Teaching diary

In the Fall 2010, I used the book *Low-dimensional geometry: from euclidean surfaces to hyperbolic knots* as the textbook in the class Math 434, Geometry and Transformations, at USC. Most of the students in the class were juniors and seniors majoring in one of our math oriented majors (BS in Mathematics, BA in Mathematics, BS in Mathematical Economics, BS in Mathematical Finance), but this was the first proof-oriented course for many of them. The only prerequisite for the class was multivariable calculus.

The goal was to cover the first eight chapters of the book. What follows is a brief summary of the topics covered on each day.

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Day 1. The geometry of complex numbers. The material is essentially taken from Section T.4 in the TOOL KIT appendix. (This was the first day of class, and the first half of the period was devoted to a brief presentation of the course and to ancillary technicalities.)

- Definition of complex numbers, addition, multiplication.
- Real part, imaginary part, modulus, complex conjugate.
- Example: If $z = 2 + 3\mathrm{i}$ and $z' = 3 - 2\mathrm{i}$, compute $z + z'$, zz' and $\frac{z}{z'}$.

Day 2. The geometry of complex numbers (continued).

- Definition of $e^{i\theta} = \cos\theta + i\sin\theta$. Mention $e^{\pi i} = -1$ for the sake of cuteness. Explain the exponential notation by the power series for e^x, $\sin x$ and $\cos x$. Show that $e^{i(\theta+\theta')} = e^{i\theta}e^{i\theta'}$ by the addition formulas for sin and cos.
- If $z \in \mathbb{C}$ has polar coordinates $[r, \theta]$, then $z = re^{i\theta}$. Explain that this makes complex coordinates and exponentials very convenient for computations in euclidean geometry.
- The rotation of angle θ around 0 is nicely expressed in complex coordinates as $\varphi(z) = ze^{i\theta}$. Compare to its ugly expression in cartesian coordinates.
- Let L be the line obtained by rotating $\mathbb{R}$ around 0 by an angle of θ in $\mathbb{C}$, and let ψ be the reflection across this line L. Then $\psi(z) = \bar{z}e^{2i\theta}$.

Day 3. We begin Chapter 1.

- The definition of the euclidean plane $\mathbb{R}^2$. (The language of set theory is progressively introduced, here by using brackets $\{\ \}$ to define a set. We had already started using the symbol $\in$.)
- Formula for the euclidean arc length.
- The euclidean distance defined as an infimum of euclidean arc length. Digression about the notion of infimum, as in Section T.2 of the Tool Kit appendix.
- The line segment $[P, Q]$ is the shortest curve from P to Q. Explicit formula for the distance between two points.

Day 4.

- Definition of metric spaces. Fundamental example: the euclidean plane with the euclidean metric. Motivate by calculus: convergence of a sequence. Balls in a metric space.
- Example of a metric space: the plane with the taxicab metric, where the metric from P to Q is defined as the infimum

of the taxicab arc lengths

$$l_{\text{taxi}}(\gamma) = \int_a^b |x'(t)| + |y'(t)| \, dt$$

of all curves γ going from P to Q.

Day 5.

- The taxicab metric (continued). Check that this is a metric. Shortest curves and explicit formula for the distance. Ball of radius r.

Day 6.

- Isometries of a metric space.
- Isometries of the euclidean plane, expressed in terms of complex numbers.
- Quick remark: isometries of $\mathbb{R}^2$ with the taxicab metric.
- The euclidean plane is homogeneous, and isotropic.

Day 7. We begin discussing the hyperbolic plane.

- Intro: the hyperbolic plane is another space that is homogeneous and isotropic. Brief discussion of Euclid's axioms, and the 5th postulate. Non-euclidean geometries were shown to be as consistent as euclidean geometry.
- The hyperbolic plane $\mathbb{H}^2$, the hyperbolic arc length $l_{\text{hyp}}(\gamma)$, the hyperbolic distance $d_{\text{hyp}}(P, Q)$ as an infimum of hyperbolic arc lengths.
- Appetite wetting: picture of the shortest curve between two points.
- Rescaling a curve changes the euclidean length, but not the hyperbolic length.

Day 8.

- The hyperbolic plane $(\mathbb{H}^2, d_{\text{hyp}})$ is a metric space.

Day 9.

- The hyperbolic plane $(\mathbb{H}^2, d_{\text{hyp}})$ is a metric space (continued).
- Some isometries of $(\mathbb{H}^2, d_{\text{hyp}})$: horizontal translations, homotheties.

Day 10.

- Some isometries of $(\mathbb{H}^2, d_{\text{hyp}})$: the standard inversion.
- Shortest curves in $\mathbb{H}^2$: the case where the two points are on the same vertical line.

Day 11.

- Shortest curves in $\mathbb{H}^2$: general case. Brief comments/history about Euclid's fifth postulate.

Day 12.

- More isometries of $\mathbb{H}^2$: linear and antilinear fractional maps.

Day 13.

- Find all isometries of the hyperbolic plane: the proof (mostly Lemma 2.4).

Day 14.

- Find all isometries of the hyperbolic plane: more proof.

Day 15.

- Find all isometries of the hyperbolic plane: end of proof.
- The differential of a map $f\colon U \to \mathbb{R}^2$ with $U \subset \mathbb{R}^2$.

Day 16.

- Computation of the differential of a hyperbolic isometry.
- The hyperbolic norm of a vector.
- The hyperbolic plane is isotropic.

Day 17. The 2–dimensional sphere.

- The spherical distance function d_{sph}.

- Shortest curves (no proof).
- Spherical isometries (no proof).
- The sphere is homogeneous and isotropic.

The intention is to have groups of students give oral presentations on these proofs at the end of the semester, as a research project.

Day 18.

- Quick review of the practice exam for the midterm.
- Constructing spaces by gluing (and by hand waving paper, scissors and Scotch tape): the cylinder, the torus.

Day 19. Midterm exam. No material covered.

Day 20. Midterm post-mortem.

Day 21. Quotient spaces. (Warning to the students: this is the most abstract part of the course.)

- Little bugs walking on the torus. How to intuitively measure distances in this case.
- Partitions.
- Discrete walks. Definition of the quotient semi-metric.

Day 22. Proof that the quotient semi-metric is a semi-metric.

Day 23. Gluing the faces of a polyhedron.

- General setup.
- Gluing is proper (no proof).

Day 24. Gluing the faces of a polyhedron.

- Quotient space is locally euclidean.
- Example: gluing the sides of a rectangle.
- Definition of homeomorphisms.
- Gluing the sides of a rectangle provides a space which is homeomorphic to the standard torus in $\mathbb{R}^3$.

Day 25. Gluing the faces of a euclidean polygon (more examples).

- Gluing opposite sides of a parallelogram.
- Gluing the sides of an infinite strip: the cylinder.
- Another way to glue the sides of an infinite strip; the Möbius strip.
- Another way to glue the sides of a rectangle: the Klein bottle

Day 26. Gluing the faces of a euclidean polygon (more examples).

- Gluing opposites sides of a regular hexagon: another way to get the torus
- Gluing opposites sides of a regular octagon: not locally euclidean any more.
- There is a hyperbolic triangle with angles $\frac{\pi}{2}$, $\frac{\pi}{8}$, $\frac{\pi}{8}$.

Day 27. Gluing opposite sides of an octagon.

- There is a hyperbolic octagon with angles $\frac{\pi}{4}$ and with all edges with the same length.
- The gluing gives a surface that is locally isometric to the hyperbolic plane, and is homeomorphic to the surface of genus 2.

Day 28. Cylinders.

- Euclidean cylinder: union of closed geodesics, all of the same length.
- Gluing the sides of a vertical half-strip in the hyperbolic plane: a cylinder swept by closed curves whose lengths grows exponentially at one end, and decays exponentially at the other end.
- Gluing the sides of a hyperbolic strip delimited by two concentric semicircles (= complete geodesics). Use the homothety to glue.

Day 29. Hyperbolic cylinder and punctured torus.

- Gluing the sides of a hyperbolic strip delimited by two concentric semicircles (continued): geodesic neck, exponential growth at both ends.
- Gluing the sides of the hyperbolic square with vertices at -1, 0, $+1$, ∞.

Day 30. The punctured torus.

- Gluing the sides of the hyperbolic square with vertices at -1, 0, $+1$, ∞.
- Decomposition into a compact part and an end with exponential decay.

Day 31. Tessellations

- Definitions and a few examples.

Day 32. Tessellations

- Statement of the Tessellation Theorem for bounded tiles.
- General idea of the proof by tile setting. No rigorous argument (and no mention of completeness or compactness).

Day 33. Tessellations

- Examples of tessellations of the euclidean plane: squares, parallelograms, tessellations by parallelograms coming from the Klein bottle.
- Tessellations of the euclidean plane by triangles.

Day 34. Tessellations

- Extension of the Tessellation Theorem to $\mathbb{H}^2$ and $\mathbb{S}^2$.
- Tessellation of $\mathbb{H}^2$ by octagons.
- Tessellation of $\mathbb{S}^2$ by triangles. The platonic solids.
- Tessellations of $\mathbb{H}^2$ by triangles.

Day 35. The Farey tessellation

- General idea and pictures.

- Farey pairs and Farey triples.

Day 36. From this day on, the first half of the lecture is devoted to student presentations (in groups of 2 or 3).

Student presentation:

- The disk model for $\mathbb{H}^2$ (§2.7).

Lecture: The Farey tessellation.

- The general form of a Farey triple. Farey sum.
- A linear fractional map with integer coefficients sends Farey pair to Farey pair.

Day 37.

Student presentation:

- Hyperbolic balls in $\mathbb{H}^2$ (Exercise 2.13).

Lecture: The Farey tessellation.

- The Farey tessellation is really a tessellation of the hyperbolic plane.

Day 38.

Student presentation:

- Area of spherical triangles (Exercise 3.6).

Lecture: The Farey tessellation.

- Traveling in the Farey tessellation (essentially Exercise 8.5).

Day 39.

Student presentation:

- Angles of spherical triangles (Exercise 3.5).

Lecture: The Farey tessellation.

- Continuation of Exercise 8.5.

Day 40.

Student presentation:

- Hyperbolic area (Exercise 2.14).

Lecture: The Farey tessellation.

- Continued fractions, and traveling in the Farey tessellation (essentially Exercise 8.6).

Day 41.

Student presentation:

- Area of hyperbolic triangles (Exercise 2.15).

Lecture: The Farey tessellation.

- Domino diagrams. The 10 and 60 freeways in Los Angeles (how many ways are there to drive from USC to the Claremont Colleges, or to Cal State San Bernardino?).

Day 42.

Student presentation:

- The Gauss-Bonnet formula (Exercise 5.16).

Lecture: The Farey tessellation.

- Domino diagrams (continued).