text{euc}}(\bar{P}_1, \bar{P}_2) \leqslant d_{\text{euc}}(P_1, P_2)$, for every $P_1 \in H_{c_1}$ and $P_2 \in H_{c_2}$. (Hint: Did we see this in class?)

(ii) Conclude that $\bar{d}_{\text{euc}}(\bar{P}_1, \bar{P}_2) = 0$ for every $\bar{P}_1$, $\bar{P}_2 \in \bar{X}$. Hint: Part d.

Additional Exercise 28. Consider the partition $\bar{X}$ of the plane $X = \mathbb{R}^2$ consisting of all subsets $A \subset X$ of the form

$$A = \begin{cases} \{(x,y), (-x,y)\} & \text{if } x \neq 0 \\ \text{or} \\ \{(0,y)\}. \end{cases}$$

Let $\bar{d}_{\text{euc}}$ be the quotient semi-metric on $\bar{X}$ induced by the euclidean metric d_{euc} of $X = \mathbb{R}^2$. We want to analyze $(\bar{X}, \bar{d}_{\text{euc}})$.

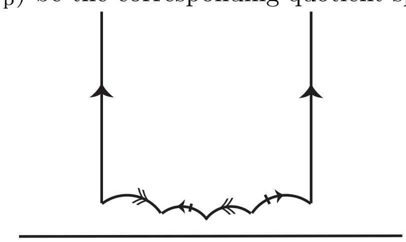
- **a.** Show that $\bar{X}$ is a partition of $X = \mathbb{R}^2$, namely that every point $P \in X$ belongs to exactly one subset $\bar{P} \in \bar{X}$.
- **b.** Let X_+ denote the closed half-space $X_+ = \{(x,y) \in \mathbb{R}^2; x \geqslant 0\}$, and consider the map $\varphi\colon X_+ \to \bar{X}$ which to the point $P \in X_+$ associates the subset $\bar{P} \in \bar{X}$ that contains it. Show that φ is bijective (namely show that it is injective and surjective).
- **c.** If $P = (x,y)$ and $Q = (u,v)$, set $P' = (|x|, y)$ and $Q' = (|u|, v)$. Show that $d_{\text{euc}}(P', Q') \leqslant d_{\text{euc}}(P, Q)$.
- **d.** For P, $Q \in X^+$, consider a discrete walk w from $\bar{P}$ to $\bar{Q}$ consisting of the points $P = P_1, Q_1 \sim P_2, Q_2 \sim P_3, \ldots, Q_{n-1} \sim P_n, Q_n = Q$ of X. Show that there is another discrete walk w' from $\bar{P}$ to $\bar{Q}$ consisting of points $P = P_1'$, $Q_1' = P_2'$, $Q_2' = P_3'$, $\ldots$, $Q_{n-1}' = P_n'$, $Q_n' = Q$, such that $\ell_{d_{\text{euc}}}(w') \leqslant \ell_{d_{\text{euc}}}(w)$ and such that all points P_i' and Q_i' are in X_+. Hint: Part c.
- **e.** For the discrete walk w' of Part d, show that $\ell_{d_{\text{euc}}}(w') \geqslant d_{\text{euc}}(P, Q)$.
- **f.** Show that $\bar{d}_{\text{euc}}(\bar{P}, \bar{Q}) = d_{\text{euc}}(P, Q)$ for every P, $Q \in X_+$. Conclude that $\varphi\colon X_+ \to \bar{X}$ is an isometry from (X_+, d_{euc}) to $(\bar{X}, \bar{d}_{\text{euc}})$.

Additional Exercise 29. Let X be a regular decagon (= polygon with 10 edges and 10 vertices) in the euclidean plane $(\mathbb{R}^2, d_{\text{euc}})$, and let $(\bar{X}, \bar{d}_{\text{euc}})$ be the quotient space obtained by gluing by euclidean translations opposite edges of the decagon X.

- **a.** The vertices of X correspond to how many points of $\bar{X}$?
- **b.** Is the quotient space $(\bar{X}, \bar{d}_{\text{euc}})$ localy isometric to the euclidean plane $(\mathbb{R}^2, d_{\text{euc}})$? Explain.
- **c.** Give a "proof by pictures", like the ones we have used in class in recent weeks, suggesting that the quotient space $(\bar{X}, \bar{d}_{\text{euc}})$ is homeomoprhic to the surface of genus 2 (namely the surface we already obtained by gluing opposite edges of an octagon).
- **d.** (No credit) If we glue opposite sides of a $2n$–gon X in $\mathbb{R}^2$, what do you think the quotient space X is homeomorphic to? (Hint: do you see a pattern in the cases $n = 2$, 3, 4, 5?

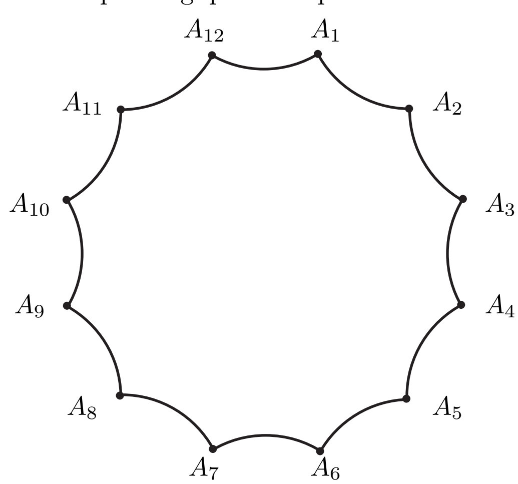
Additional Exercise 30. Consider the gluing construction defined by a partition $\bar{X}$ of a metric space (X, d). Give the definition of the quotient semi-metric $\bar{d}$. Namely, for every $\bar{P}$, $\bar{Q} \in \bar{X}$, explain how $\bar{d}(\bar{P}, \bar{Q})$ is defined.

Additional Exercise 31. In the hyperbolic plane $(\mathbb{H}^2, d_{\text{hyp}})$, let X be the hyperbolic polygon indicated on the figure. Glue the edges of X as indicated on the picture, and let $(\bar{X}, \bar{d}_{\text{hyp}})$ be the corresponding quotient space.



- **a.** How many points of $\bar{X}$ correspond to the vertices of X?
- **b.** If the angles of X at its 5 vertices are all equal to θ, is there a value of θ for which the quotient space $(\bar{X}, \bar{d}_{\text{hyp}})$ is locally isometric to the hyperbolic plane $(\mathbb{H}^2, d_{\text{hyp}})$?
- **c.** Give a "proof by picture" to show that $(\bar{X}, \bar{d}_{\text{hyp}})$ is homeomorphic to a torus minus a point.

Additional Exercise 32. Let X be a regular dodecagon in the hyperbolic plane $(\mathbb{H}^2, d_{\text{hyp}})$, with all 12 sides of equal lengths and all 12 angles equal to θ. Label the vertices of X as $V_1, V_2, \ldots, V_{12}$ in this order around X, and glue the edge A_1A_2 to A_8A_7, the edge A_2A_3 to A_1A_{12}, the edge A_3A_4 to A_6A_5, the edge A_4A_5 to $A_{11}A_{10}$, the edge A_6A_7 to A_9A_8, and the edge A_9A_{10} to $A_{12}A_{11}$. (It may help to draw arrows on the picture below, which represents X is the disk model for symmetry.) Let $(\bar{X}, \bar{d}_{\text{hyp}})$ be the corresponding quotient space.



- **a.** How many points of $\bar{X}$ correspond to the vertices of X?
- **b.** For which value of θ is the quotient space $(\bar{X}, \bar{d}_{\text{hyp}})$ locally isometric to the hyperbolic plane $(\mathbb{H}^2, d_{\text{hyp}})$?

Additional Exercise 33. Let X be a polygon in the euclidean plane $\mathbb{R}^2$, and let $\bar{X}$ be the quotient space obtained by gluing edges of X together. Given two points $\bar{P}, \bar{Q} \in \bar{X}$ in this quotient space, give the precise definition of a discrete walk from $\bar{P}$ to $\bar{Q}$.

Exercises on Chapter 6.

Additional Exercise 34. It can be shown that, in the hyperbolic plane $\mathbb{H}^2$, there is a regular quadrilateral X whose four sides have the same length and whose four angles are equal to $\frac{3}{7}\pi$. Is there a tessellation of $\mathbb{H}^2$ whose tiles are all isometric to X?

Additional Exercise 35. Sketch a tessellation of the euclidean plane $\mathbb{R}^2$ by triangles whose angles are $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{6}$.