Geodesic laminations on surfaces

Francis Bonahon

Contents

Part I. The dynamical viewpoint  3
Definitions and first properties  3
Examples of geodesic laminations  4
  Geodesic laminations with finitely many leaves  4
  Interval exchange maps  4
The topology of geodesic laminations  6
The higher dimensional case  8
A more explicit example  9
Local properties of geodesic laminations  10
Transverse structures  11
  Transverse measures  11
  Transverse distributions(?)  13
  Transverse Hölder distributions  14
  Transverse cocycles  15

Part II. The topological viewpoint  18
The topology and piecewise linear structure of $\mathcal{ML}(S)$  19
Change of metric  20
The length function  20
Tangent vectors to $\mathcal{ML}(S)$  21
The derivative of the length function  24

Part III. The geometric viewpoint  25
The convex core of a hyperbolic 3–manifold  25
Pleated surfaces in hyperbolic 3–manifolds  27
Variations of the geometry of convex cores  31
Rotation angles, bending cocycles and Thurston’s intersection
  form  33
References  36
Index  37

1991 Mathematics Subject Classification. 53C25, 30F40, 57N05.
Key words and phrases. geodesic laminations, transverse structures.
This research was partially supported by N.S.F. grants DMS-9504282 and DMS-9803445.

©1997 American Mathematical Society
Geodesic laminations on surfaces were introduced by W.P. Thurston a little over 20 years ago, and have since been a very powerful tool in hyperbolic geometry, low-dimensional topology and dynamical systems. In particular, there are several different contexts where geodesic laminations now routinely occur. Geodesic laminations can be considered as:

- topological objects, occurring as generalizations of simple closed curves on surfaces;
- geometric objects, such as bending laminations of hyperbolic convex cores, shearing loci of earthquakes on hyperbolic surfaces, stable laminations of pseudo-Anosov diffeomorphisms, maximal stretch laminations between hyperbolic surfaces;
- interesting dynamical objects, in particular because of their connection with interval exchange maps.

This versatility can be somewhat confusing for the non-expert. To the expert, it provides interesting challenges when these very different points of view need to interact with each other. The minicourse given at the Workshop was devoted to illustrations of the above three aspects of geodesic laminations and of their interactions. Each of the three lectures focused on one of these viewpoints.

This article is similarly divided into three parts.

The first part is devoted to generalities on geodesic laminations, gives some examples, and discusses dynamically interesting transverse structures for geodesic laminations. This includes the analytic notion of a transverse Hölder distribution, and the more combinatorial notion of a transverse cocycle; these two transverse structures are later shown to be equivalent.

The second part discusses topological applications of geodesic laminations. In particular, we consider the space of measured geodesic laminations, as a completion of the space of simple closed curves on the surface. We mention the piecewise linear structure $\mathcal{ML}(S)$, and indicate how the combinatorial tangent vectors of this piecewise linear manifold have a geometric interpretation as geodesic laminations with transverse Hölder distributions.

Finally, the third part is devoted to some geometric applications of geodesic laminations. We chose to focus on geodesic laminations as bending loci of boundaries of convex cores of hyperbolic 3–dimensional manifolds, and as pleating loci of pleated surfaces. In particular, we show how the bending of a pleated surface along its pleating locus can be measured by a transverse cocycle. We also connect the bending of pleated surfaces to the rotation angle of closed geodesics in the hyperbolic 3–manifold.

We made the deliberate effort of closely following the style and structure of the minicourse. Anybody who has attended a talk where the lecturer directly reads from a preprint knows that the translation between mathematical writing and mathematical lecturing often requires a serious reorganization. However, we decided to try the experiment, hoping that this informal and pictorial\textsuperscript{1} style will be more effective at communicating the main ideas, while the reader can always turn to the published references in the bibliography for a more rigorous treatment of these topics.

We are grateful to the referee for a careful reading of the manuscript, and for the suggestion of many improvements.

\textsuperscript{1}Many pictures are taken from the monograph \cite{Bo6} in preparation.
PART I. THE DYNAMICAL VIEWPOINT

Definitions and first properties

One of the usual frameworks for geodesic laminations is that of a surface $S$ with no boundary, endowed with a complete metric whose curvature is constant equal to $-1$ (namely a hyperbolic metric) and whose area is finite. There are possible variations of this setting, where we can allow the curvature of the metric to vary between two negative constants, and/or where the surface $S$ can be required to be compact with convex boundary. However, these variations only lead to minor technical differences, and we will restrict ourselves to this first setting.

An important consequence of the existence of such a hyperbolic metric is that $S$ is a surface of finite topological type, namely is homeomorphic to the complement of finitely many points in a compact surface. In addition, the Gauss-Bonnet formula implies that the Euler characteristic $\chi(S)$ is strictly negative. Conversely, it is well known that a surface $S$ of finite topological type and negative Euler characteristic always admits a hyperbolic metric of this type.

![Diagram of a geodesic lamination](image)

**FIG. 1.** The local type of a geodesic lamination

A geodesic lamination on the surface $S$ is a lamination $\lambda$ of $S$ whose leaves are geodesic. Recall that the fact that $\lambda$ is a lamination means that $\lambda$ is a closed subset of $S$ and is decomposed into subsets called its leaves, so that locally the situation is as follows: Every point of $\lambda$ has a neighborhood $U$ homeomorphic to the product $]a, b[ \times ]c, d[$ of two open intervals in such a way that $U \cap \lambda$ corresponds to $K \times ]c, d[$ for some compact subset $K$ of $]a, b[$ and, for every leaf $g$ of $\lambda$, $U \cap g$ corresponds to $A_g \times ]c, d[$ for some subset $A_g$ of $]a, b[$. Note that this local description implies that, for a geodesic lamination, the leaves are complete geodesics in the sense that each leaf is either closed or has infinite length in both of its ends.

In practice, checking whether a subset of $S$ is a geodesic lamination is much easier than one might think from this elaborate description of its local type. Indeed, the following property holds,

**Proposition 1.** If $\lambda$ is a closed subset of $S$ which is a disjoint union of simple geodesics, then $\lambda$ is a geodesic lamination.

Here, a geodesic is simple if it does not cross itself. It may be closed or infinite.

The key point in the proof of Proposition 1 is the following estimate in hyperbolic geometry, whose proof can be found in [CEG, §5.2.6].

**Lemma 2.** In the hyperbolic plane $\mathbb{H}^2$, there is a constant $C$ with the following property: If the geodesic $g$ has unit tangent vector $v$ at $x$, if the geodesic $h$ has unit
Fig. 2. A geodesic lamination

tangent vector \( w \) at \( y \), and if the geodesics \( g \) and \( h \) are disjoint, then,

\[
d(v, \pm w) \leq C d(x, y).
\]

In particular, if \( \lambda \) is a closed subset of \( S \) which is a disjoint union of simple geodesics, the tangent directions of these geodesics form a Lipschitz direction field on \( \lambda \). Extending it to a Lipschitz direction field over a neighborhood of \( \lambda \) and using the standard existence results for solutions to ordinary differential equations, one obtains local charts which show that \( \lambda \) is a lamination, in the sense indicated above.

Fig. 3. Disjoint geodesics have nearby directions at nearby points

Examples of geodesic laminations

Geodesic laminations with finitely many leaves. The first example is provided by a family of disjoint simple closed geodesics \( \gamma_1, \gamma_2, \ldots, \gamma_n \).

Fig. 4. A geodesic lamination with all leaves closed.

This example can be enlarged by adding a finite family of infinite geodesics which spiral along these closed geodesics, as in Figure 5.

Interval exchange maps. More complex examples are associated to interval exchange maps, which we now define. Consider a compact interval \( I \) in \( \mathbb{R} \), which
FIG. 5. A geodesic lamination with finitely many leaves.

we decompose as a finite union \( I = I_1 \cup I_2 \cup \ldots \cup I_n \) of intervals \( I_i \) with disjoint interiors. Choose another decomposition of \( I = J_1 \cup J_2 \cup \ldots \cup J_n \) into intervals \( J_j \) with disjoint interiors such that, for every \( i \), there is an isometry \( \varphi_i : I_i \to J_i \). The collection of the \( \varphi_i \) defines a ‘map’ \( \varphi : I \to I \). This map is in general 1-to-2 at the end points of the \( I_i \), but is well-defined everywhere else. Such a \( \varphi \) is an interval exchange map.

FIG. 6. An interval exchange map.

Let us restrict attention to the case where \( \varphi (\partial I) \cap \partial I \neq \emptyset \). Otherwise, \( \varphi \) restricts to an exchange of fewer intervals.

To the interval exchange map \( \varphi : I \to I \) is associated its suspension \( \Sigma_\varphi \), obtained by gluing \( n \) rectangles \( I_i \times [0,1] \) to the interval \( I \) as follows: \( I_i \times \{0\} \) is identified to the interval \( I_i \subset I \) by the obvious map, and \( I_i \times \{1\} \) to the interval \( J_i \subset I \) by the (isometric) restriction \( \varphi_i : I_i \to J_i \) of \( \varphi \). In particular, \( \Sigma_\varphi \) looks like a freeway interchange without entry or exit ramps.

Foliate each rectangle \( I_i \times [0,1] \) by the arcs \( \{*\} \times [0,1] \). This defines a foliation of the suspension \( \Sigma_\varphi \) by lines and simple closed curves. The leaves of this foliation are in one-to-one correspondence with the 2-sided orbits of \( \varphi \), namely with the bi-infinite sequences \( \ldots, x_{-N}, \ldots, x_{-1}, x_0, x_1, \ldots, x_N, \ldots \) where \( x_{n+1} = \varphi (x_n) \) for every \( n \in \mathbb{Z} \) and where the indexing is only defined modulo translation in \( \mathbb{Z} \). Let a leaf be regular if it does not pass through any of the end points of the \( I_i \).

Note that the boundary of the suspension \( \Sigma_\varphi \) consists of a certain number of cycles. Gluing a semi-open annulus along each of these cycles, we obtain a surface \( S \) of finite type, whose Euler characteristic \( \chi (S) = 1 - n \) is negative. Endow \( S \) with a finite area hyperbolic metric.

Pulling tight each leaf \( l \) of \( \Sigma_\varphi \) defines a geodesic \( g_l \) of \( S \). The key technical property here is that each leaf of \( \Sigma_\varphi \) is quasi-geodesic in \( \Sigma_\varphi \) and (by a small argument using the fact that \( \varphi (\partial I) \cap \partial I \neq \emptyset \)) in \( S \), so that there is a unique geodesic \( g_l \).
of $S$ for which there is a homotopy from $l$ to $g_l$ which moves points by a uniformly bounded distance.

Consider the family of those geodesics $g_l$ which are associated in this way to the regular leaves of $\Sigma_\varphi$. Since distinct regular leaves $l$ and $l'$ are disjoint, the boundedness of the homotopies from $l$ to $g_l$ and from $l'$ to $g_{l'}$ shows that the associated geodesics cannot cross each other, and therefore are either disjoint or equal. Similarly, since a regular leaf does not cross itself, the geodesic $g_l$ must be simple (closed or infinite). Let $\lambda'$ be the union of the geodesics $g_l$ associated to the regular leaves $l$ of $\Sigma_\varphi$, and let $\lambda$ be the closure of $\lambda'$. An application of Lemma 2 and Proposition 1 shows that $\lambda$ is a geodesic lamination.

If the lengths of all the intervals $I_i$ are rational, then every regular leaf of $\Sigma_\varphi$ is closed. Consequently, $\lambda$ consists of finitely many simple closed geodesics, and we are back to our first example.

The situation is much more interesting in the other extreme case where the lengths of the intervals $I_i$ are linearly independent over $\mathbb{Q}$. Then, assuming in addition that $\varphi$ does not restrict to an exchange of fewer intervals, all the regular leaves of $\Sigma_\varphi$ are infinite. In addition, two distinct regular leaves $l, l'$ must diverge at some point and follow different routes on the 'freeway' $\Sigma_\varphi$; it easily follows that the associated geodesics $g_l$ and $g_{l'}$ must be distinct. As a consequence, the geodesic lamination $\lambda$ so constructed now has uncountably many leaves, all infinite.

The topology of geodesic laminations

Consider the complement $S - \lambda$ of a geodesic lamination $\lambda$ in the hyperbolic surface $S$. On the open subset $S - \lambda$, we can consider the path metric induced by
the metric of $S$, for which the distance between $x$ and $y \in S - \lambda$ is the infimum of the lengths of all paths going from $x$ to $y$ in $S - \lambda$ (and is infinite if $x$ and $y$ are in different components of $S - \lambda$).

![Diagram of a geodesic lamination $\lambda$.](image)

**Fig. 8.** A geodesic lamination $\lambda$.

Let $\overline{S - \lambda}$ be the completion of $S - \lambda$ for this metric. From the local picture of $\lambda$, we see that $\overline{S - \lambda}$ locally is a hyperbolic surface with totally geodesic boundary. More precisely, $\overline{S - \lambda}$ is obtained by abstractly adding to $S - \lambda$ a boundary made up of leaves of $\lambda$.

![Completion of $S - \lambda$.](image)

**Fig. 9.** The completion $\overline{S - \lambda}$ of the complement $S - \lambda$.

Since $S$ has finite area, it follows from the above description that $\overline{S - \lambda}$ also has finite area. From this, we conclude that $\overline{S - \lambda}$ is the union of a compact part, of a finite number of cusps corresponding to cusps of $S$, and of finitely many spikes, each delimited by two (possibly equal) boundary components of $\overline{S - \lambda}$. In particular, the topological type of $\overline{S - \lambda}$ is finite and its boundary consists of finitely many geodesics.

We can be a little more precise. Using Lemma 2, we can construct a direction field which is non-singular and is transverse to the leaves on $\lambda$, and has finitely many isolated singularities on $\overline{S - \lambda}$. Using the Poincaré-Hopf formula to express the Euler characteristic $\chi(\ )$ in terms of the indices of these singularities, we conclude that

$$\chi \left( \overline{S - \lambda} \right) = \chi(S) + \frac{1}{2}s$$

where $s$ is the number of spikes of $\overline{S - \lambda}$.

A corollary of this formula is that the number of components of $\lambda$ is uniformly bounded. In particular, $\lambda$ can contain at most finitely many sub-laminations, namely closed subsets which are the union of leaves of $\lambda$. As a consequence, $\lambda$
contains finitely many sub-laminations which are minimal for inclusion. Analyzing the leaves of $\lambda$ which are not contained in minimal sub-laminations and using the above Euler characteristic formula to guarantee finiteness, one can prove (see [Th1, §8], [CEG, §4.2.8] or [CaB, §4]):

**Proposition 3.** A geodesic lamination is the union of finitely many minimal sub-laminations and of finitely many infinite isolated leaves, whose ends spiral along the minimal sub-laminations or converge to a cusp.

Note that a sub-lamination $\lambda'$ is minimal if and only if every half-leaf of $\lambda'$ is dense in $\lambda'$. A simple example consists of a closed leaf of $\lambda$. However, interval exchange maps provide many examples where $\lambda$ is minimal and has uncountably many leaves.

A leaf $g$ of $\lambda$ is isolated if every point of $g$ has a neighborhood whose intersection with $\lambda$ is a single arc (contained in $g$).

As a corollary of the Proposition 3, we have:

**Proposition 4.** Each cusp of $S$ has a neighborhood whose intersection with $\lambda$ consists of finitely many isolated half-leaves converging towards the cusp.

**The higher dimensional case**

Before going any further, we might wonder if we would not get more insight by considering geodesic laminations in all dimensions, namely laminations whose leaves are totally geodesic. Compact totally geodesic submanifolds provide examples of such laminations, although somewhat trivial. However, Abdelghani Zeghib essentially showed that this is the only example:

**Theorem 5.** (Zeghib [Ze]) Let $M$ be a compact Riemannian manifold of dimension $n \geq 3$ and of negative curvature. If $\lambda$ is a codimension 1 lamination of $M$ whose leaves are totally geodesic, then all the leaves of $\lambda$ are compact.

The idea of the proof is based on a volume argument: If a boundary component $F$ of $\overline{S - \lambda}$ has infinite $(n - 1)$-volume, then the other boundary components cannot be packed close enough to $F$ to enclose a finite $n$-dimensional volume; in particular, $\overline{S - \lambda}$ will then have infinite volume, contradicting the compactness of $M$. Namely, the existence of finite area spikes in $\overline{S - \lambda}$ is a phenomenon which is typical of the dimension 2.

![Diagram](fig10.png)

**Fig. 10.** In dimension $\geq 3$, totally geodesic surfaces cannot be packed close together.
A more explicit example

Traditionally, geodesic laminations are often represented by rough pictures analogous to the ones which have appeared above, such as Figures 2 or 8. These pictures are convenient and are topologically correct, but they often are metrically incorrect. This fact had been overlooked for a long time by the author until, recently, he had the curiosity of drawing computer pictures of explicit geodesic laminations. Such an example appears on Figure 11. Here, the underlying surface $S$ is a once punctured torus, obtained by identifying opposite sides in a quadrilateral in (the Poincaré model for) the hyperbolic plane $\mathbb{H}^2$ with its vertices on the circle at infinity. The identifications between sides respect the intersection of the geodesic lamination $\lambda$ with these sides.

In this example, $\lambda$ is the geodesic lamination associated to the suspension of an exchange of two intervals, whose length are rationally independent. In particular, $\lambda$ has uncountably many leaves.

![Diagram of a geodesic lamination on a punctured torus.](image)

**Fig. 11.** A geodesic lamination on the punctured torus.

Two striking features of this picture are that $\lambda$ occupies remarkably little space on the surface $S$, and that there is not much of the structure of $\lambda$ which is really visible. Actually, drawing this picture required a careful choice of $\lambda$ to make it more
interesting: Most geodesic laminations on \(S\) appear only as the union of three fat lines on the quadrilateral.

However, if we zoom on the little rectangle shown on Figure 11, the enlarged rectangle is the one represented on Figure 12(a). Similarly, Figure 12(b) is an enlargement of the small rectangle indicated on Figure 12(a), and Figure 12(c) represents the small rectangle of Figure 12(b). These pictures suggest that the leaves of \(\lambda\) are grouped in patterns reminiscent of Cantor sets.

\[
\begin{align*}
(a) & \\
(b) & \\
(c) & 
\end{align*}
\]

Fig. 12. Zooming in on Figure 11.

Local properties of geodesic laminations

These heuristic observations can be justified as follows.

First, a surprising result of Joan Birman and Caroline Series [BiS] (see also [Th3, §10] for a proof which is technically equivalent, but perhaps more conceptual) shows that very few geodesics of \(S\) are simple.

**Theorem 6.** (J. Birman, C. Series) The union of all simple geodesics of \(S\) has Hausdorff dimension 1.

Note that the union of all simple geodesics is also the union of all geodesic laminations.

An immediate corollary of Theorem 6 is that an individual geodesic lamination \(\lambda\) has Hausdorff dimension 1. In particular, if \(k\) is a differentiable arc which is transverse to the leaves of \(\lambda\), then \(k \cap \lambda\) has Hausdorff dimension 0, and in particular is totally disconnected. This proves:

**Proposition 7.** Suppose that the geodesic lamination \(\lambda\) has no isolated leaves. Then, for every arc \(k\) transverse to the leaves of \(\lambda\), the intersection \(k \cap \lambda\) is a Cantor set.
However, metrically, \( k \cap \lambda \) does not look like the typical middle third Cantor set at all.

**Proposition 8.** If the arc \( k \) is transverse to the geodesic lamination \( \lambda \) and if we index the components \( d_1, d_2, \ldots, d_n, \ldots \) of the complement \( k - \lambda \) by order of non-increasing size, then

\[
|\log \text{length} \,(d_n)| \approx n
\]

Here the symbol \( a_n \asymp b_n \) means that the ratio \( a_n/b_n \) is bounded between two positive constants for \( n \) large enough. It should be noted that

\[
|\log \text{length} \,(d_n)| \asymp \log n
\]

for the middle third Cantor set.

To explain why Proposition 8 holds, we should observe that all but finitely many \( d_n \) are in spikes of \( \overline{S - \lambda} \). For a given spike, list the \( d_n \) contained in this spike as \( d'_1, d'_2, \ldots, d'_p, \ldots \) in the order of their progression towards the end of the spike.

![Diagram](image)

**Fig.13.** Components of \( k - \lambda \) in a spike of \( \overline{S - \lambda} \).

For any two consecutive components \( d'_p, d'_{p+1} \), an estimate using hyperbolic geometry gives that the ratio

\[
\frac{\text{length} \,(d'_{p+1})}{\text{length} \,(d'_p)} \approx e^{-\text{(distance from } d'_p \text{ to } d'_{p+1})}
\]

By compactness of \( k \cap \lambda \), the distance between \( d'_p \) and \( d'_{p+1} \) is bounded between two constants. It follows that

\[
|\log \text{length} \,(d'_p)| \approx p.
\]

Since \( \overline{S - \lambda} \) has only finitely many spikes, the estimate of Proposition 8 follows.

**Transverse structures**

**Transverse measures.** The first type of transverse structure which we can consider for a geodesic lamination is provided by the classical notion of transverse measure.

A transverse measure for a lamination is locally a measure on the space of leaves of the lamination. For a geodesic lamination \( \lambda \), the leaves of \( \lambda \) are locally determined by their intersection points with a differentiable arc transverse to \( \lambda \). We can therefore define a *transverse measure* for the geodesic lamination \( \lambda \) as the
assignment of a measure on each arc $k$ transverse to $\lambda$, subject to the following two conditions: If the arc $k'$ is contained in the transverse arc $k$, then the measure assigned to $k'$ is the restriction of the measure assigned to $k$; if the two arcs $k$ and $k'$ are homotopic through a family of arcs transverse to $\lambda$, the homotopy sends the measure assigned to $k$ to the measure assigned to $k'$.

We are here using the traditional geometric analysis convention that a measure actually is a Radon measure. Namely, it assigns a finite non-negative mass to each relatively compact Borelian subset, in a countably additive manner.

An immediate consequence of the condition of invariance under homotopy of arcs transverse to $\lambda$ is that the support of the measure assigned to the transverse arc $k$ must be contained in $k \cap \lambda$.

![Fig. 14. A homotopy of arcs transverse to $\lambda$.](image)

The simplest example of such a transverse measure occurs for a geodesic lamination $\lambda$ which admits a certain number of closed leaves $l_1, \ldots, l_n$. Pick positive numbers $a_1, \ldots, a_n$. For each arc $k$ transverse to $\lambda$, consider the measure $\alpha$ defined by the property that

$$\alpha(A) = \sum_{i=1}^{n} a_i \# A \cap l_i$$

for every subset $A$ of $k$, where $\# X$ denotes the cardinal of $X$. This $\alpha$ is clearly invariant under homotopy of arcs transverse to $\lambda$, and consequently defines a transverse measure for $\lambda$.

A more elaborate construction uses the fact that the leaves of $\lambda$ are 1-dimensional, and therefore have polynomial growth. Then, if we apply a construction similar to the above ones to arbitrary large pieces of leaves of $\lambda$, an averaging process provides a non-trivial transverse measure for $\lambda$; compare [P]t.

**Proposition 9.** Every geodesic lamination $\lambda$ admits a transverse measure whose support consists of all the minimal sub-laminations of $\lambda$.

It is easy to check that infinite isolated leaves of $\lambda$ cannot be in the support of any transverse measure $\alpha$ for $\lambda$, as they would otherwise provide arcs transverse to $\lambda$ and with infinite $\alpha$-mass. Therefore, Proposition 9 is optimal.

Interval exchange maps also provide interesting examples of geodesic laminations with transverse measures. Indeed, the regular 2-sided orbits of an interval exchange map $\varphi : I \to I$ defines leaves of the associated geodesic lamination $\lambda$. 
Since $\varphi$ preserves the Lebesgue measure of $I$, the space of 2-sided orbits of $\varphi$ inherits a natural measure from the Lebesgue measure of $I$. This measure pushes forward to locally define a measure on the space of leaves of $\lambda$, and therefore defines a transverse measure for $\lambda$.

**Transverse distributions**. Transverse measures are classical objects in the theory of foliations and laminations. A natural generalization, with a wider analytical scope, is to consider transverse distributions.

At this point, it is probably worth recalling that, by the Riesz Representation Theorem, a Radon measure on a locally compact space $X$ is the same thing as a positive linear functional on the space of all continuous functions $\varphi : X \to \mathbb{R}$ with compact support. A *distribution* on a manifold $X$ is a continuous linear form on the space of infinitely differentiable functions $\varphi : X \to \mathbb{R}$ with compact support. See for instance [Sc].

In the case of a geodesic lamination $\lambda$, one can try to define a transverse distribution as a distribution defined on each arc transverse to $\lambda$, such that, when the arcs $k$ and $k'$ are homotopic through a family of arcs transverse to $\lambda$, the homotopy sends the distribution defined on $k$ to the distribution defined on $k'$. However, there is a serious technical problem here. To be able to compare in full generality distributions on $k$ to distributions on $k'$, the homotopy from $k$ to $k'$ must be differentiable. This almost never happens.

Indeed, if $k$ is an arc transverse to $\lambda$, for all but finitely many components of $k - \lambda$, the leaves of $\lambda$ passing through the end points of this component of $k - \lambda$ are asymptotic and bound a spike of the completed complement $S - \lambda$. If we homotop $k$ to $k'$ through a family of arcs transverse to $\lambda$, this homotopy establishes a one-to-one correspondence between components of $k - \lambda$ and components of $k' - \lambda$. However, if the homotopy moves each point of $k$ by a distance of approximately $D > 0$, we notice two possible behaviors for a component $d$ of $k - \lambda$: if the two leaves passing through the end points of $d$ are asymptotic in the direction of $k'$, then the corresponding component $d'$ of $k' - \lambda$ has length $l(d')$ which is approximately $e^{-Dl(d)}$, by an easy hyperbolic geometry estimate; on the other hand, if these two leaves are asymptotic in the direction opposite to $k'$, then $l(d')$ is approximately $e^{+Dl(d)}$. For a typical geodesic lamination, every neighborhood of a point of $k \cap \lambda$ contains components of $k - \lambda$ which are of each of these two types. This clearly prevents the homotopy from $k$ to $k'$ from being differentiable. See Figure 15.

In particular, a typical geodesic lamination admits no transverse differentiable structure. As a consequence, it does not in general make sense to talk of transverse distributions for a geodesic lamination $\lambda$, which explains the question mark in the heading of this subsection.

However the above argument, which shows that a homotopy of arcs transverse to $\lambda$ cannot be differentiable, also shows that the homotopy can be chosen to be Lipschitz, and in particular can be chosen to be Hölder continuous.

Recall that a function $\varphi$ between metric spaces is Hölder continuous with Hölder exponent $\nu \leq 1$ if

$$d(\varphi(x) , \varphi(y)) \leq \text{constant} \cdot d(x, y)^\nu$$

for all $x, y$. It is Lipschitz continuous if it is Hölder continuous with Hölder exponent $\nu = 1$.

Having thus showed that geodesic laminations admit transverse Hölder structures, this leads us to consider:
these components expand by $\approx e^{D}$

these components shrink by $\approx e^{-D}$

Fig. 15. A homotopy of arcs transverse to $\lambda$ cannot be differentiable.

**Transverse Hölder distributions.** Let us define a *Hölder distribution* on a compact metric space $X$ as a continuous linear functional on the space of all Hölder continuous functions $\varphi : X \to \mathbb{R}$ with compact support.

A *transverse Hölder distribution* $\alpha$ for a geodesic lamination $\lambda$ is a Hölder distribution defined on each arc $k$ transverse to $\lambda$, which is invariant under every Hölder continuous homotopy of arcs transverse to $\lambda$. A consequence of the invariance property is that, for a transverse arc $k$, the Hölder distribution induced on $k$ by $\alpha$ has support contained in $k \cap \lambda$ (namely $\alpha(\varphi) = 0$ for every Hölder continuous function whose support is contained in $k - \lambda$).

Here is an example of such a transverse Hölder distribution. Consider the geodesic lamination $\lambda$ of Figure 16, consisting of one closed geodesic $\gamma$ and of one infinite leaf whose ends spiral along $\gamma$.

Fig. 16. A geodesic lamination with a transverse Hölder distribution

For the arc $k$ shown on Figure 16,

$$k \cap \lambda = \{x_{\infty}, x_1, x_2, \ldots, x_n, \ldots\}$$
where $x_\infty$ is the point $k \cap \gamma$. Note that the other points $x_1, x_2, \ldots, x_n$ accumulate on $x_\infty$. For $a, b \in \mathbb{R}$, define a Hölder distribution $\alpha$ on $k$ by

$$\alpha (\varphi) = a \varphi (x_\infty) + b \sum_{1}^{\infty} (\varphi (x_n) - \varphi (x_\infty)),$$

for every Hölder continuous function $\varphi : k \to \mathbb{R}$. To check that the infinite sum does converge note that, if we index the $x_n$ so that the distance $d(x_n, x_\infty)$ monotonically converges to 0, then $d(x_n, x_\infty) \approx (e^{-L})^n$ where $L$ is the length of $\gamma$; indeed, this easily follows from an estimate in hyperbolic geometry. This implies that $|\varphi (x_n) - \varphi (x_\infty)| \approx (e^{-\nu L})^n$, where $\nu$ is the Hölder exponent of $\varphi$, so that the infinite sum is absolutely bounded by a converging geometric series.

Note that this formula is easily transported by a homotopy of arcs transverse to $\lambda$. In general, every transverse arc $k'$ can be decomposed into smaller arcs, each homotopic to a subarc of $k$ through a family of transverse arcs. The formula for $k$ then enables us to define a Hölder distribution on this transverse arc $k'$.

In this way, we define a transverse Hölder distribution for the geodesic lamination of Figure 16. This construction depends on the choice of the two reals $a, b \in \mathbb{R}$, so that we actually have a 2-parameter family of such transverse Hölder distributions.

**Transverse cocycles.** Transverse measures and transverse Hölder distributions are analytic objects. We now consider simpler transverse structures, of a more combinatorial nature.

A *transverse cocycle* $\alpha$ for the geodesic lamination $\lambda$ associates to each arc $k$ transverse to $\lambda$ a number $\alpha (k)$ in a way which is:

- invariant under homotopy of arcs transverse to $\lambda$;
- finitely additive, in the sense that, if we split $k$ as the union of two subarcs $k_1$ and $k_2$ with disjoint interiors, then $\alpha (k) = \alpha (k_1) + \alpha (k_2)$.

We can give a more analytic flavor to this definition by observing that a transverse cocycle is equivalent to the data, for each arc $k$ transverse to $\lambda$, of a linear function on the space of all locally constant functions $\varphi : K \cap \lambda \to \mathbb{R}$, in a way which is invariant under homotopy of arcs transverse to $\lambda$.

The combinatorial nature of transverse cocycles makes it very easy to analyze them. In particular, using train tracks, one easily proves [Bo2]:

**Proposition 10.** For a geodesic lamination $\lambda$, the space $\mathcal{H} (\lambda; \mathbb{R})$ of transverse cocycles for $\lambda$ is a vector space of dimension

$$n_{\partial} (\lambda) + \frac{1}{2} \# \text{spikes of } S - \lambda,$$

where $n_{\partial} (\lambda)$ is the number of orientable components of $\lambda$.

These transverse cocycles look very different from transverse Hölder distributions. However, it turns out that the two notions are actually equivalent. Indeed, a transverse Hölder distribution defines a transverse cocycle $\alpha$ by the property that

$$\alpha (k) = \alpha (\text{constant function 1 on } k).$$
THEOREM 11. [Bo2] This establishes as one-to-one correspondence between transverse Hölder distributions and transverse cocycles for the geodesic lamination \( \lambda \).

The key part of the proof is to show that two Hölder distributions which induce the same transverse cocycle are actually equal. This is provided by the following lemma, which, for a transverse Hölder distribution \( \alpha \) and a Hölder continuous function \( \varphi : k \rightarrow \mathbb{R} \) defined on an arc transverse to \( \lambda \), expresses \( \alpha (\varphi) \) in terms of the transverse cocycle associated to \( \alpha \) and of the values of the function \( \varphi \) on \( k \cap \lambda \).

PROPOSITION 12. (Gap Formula) Let \( \alpha \) be a transverse Hölder distribution for the geodesic lamination \( \lambda \). For every Hölder continuous function \( \varphi : k \rightarrow \mathbb{R} \) defined on an oriented arc \( k \) transverse to \( \lambda \),

\[
\alpha (\varphi) = \alpha (k) \varphi (x_k^+) + \sum_d \alpha (k_d) \left( \varphi (x_d^-) - \varphi (x_d^+) \right)
\]

where: \( x_k^+ \) is the positive end point of the oriented arc \( k \); \( d \) ranges over all components of \( k - \lambda \) (= gaps); \( x_d^+ \) and \( x_d^- \) are the positive and negative end points of \( d \); \( k_d \) is the sub-arc of \( k \) joining the negative end point \( x_d^- \) of \( k \) to an arbitrary point in \( d \).

\[x_k^+\]
\[x_d^+\]
\[x_d^-\]
\[x_k^-\]

\[k_d\]

\[d\]

Fig. 17. The situation of the Gap Formula

Note that this formula holds in particular if \( \alpha \) is a transverse measure, but that it is necessary for \( \varphi \) to be Hölder continuous even in this case (when \( k \cap \lambda \) is a Cantor set, think of the Lebesgue function \( \varphi : k \rightarrow \mathbb{R} \) which is continuous, non-constant, but constant on each component of \( k - \lambda \)).

Although the result is apparently local, the proof of Proposition 12 uses the global structure of \( \lambda \) in a crucial way. We can illustrate this by showing why the infinite sum in the Gap Formula converges for a Hölder continuous function
\( \varphi : k \to \mathbb{R} \). Recall from Proposition 8 that we can index the components \( d_1, d_2, \ldots, d_n, \ldots \) of \( k - \lambda \) so that \( |\log \text{length}(d_n)| \asymp n \). In particular,
\[
\text{length}(d_n) \sim e^{-An}
\]
for some constant \( A > 0 \), and
\[
|\varphi(x^-_{d_n}) - \varphi(x^+_{d_n})| \sim e^{-\nu n}
\]
if \( \varphi \) has Hölder exponent \( \nu \). In addition, the index \( n \) measures how deep \( d_n \) converges towards the spikes of \( S - \lambda \). By a careful analysis of the combinators of transverse cocycles, one can show that
\[
|\alpha(d_n)| \asymp n.
\]
It follows that the infinite sum in the Gap Formula is bounded by a geometric series, and therefore converges.

What the Gap Formula shows is that the map which associates a transverse cocycle to each transverse Hölder distribution is injective. Conversely, given a transverse cocycle for \( \lambda \), we can use the Gap Formula to define a Hölder distribution transverse to \( \lambda \). A little argument shows the invariance under homotopy of arcs transverse to \( \lambda \), and concludes the proof of Theorem 11.

As an aside, we should mention another important ingredient of the proof of Proposition 12, namely the following general (and elementary) lemma. Recall that the support of a distribution \( \alpha \) on the manifold \( X \) is the set of those points \( x \in X \) which satisfy the following property: For every neighborhood \( U \) of \( x \), there is a differentiable function \( \varphi : X \to \mathbb{R} \) whose support is contained in \( U \) and such that \( \alpha(\varphi) \neq 0 \); see for instance [Sc]. Replacing “differentiable” by “Hölder continuous”, this definition immediately extends to the case of a Hölder distribution \( \alpha \) on a metric space \( X \).

**Proposition 13.** (Support Lemma) If \( \alpha \) is a Hölder distribution in a metric space \( X \) and if \( \varphi : X \to \mathbb{R} \) is Hölder continuous, then \( \alpha(\varphi) \) depends only on the restriction of \( \varphi \) to the support of \( \alpha \).

This property is of course satisfied by measures. However, it is false for arbitrary distributions. For instance, the distribution on \( \mathbb{R} \) which to a function \( \varphi : \mathbb{R} \to \mathbb{R} \) associates its derivative at 0 does not satisfy this property. In some sense the Support Lemma shows that, although more general, Hölder distributions still have much in common with measures.

We can conclude this first part with an open question. We observed that a geodesic lamination \( \lambda \) admits, not only a Hölder transverse structure, but also a Lipschitz transverse structure. We can therefore generalize transverse Hölder distributions to transverse Lipschitz distributions for \( \lambda \). A natural question is to ask whether this is a generalization at all, namely if a geodesic lamination can admit a transverse Lipschitz distribution which is not a transverse Hölder distribution. (Conjecture: No). The main obstacle to answering this question is that we do not know if the Support Lemma of Proposition 13 also applies to Lipschitz distributions. Of course, this leads to another question, which is to ask why anybody should care about this first question. Indeed, we will see in the second part that geodesic laminations can be defined as purely topological objects, and that their transverse Hölder structures are independent of the choice of a negatively curved metric on the surface \( S \). However, the transverse Lipschitz structure of a geodesic lamination
strongly depends on the metric; for instance, for the geodesic lamination of Figure 5, the lengths of the closed leaves of the geodesic lamination are completely determined by the transverse Lipschitz structure (compare Figure 15). Therefore, for someone who was hardwired as a topologist, such as the author of these notes, only geodesic laminations with transverse Hölder distributions really make sense, and transverse Lipschitz distributions are just a curiosity. On the other hand, for a geometer, if there is indeed a difference between Hölder and Lipschitz distributions, this could be a way to introduce very subtle invariants of a negatively curved metric on the surface $S$.

**PART II. THE TOPOLOGICAL VIEWPOINT**

From the point of view of the topologist, one of the motivations for the study of geodesic laminations on a surface $S$ (with negative Euler characteristic $\chi(S)$) is to analyze the set of simple closed curves on $S$, considered up to deformation. By a deformation, we mean here a homotopy or an isotopy; since these two notions are equivalent for simple closed curves on surfaces [Ba][Ep][Bu, Appendix].

The idea is to endow $S$ with an arbitrary metric of constant negative curvature $-1$ and of finite area, which always exists since $\chi(S)$ is negative.

Then, every simple closed curve can be deformed to a unique simple closed geodesic, unless it can be shrunk to a point or pushed to one of the cusps of the surface. So, essentially, this establishes a one-to-one correspondence between the set of deformation classes of simple closed curves and the set of closed geodesics, since we only lose finitely many uninteresting isotopy classes of simple closed curves. Let a simple closed curve in $S$ be *essential* if it cannot be shrunk to a point or pushed to a cusp.

As observed in Part I, a simple closed geodesic can be interpreted as a geodesic lamination. This embeds the set

$$S(S) = \{\text{essential simple closed curves in } S\}/\text{deformation}$$

in the set

$$\mathcal{L}(S) = \{\text{geodesic laminations in } S\}.$$ 

The set $\mathcal{L}(S)$ has a natural topology, coming from the Hausdorff metric on the space of closed non-empty subsets of $S$. Recall that the space of closed non-empty subsets of a metric space is compact in the topology induced by the Hausdorff metric, and it is relatively easy to check that $\mathcal{L}(S)$ is closed in this space. As a consequence, $\mathcal{L}(S)$ is compact. In particular, every sequence of simple closed curves has a subsequence which converges in $\mathcal{L}(S)$.

However, the following result, originally announced by Thurston in [Th3, §10] and proved by Xiaodong Zhu and the author in [ZhB], shows that the space $\mathcal{L}(S)$ with this topology is totally disconnected, and in particular is not very convenient.

**Theorem 14.** (Thurston, Zhu-Bonahon) The space $\mathcal{L}(S)$ has Hausdorff dimension 0 with respect to the Hausdorff metric. In particular, the space $\mathcal{L}(S)$ is totally disconnected.

A much better idea is to use geodesic laminations with transverse measures.

As a geodesic lamination, a simple closed geodesic $\gamma$ admits a natural transverse measure $\alpha$ defined as follows: If $A$ is a subset of an arc $k$ transverse to $\gamma$, $\alpha(A)$ is
equal to the number of points of $A \cap \gamma$. This embeds the set

$$S(S) = \{ \text{essential simple closed curves in } S \} / \text{deformation}$$

in

$$\mathcal{ML}(S) = \{ \text{measured geodesic laminations in } S \}$$

where a measured geodesic lamination $\alpha$ consists of a compact geodesic lamination $\lambda_\alpha$ endowed with a transverse measure $\alpha$ whose support is equal to all of $\lambda_\alpha$. It easily follows from the existence of $\alpha$ that $\lambda_\alpha$ has no infinite isolated leaf, namely that every connected component of $\lambda_\alpha$ is minimal.

**The topology and piecewise linear structure of $\mathcal{ML}(S)$**

An arc $k$ is generic (with respect to simple geodesics) if it is transverse to every simple geodesic of $S$. We saw in Theorem 6 that the union of all simple geodesics has Hausdorff dimension 1. It follows that almost every geodesic arc is generic, and that every arc can be approximated by a generic arc.

If the arc $k$ is generic, then every measured geodesic lamination $\alpha \in \mathcal{ML}(S)$ defines a measure on $k$, which we will still denote by $\alpha$. In particular, for each continuous function $\varphi : k \to \mathbb{R}$, we can consider its integral $\alpha(\varphi) = \int \varphi \, da$ with respect to $\alpha$.

Endow $\mathcal{ML}(S)$ with the topology defined by the family of semi-norms $\alpha \mapsto |\alpha(\varphi)|$, where $\varphi : k \to \mathbb{R}$ ranges over all continuous functions defined on transverse arcs. In particular, a sequence $\alpha_n \in \mathcal{ML}(S)$ converges to $\alpha$ if and only if $\alpha_n(\varphi)$ converges to $\alpha(\varphi)$ for every continuous function $\varphi$ defined on a generic arc $k$. (By the standard approximation techniques, we can restrict attention to countably many functions $\varphi$ to define the topology of $\mathcal{ML}(S)$, so that the topology is metrizable.)

**Proposition 15.** (Thurston) The positive real multiples of simple closed geodesics are dense in $\mathcal{ML}(S)$.

In some sense, we can think of the passage from the set $S(S)$ of simple closed curves to the topological space $\mathcal{ML}(S)$ as analogous to the passage from the lattice $\mathbb{Z}$ to the euclidean space $\mathbb{R}$. In particular, one of the motivations to introduce the space $\mathcal{ML}(S)$ is to analyze asymptotic directions when going to infinity in $S(S)$.

If $k$ is a generic arc and if $\alpha \in \mathcal{ML}(S)$, let $\alpha(k) \in \mathbb{R}$ be the $\alpha$-integral of the constant function 1 on $k$.

**Theorem 16.** (Thurston) There exists a finite family $k_1, k_2, \ldots, k_n$ such that the map $\mathcal{ML}(S) \to \mathbb{R}^n$ defined by $\alpha \mapsto (\alpha(k_i))_{i=1,\ldots,n}$ induces a homeomorphism between $\mathcal{ML}(S)$ and a piecewise linear submanifold of $\mathbb{R}^n$. In addition, for every generic arc $k$, there exists a piecewise linear function which expresses $\alpha(k)$ in terms of the $\alpha(k_i)$ for every measured lamination $\alpha \in \mathcal{ML}(S)$.

Variations of this result can be found in [FLP] or [PeH]. Recall that a map $\varphi : \mathbb{R}^p \to \mathbb{R}^q$ is piecewise linear if one can decompose $\mathbb{R}^p$ as the union of finitely many polyhedra so that the restriction of $\varphi$ to each polyhedra is affine. A piecewise linear submanifold of $\mathbb{R}^n$ is a subset $M$ such that, for every point $x \in M$, there is a piecewise linear homeomorphism from a neighborhood $U$ of $M$ to an open subset $V$ of $\mathbb{R}^n$ which sends $U \cap M$ to $V \cap \mathbb{R}^p$. A piecewise linear manifold is a space $M$ which
is endowed with a family of charts which locally model \( M \) over open subsets of \( \mathbb{R}^n \), such that the changes of charts are restrictions of piecewise linear maps \( \mathbb{R}^n \to \mathbb{R}^n \).

An immediate consequence of Theorem 16 is that \( \mathcal{ML}(S) \) admits the structure of a piecewise linear manifold.

**Theorem 17.** (Thurston) *As a piecewise linear manifold, \( \mathcal{ML}(S) \) is isomorphic to \( \mathbb{R}^{3\chi(S)} \), where \( b \) is the number of ends of \( S \) and where \( \chi(S) \) is its Euler characteristic.*

Again, references include [FLP] or [PeH].

**Change of metric**

To define measured geodesic laminations, we used a finite area hyperbolic metric \( m \) on the surface \( S \). A natural question is to wonder what will happen if we use a different finite area hyperbolic metric \( m' \).

**Lemma 18.** For every \( m \)-geodesic \( g \), there is a unique \( m' \)-geodesic \( g' \) such that \( g \) can be deformed to \( g' \) by a homotopy which moves points by a uniformly bounded distance. In addition, if the two \( m \)-geodesics \( g, h \) are simple and disjoint, then the corresponding \( m' \)-geodesics \( g', h' \) are also simple and disjoint.

See [CEG, §4.1.4]. The key property here is that an \( m \)-geodesic is quasi-geodesic for the metric \( m' \) which, because the curvature of \( m' \) is negative, implies that it stays within bounded distance of an \( m \)-geodesic.

Lemma 18 establishes a one-to-one correspondence between \( m \)-geodesic laminations and \( m' \)-geodesic laminations. In addition, if we interpret a transverse measure for a geodesic laminations \( \lambda \) as a locally defined measure on the space of leaves of \( \lambda \), this correspondence also induces a one-to-one correspondence between measured \( m \)-geodesic laminations and measured \( m' \)-geodesic laminations.

With somewhat more work, it can also be shown that the piecewise linear structure of \( \mathcal{ML}(S) \) is independent of the metric \( m \).

As an addendum which will be important later on, we should mention that a consequence of Lemma 18 and of an estimate of hyperbolic geometry is that the correspondence between \( m \)-geodesics and \( m' \)-geodesics is Hélder continuous. In particular, each transverse Hélder distribution for the \( m \)-geodesic laminations \( \lambda \) gets transported to a transverse Hélder distribution for the corresponding \( m' \)-geodesic laminations \( \lambda' \). In other words, geodesic laminations with transverse Hélder distributions are metric independent objects.

**The length function**

Many notions defined for deformation classes of simple closed curves extend to measured geodesic laminations. A typical example is the length function \( l_m \) associated to a hyperbolic metric \( m \).

Fix a hyperbolic metric \( m \) on \( S \). (This hypothesis can be relaxed to allow \( m \) to be a complete metric of negative curvature). For a simple closed curve \( \gamma \), define

\[
l_m(\gamma) = \inf_{\gamma' \sim \gamma} m - \text{length}(\gamma'),
\]

where the infimum is taken over all simple closed curves \( \gamma' \) homotopic to \( \gamma \).
Proposition 19. (Thurston) The length function uniquely extends to a continuous function

\[ l_m : \mathcal{ML}(S) \to \mathbb{R}^+ \]

that is positively homogeneous, namely such that \( l_m(a\alpha) = a l_m(\alpha) \) for every \( a \in \mathbb{R}^+ \) and \( \alpha \in \mathcal{ML}(S) \), where \( a\alpha \) is obtained by multiplying the transverse measure of \( \alpha \) by \( a \).

**Fig. 18.** Measuring the length \( l_m(\alpha) \) of a measured geodesic lamination \( \alpha \)

The length \( l_m(\alpha) \) of a measured geodesic lamination \( \alpha \in \mathcal{ML}(S) \) is defined as follows. Pick a finite family of arcs \( k_1, \ldots, k_n \) transverse to the geodesic lamination \( \lambda_\alpha \) underlying \( \alpha \), so that \( \lambda_\alpha - \bigcup_{i=1}^n k_i \) consists of arcs of finite length. Each component of \( \lambda_\alpha - \bigcup_{i=1}^n k_i \) is characterized by (any one of) its end points in \( \bigcup_{i=1}^n k_i \), and the measure defined by \( \alpha \) on the \( k_i \) consequently induces a measure on the space of these finite length components of \( \lambda_\alpha - \bigcup_{i=1}^n k_i \). Then \( l_m(\alpha) \) is defined as the integral of the lengths of these arcs with respect to the measure defined by \( \alpha \). The invariance of \( \alpha \) under homotopy of arcs transverse to \( \lambda_\alpha \) easily implies that \( l_m(\alpha) \) is independent on the choice of the \( k_i \). If we choose the \( k_i \) generic with respect to simple geodesics and so that every simple geodesic meets at least one of them, the continuity of \( l_m : \mathcal{ML}(S) \to \mathbb{R}^+ \) immediately follows from the definition of the topology of \( \mathcal{ML}(S) \).

This length function \( l_m \) plays an important rôle in many branches of low-dimensional topology and geometry.

In this context, a natural question is to ask whether the function \( l_m : \mathcal{ML}(S) \to \mathbb{R}^+ \) might be differentiable. If \( l_m \) is differentiable at \( \alpha \in \mathcal{ML}(S) \), it should have a differential \( T_\alpha \alpha : T_\alpha \mathcal{ML}(S) \to \mathbb{R} \) defined on the tangent space \( T_\alpha \mathcal{ML}(S) \) of \( \mathcal{ML}(S) \) at \( \alpha \).

However, we now encounter a conceptual problem in defining this tangent space \( T_\alpha \mathcal{ML}(S) \). Indeed, we only saw that \( \mathcal{ML}(S) \) is a piecewise linear manifold, not a differentiable manifold. In fact, it can be shown that, except when \( S \) is the three times punctured sphere, there is no homeomorphism between \( \mathcal{ML}(S) \) and a differentiable manifold for which the natural action of the diffeomorphism group of \( S \) corresponds to a differentiable action.

**Tangent vectors to \( \mathcal{ML}(S) \)**

Fortunately, a piecewise linear manifold classically admits a well-defined tangent space at each point, although its properties are weaker than that of the tangent space of a differentiable manifold. The definition mimics the differentiable context. A tangent vector of \( \mathcal{ML}(S) \) at \( \alpha \) is an equivalence class of curves
\[ \gamma : [0, \varepsilon[ \to \mathcal{ML}(S) \] with \( \gamma(0) = \alpha \) and with the following property: If \( \varphi : U \to V \) is a chart of the piecewise linear structure of \( \mathcal{ML}(S) \), identifying a neighborhood \( U \) of \( \alpha \) in \( \mathcal{ML}(S) \) to an open subset \( V \) of \( \mathbb{R}^n \), then the curve \( \varphi \circ \gamma \) admits a tangent vector at 0; two such curves \( \gamma \) and \( \gamma' \) are equivalent if and only if \( \varphi \circ \gamma \) and \( \varphi \circ \gamma' \) have the same tangent vector at 0 in \( \mathbb{R}^n \). Since the change of charts are piecewise linear, these properties are clearly independent of the choice of the chart \( \varphi \).

The space \( T_\alpha \mathcal{ML}(S) \) of tangent vectors of \( \mathcal{ML}(S) \) at \( \alpha \) is homeomorphic to \( \mathbb{R}^n \), and admits a natural multiplicative action of the group \( \mathbb{R}^+ \) of positive integers. However, in general, it admits no natural structure of vector space, or even of multiplication by \(-1\).

This combinatorial definition of tangent vectors is easy to introduce but, for applications, we would like to have a more geometric interpretation of them. Namely, given a curve \( t \mapsto \alpha_t \in \mathcal{ML}(S) \), we want to analyze the tangent vector \( \alpha_0 = d\alpha_t/dt \bigg|_{t=0} \) in a very geometric way.

For this, we can heuristically study a very specific example. Let \( \alpha_t \) be the geodesic laminations, which is associated to the suspension of the exchange of two intervals of length 1 and \( t \), namely the interval exchange \( [0, 1+t] \to [0, 1+t] \) which sends \([0, 1] \) to \([t, 1+t] \) and \([1, 1+t] \) to \([0, t] \). The suspension of this interval exchange is a punctured torus.

The structure of \( \alpha_t \) and of its underlying geodesic lamination \( \lambda_t \) is in general relatively complex. However, it is easy to describe for \( t = 1/n \) with \( n \) a positive integer. Indeed, all orbits of the interval exchange map are then closed, and \( \alpha_{1/n} \) consists of a single closed geodesic \( \lambda_{1/n} \) going \( n \) times around one direction of the punctured torus and once around the other one, and endowed with the transverse measure which is \( 1/n \) times the Dirac transverse measure. For \( t = 0 \), \( \alpha_0 \) consists of the closed geodesic \( \lambda_0 \) going once around the first direction of the punctured torus. However, as \( n \) tends to \( \infty \), the \( \alpha_t \) converges for the Hausdorff topology to the geodesic lamination \( \lambda_0^\infty \) consisting of the closed geodesic \( \lambda_0 \) and of an additional infinite geodesic whose two ends spiral around \( \lambda_0 \). See Figure 20. Note that we already encountered the geodesic lamination \( \lambda_0^\infty \) in Figure 16.

Consider the arc \( k \) represented on Figure 19. Its intersection with \( \lambda_0 \) consists of a single point \( x_0^\infty \). Its intersection with \( \lambda_0^\infty \) consists of \( x_0^\infty \) and of infinitely many points \( x_1^\infty, x_2^\infty, \ldots, x_i^\infty, \ldots \) which accumulate on \( x_0^\infty \). For every \( n \), \( k \cap \lambda_{1/n} \) consists of \( n \) points \( x_1^n, x_2^n, \ldots, x_n^n \). In addition, it is possible to choose the indexings so that, as \( n \) tends to \( \infty \), the point \( x_i^n \) converges to \( x_i^\infty \).

Let us perform a formal computation. For a continuous function \( \varphi : k \to \mathbb{R} \), its integrals with respect to the measures defined by \( \alpha_0 \) and \( \alpha_{1/n} \) on \( k \) are

\[
\alpha_0(\varphi) = \varphi(x_0^\infty)
\]

and

\[
\alpha_{1/n}(\varphi) = \frac{1}{n} \sum_{i=1}^{n} \varphi(x_i^n).
\]
Fig. 19. A formal computation

If the derivative $\dot{\alpha}_0 = d\alpha_t/dt^+_{t=0}$ exists, it should associate to $\varphi$ the quantity

$$
\dot{\alpha}_0 (\varphi) = \lim_{t \to 0^+} \frac{\alpha_t (\varphi) - \alpha_0 (\varphi)}{t} \\
= \lim_{n \to \infty} n \left( \alpha_1/n (\varphi) - \alpha_0 (\varphi) \right) \\
= \lim_{n \to \infty} \sum_{i=1}^{n} \varphi (x^n_i) - n \varphi (x^n_{\infty}) \\
= \lim_{n \to \infty} \sum_{i=1}^{n} (\varphi (x^n_i) - \varphi (x^n_{\infty})) \\
= \sum_{i=1}^{\infty} (\varphi (x^n_i) - \varphi (x^n_{\infty}))
$$

since $x^n_i$ tends to $x^n_{\infty}$ as $n$ tends to $\infty$. Of course, this is only a formal computation. In particular, for a general continuous $\varphi$, there is no reason why the infinite sum of the last expression would even converge. However, a geometric estimate shows that the distances $d(x^n_{\infty}, x^n_{\infty})$ are asymptotic to $e^{-L_i}$, where $L_i > 0$ is the length of the closed geodesic $\lambda_0$. Compare Proposition 8. If we impose in addition that $\varphi : k \to \mathbb{R}$ is Hölder continuous, then the above infinite sum is consequently bounded by a converging geometric series. A little more work enables one to rigorously justify the above computation, and to prove that

$$
\dot{\alpha}_0 (\varphi) = \lim_{t \to 0^+} \frac{\alpha_t (\varphi) - \alpha_0 (\varphi)}{t} = \sum_{i=1}^{\infty} (\varphi (x^n_i) - \varphi (x^n_{\infty}))
$$
when \( \varphi : k \to \mathbb{R} \) is Hölder continuous. Note that we already encountered this transverse Hölder distribution for the geodesic lamination of Figure 16.

A similar argument can be made on every arc transverse to \( \lambda_0^+ \), and provides a transverse Hölder distribution for \( \lambda_0^+ \).

This heuristic argument in one specific example can actually be extended to the full setting of tangent vectors to \( \mathcal{ML}(S) \).

**Theorem 20.** [Bo1] Let the curve \( t \mapsto \alpha_t \in \mathcal{ML}(S) \) have a tangent vector \( \dot{\alpha}_0 \in T_{\alpha_0} \mathcal{ML}(S) \) at \( \alpha_0 \). Then, for every Hölder continuous function \( \varphi : k \to \mathbb{R} \) defined on a generic arc \( k \), the right derivative \( d\alpha_t(\varphi)/dt|_{t=0} \) exists and defines a Hölder distribution on \( k \).

In addition, as \( k \) ranges over all generic arcs, the support of the corresponding Hölder distributions form a geodesic lamination \( \lambda_0^+ \), and \( \dot{\alpha}_0 \) defines a transverse Hölder distribution for \( \lambda_0^+ \).

In this way, we have associated to each tangent vector \( \dot{\alpha}_0 \in T_{\alpha_0} \mathcal{ML}(S) \) a geodesic lamination \( \lambda_0^+ \) endowed with a transverse Hölder distribution. The following proposition provides a converse, by characterizing which geodesic laminations with transverse distributions correspond to tangent vectors to \( \mathcal{ML}(S) \).

**Proposition 21.** [Bo1] Let \( \alpha \) be a transverse Hölder distribution for the geodesic lamination \( \lambda \), and assume that the support of \( \alpha \) is all of \( \lambda \). This data corresponds to a tangent vector \( \dot{\alpha}_0 \in T_{\alpha_0} \mathcal{ML}(S) \) at \( \alpha_0 \in \mathcal{ML}(S) \) if and only if:

(i) no leaf of \( \lambda \) crosses the geodesic lamination \( \lambda_0^+ \) underlying \( \alpha_0 \), namely their union is a geodesic lamination;

(ii) every infinite isolated half-leaf \( g \) of \( \lambda \) is asymptotic to a component of \( \lambda_0^+ \), and \( \alpha(k) \geq 0 \) for an (arbitrary) transverse arc \( k \) with \( k \cap \lambda = k \cap g = 1 \) point;

(iii) for every minimal sublamination \( \lambda_1 \) of \( \lambda \) that is not in the geodesic lamination \( \lambda_0^+ \) underlying \( \alpha_0 \), the restriction of \( \alpha \) to \( \lambda_1 \) is a transverse measure.

The conditions of Proposition 21 just reflect the fact that, if \( t \mapsto \alpha_t \) is tangent to \( \alpha \) at \( \alpha_0 \), then \( \alpha_t(k) \geq 0 \) for every transverse arc \( k \) and every \( t \).

For almost every \( \alpha_0 \in \mathcal{ML}(S) \), the geodesic lamination \( \lambda_0^+ \) underlying \( \alpha_0 \) is maximal for inclusion among all geodesic laminations. In this case, the conditions of Proposition 21 are automatically satisfied, and it follows that \( T_{\alpha_0} \mathcal{ML}(S) \) is equal to the vector space \( \mathcal{H}(\lambda_0^+) \) of all transverse Hölder distributions or, equivalently, of all transverse cocycles for \( \lambda_0^+ \).

On the other hand, when the geodesic lamination \( \lambda_0^+ \) underlying \( \alpha_0 \) is not maximal, every chain recurrent geodesic lamination \( \lambda \) containing \( \lambda_0^+ \) determines a linear face of \( T_{\alpha_0} \mathcal{ML}(S) \), for the piecewise linear structure of \( \mathcal{ML}(S) \). This face is naturally isomorphic to a convex cone in \( \mathcal{H}(\lambda) \), bounded by finitely many hyperplanes.

**The derivative of the length function**

We are now ready to compute the tangent map

\[
T_{\alpha_0} l_m : T_{\alpha_0} \mathcal{ML}(S) \to \mathbb{R}
\]

of the length function

\[
l_m : \mathcal{ML}(S) \to \mathbb{R}^+.
\]
at $\alpha_0 \in \mathcal{ML}(S)$. By definition, if this tangent map exists, $T_{\alpha_0}l_m(\hat{\alpha}_0)$ should be equal to $dl_m(\alpha_t)/dt^+|_{t=0}$ for every curve $t \mapsto \alpha_t$ tangent to the vector $\hat{\alpha}_0 \in T_{\alpha_0}\mathcal{ML}(S)$ at $\alpha_0$.

When we defined the length of a measured geodesic lamination $\alpha$, we cut the underlying geodesic lamination $\lambda_\alpha$ along a family of transverse arcs $k_1, \ldots, k_n$ so that $\lambda_\alpha - \bigcup_i k_i$ consisted of arcs of finite length, noted that each component of $\lambda_\alpha - \bigcup_i k_i$ is characterized by (any one of) its end points in $\bigcup_i k_i$, and defined the length $\lambda_m(\alpha)$ as the integral of the lengths of the components with respect to the measure defined by $\alpha$ on the $k_i$. However, not only is the length of a component of $\lambda_\alpha - \bigcup_i k_i$ a continuous function of each of its end points, Lemma 2 actually shows that this length is a Lipschitz continuous function of this end points. In particular, the same process can be used to define the length $l_m(\alpha)$ when $\alpha$ is only a Hölder distribution for the geodesic lamination $\lambda_\alpha$.

Because the construction is so natural, the following immediately follows from Theorem 20.

**Theorem 22.** [Bo1] Let the curve $t \mapsto \alpha_t \in \mathcal{ML}(S)$ be tangent to the vector $\hat{\alpha}_0 \in T_{\alpha_0}\mathcal{ML}(S)$ at $\alpha_0$. If we interpret the tangent vector $\hat{\alpha}_0$ as a geodesic lamination with a transverse Hölder distribution, then

$$dl_m(\alpha_t)/dt^+|_{t=0} = l_m(\hat{\alpha}_0).$$

In other words, the tangent map $T_{\alpha_0}l_m$ of the length function

$$l_m : \mathcal{ML}(S) \to \mathbb{R}^+$$

at $\alpha_0$ coincides with the length function

$$l_m : T_{\alpha_0}\mathcal{ML}(S) \to \mathbb{R}$$

defined by interpreting the vectors of $T_{\alpha_0}\mathcal{ML}(S)$ as geodesic laminations with transverse Hölder distributions.

**Part III. The Geometric Viewpoint**

As indicated in the introduction, geodesic laminations occur in many geometric contexts. We will focus on only one example: pleated surfaces and convex cores of hyperbolic 3-manifolds.

**The convex core of a hyperbolic 3–manifold**

Let $M$ be a hyperbolic 3–manifold, namely a 3-dimensional manifold with a complete Riemannian metric of constant curvature $-1$. In addition, we assume that $M$ is non-compact.

The convex core $C(M)$ of $M$ is the smallest closed convex subset of $M$. Here a subset $A$ of $M$ is convex if every geodesic arc of $M$ whose end points are contained in $A$ is completely contained in $A$; in particular, a single point is in general not convex. The convex core $C(M)$ exists and is unique, except in a few degenerate cases which we will exclude; see for instance [Th1][EpM].

Here is another description of the convex core $C(M)$. Endowed with the lift of the metric of $M$, the universal covering $\tilde{M}$ of $M$ must be isometric to the hyperbolic
space $\mathbb{H}^3$, as this is the only simply connected hyperbolic 3-manifold. It follows that $M$ is isometric to the quotient of $\mathbb{H}^3$ under an isometric action of the fundamental group $\pi_1(M)$.

In the union of $\mathbb{H}^3$ and of its sphere at infinity $\partial_{\infty} \mathbb{H}^3$, the \textit{limit set} $\Lambda \subset \partial_{\infty} \mathbb{H}^3$ of $M$ is defined as the set of cluster points of any orbit of the action of $\pi_1(M)$ on $\mathbb{H}^3$. We can then consider the \textit{convex hull} $C(\Lambda)$ of $\Lambda$, namely the smallest convex subset of $\mathbb{H}^3 \cup \partial_{\infty} \mathbb{H}^3$ containing $\Lambda$. If we use the projective model for $\mathbb{H}^3$, in which geodesics correspond to euclidean line segments of $\mathbb{R}^3$, then $C(\Lambda)$ just corresponds to the euclidean convex hull of $\Lambda$. It is now easy to see that

$$C(M) = \left( C(\Lambda) - \partial_{\infty} \mathbb{H}^3 \right) / \pi_1(M)$$
in $M = \mathbb{H}^3 / \pi_1 (M)$, except in the very degenerate cases where $\Lambda$ consists of 0, 1 or 2 points, which we henceforth exclude.

One of the advantages of this construction is that it provides some insight on the geometry of the boundary $\partial C (M)$, which is locally the same as the boundary of $C (\Lambda)$.

First of all $\partial C (M)$ topologically is a surface.

Its local geometry is reminiscent of that of the boundary of a convex polyhedron, in the sense that it is a union of disjoint embedded geodesics of $M$ (the ‘edges’) and of embedded totally geodesic pieces bounded by geodesics (the ‘faces’). However, there are no points corresponding to the vertices of a polyhedron, as these points are now ‘at infinity’, in $\Lambda$.

In addition, exactly as on the boundary of a hyperbolic polyhedron away from the vertices, the path metric induced by the metric of $M$ on $\partial C (M)$ is a (2-dimensional) hyperbolic metric. If, in addition, the fundamental group $\pi_1 (M)$ is finitely generated, a theorem of Ahlfors implies that $\partial C (M)$ has finite area.

In sharp contrast with the case of a finite polyhedron, the family of ‘edges’ along which $\partial C (M)$ is not totally geodesic is not necessarily locally finite. However, what we know about the structure of geodesic laminations provides us with some information. Indeed, the union of these edges forms a geodesic lamination on the hyperbolic surface $\partial C (M)$, called the bending locus of $\partial C (M)$. From Theorem 6, we conclude that the bending locus has Hausdorff dimension 1, in spite of the fact that it may consist of uncountably many geodesics. We should emphasize that this local property of $\partial C (M)$ depends on global information, and that it is false if we do not require the fundamental group $\pi_1 (M)$ to be finitely generated.

The amount by which $\partial C (M)$ actually bends along the bending locus is measured by a transverse measure for the bending locus, as explained below. The bending locus together with this transverse measure is the bending measured lamination of $\partial C (M)$.

Again, [Th1, §8] and [EpM] are good references for these facts.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{pleated_surface.png}
\caption{A pleated surface}
\end{figure}

**Pleated surfaces in hyperbolic 3–manifolds**

The boundary of the convex core is a special example of a pleated surface. A pleated surface in a hyperbolic 3–manifold $M$ consists of a hyperbolic surface $S$ and of a map $f : S \to M$ such that

1. $f$ sends each continuous arc in $S$ to an arc of the same length in $M$;
2. there is a geodesic lamination $\lambda \subset S$ (called a pleating locus for $f$) such that
   - $f$ sends each leaf of $\lambda$ to a geodesic of $M$ and is totally geodesic on $S - \lambda$;
3. when $S$ is non-compact, $f$ is *cusp-preserving* in the sense that it sends each small neighborhood of a cusp of $S$ to a small neighborhood of a cusp of $M$.

![Diagram](image)

**Fig. 23.** Measuring the bending of a pleated surface

We now explain how to measure the bending of the pleated surface $f : S \to M$ along $\lambda$.

Consider an arc $k$ transverse to $\lambda$ in $S$. We want to measure the amount of bending which occurs along $k$. It is convenient to lift the situation to a map $\tilde{f} : \tilde{S} \to \tilde{M}$ between the universal coverings of $S$ and $M$, and to isometrically identify $\tilde{M}$ to the hyperbolic space $\mathbb{H}^3$. Lift $k$ to an arc $\tilde{k}$ in $\tilde{S}$, and let $\tilde{\lambda} \subset \tilde{S}$ be the pro-image of $\lambda$ in $\tilde{S}$.

We can “project $\tilde{k}$ to $\partial_\infty \mathbb{H}^3$ parallel to $\tilde{f}(\tilde{S})$ ” as follows. Pick an orientation for $\tilde{k}$, and orient the leaves of $\tilde{\lambda}$ meeting $\tilde{k}$ from left to right (using the orientations of $k$ and $S$). To each $x \in \tilde{k} \cap \tilde{\lambda}$ we can then associate the positive end point $x^\infty \in \partial_\infty \mathbb{H}^3$ of the oriented geodesic $\tilde{f}(g)$ of $\mathbb{H}^3$, where $g$ is the leaf of $\tilde{\lambda}$ passing through $x$. To extend this set of points to a curve in $\partial_\infty \mathbb{H}^3$ we need, for every component $d$ of $\tilde{k} - \tilde{\lambda}$, to connect by an arc the two points $x_d^\infty$, $y_d^\infty \in \partial_\infty \mathbb{H}^3$ associated to the two points $x_d$, $y_d \in \tilde{k} \cap \tilde{\lambda}$ of $\partial d$. Such an arc is provided by the totally geodesic plane of $\mathbb{H}^3$ which coincides with $\tilde{f}(\tilde{S})$ on a neighborhood of each point of $\tilde{f}(d)$. This plane intersects the sphere at infinity along a round circle, and consequently specifies a circle arc $d^\infty$ joining $x_d^\infty$ to $y_d^\infty$. Note that $d^\infty$ may consist of a single point, when $x_d^\infty = y_d^\infty$; this will happen very often, for instance roughly 50% of the time when $\lambda$ is maximal for inclusion, namely when its complement $S - \lambda$ consists only of infinite triangles. So far, we have implicitly assumed that $d$ does not contain one of the end points of $\tilde{k}$, so that the end points $x_d^\infty$, $y_d^\infty \in \tilde{k} \cap \tilde{\lambda}$ are well-defined; for the two components containing one of the end points of $\tilde{d}$, the same construction only associates a germ $d^\infty$ of circle arc (of arbitrary length) associated to each such component $d$.

Let $\tilde{k}^\infty$ be the curve in $\partial_\infty \mathbb{H}^3$ which is made up of the points $x^\infty$ associated to all $x \in \tilde{k} \cap \tilde{\lambda}$ and of the circle arcs $d^\infty$ associated to all components $d$ of $\tilde{k} - \tilde{\lambda}$. Intuitively, the total amount by which $f(S)$ bends at the points of $k \cap \lambda$ is equal to the amount by which $\tilde{k}^\infty$ turns at its ‘corners’, corresponding to points of $\tilde{k} \cap \tilde{\lambda}$. Since there may
be uncountably many such corners, we will use an indirect arguments to measure this amount of turning.

Extend the arc $\tilde{k}^\infty$ to a closed curve $\tilde{k}^\infty_+$ in $\partial_\infty \mathbb{H}^3$, in such a way that $\tilde{k}^\infty_+$ is smooth near the two end points of $\tilde{k}^\infty$. The closed curve $\tilde{k}^\infty_+$ bounds a cycle $\Sigma$ in the sphere $\partial_\infty \mathbb{H}^3$, and we can consider the oriented area of this cycle for the standard metric of $\partial_\infty \mathbb{H}^3$ considered as the unit sphere in $\mathbb{R}^3$. Then, define

$$\beta(k) = \text{area}(\Sigma) - \int_{k^\infty} \text{geodesic curvature of the smooth pieces} \in \mathbb{R}/2\pi \mathbb{Z}.$$ 

The finiteness of the integral is easily proved. Since the area of the sphere is $4\pi$, this $\beta(k) \in \mathbb{R}/2\pi \mathbb{Z}$ is clearly independent of the choice of the cycle $\Sigma$. Also, it easily follows from the Gauss-Bonnet formula that $\beta(k)$ is independent of the closed curve extension $\tilde{k}^\infty_+$ of the arc $\tilde{k}^\infty$.

When $\tilde{k}^\infty_+$ has only finitely many corners, the Gauss-Bonnet formula shows that $\beta(k)$ is the sum of the turning angles at these corners. In the general case, we define the sum of the turning angles at the corners of $\tilde{k}^\infty_+$ to be equal to $\beta(k)$.

One can check that $\beta(k)$ is independent of the orientation of $k$ and of the choice of the lift $\tilde{k}$ of $k$. Also, $k \mapsto \beta(k)$ is clearly finitely additive, and invariant under homotopy of arcs transverse to the pleating locus $\lambda$. As a consequence, this defines an $\mathbb{R}/2\pi \mathbb{Z}$-valued transverse cocycle $\beta \in \mathcal{H}(\lambda; \mathbb{R}/2\pi \mathbb{Z})$, called the bending transverse cocycle of the pleated surface $f : S \to M$. By definition, an $\mathbb{R}/2\pi \mathbb{Z}$-valued transverse cocycle associates an element of $\mathbb{R}/2\pi \mathbb{Z}$ to each arc transverse to $\lambda$, in a way which is finitely additive and invariant under homotopy of arcs transverse to $\lambda$. When the surface is non-compact, the bending transverse cocycle $\beta$ satisfies an additional condition for every cusp $c$ of $S$: Recall that $c$ admits a neighborhood which meets $\lambda$ only along finitely many isolated half-leaves $l_1, \ldots, l_n$ converging towards $c$; then $\sum_{i=1}^n \beta(l_i) = 0 \in \mathbb{R}/2\pi \mathbb{Z}$, where $\beta(l_i)$ is the element of $\mathbb{R}/2\pi \mathbb{Z}$ associated by $\beta$ to an arbitrary transverse arc $k_i$ with $k_i \cap \lambda = k_i \cap l_i = 1$ point. Let $\mathcal{H}_0(\lambda; \mathbb{R}/2\pi \mathbb{Z})$ denote the space of those $\mathbb{R}/2\pi \mathbb{Z}$-valued transverse cocycles which satisfy this cusp relation for every cusp $c$ of $S$.

Conversely, if we are given the hyperbolic metric $m$ of $S$, the bending locus $\lambda$ and the bending transverse cocycle in $\mathcal{H}_0(\lambda; \mathbb{R}/2\pi \mathbb{Z})$, we can completely reconstruct...
the local geometry of the pleated surface \( f : S \to M \), the lift \( \tilde{f} : \tilde{S} \to \tilde{M} = \mathbb{H}^3 \), and in particular the monodromy representation

\[
\rho : \pi_1 (S) \to \text{Isom} (\mathbb{H}^3)
\]

with respect to which \( \tilde{f} \) is equivariant. The proof is comparatively easy when \( \beta \) is a transverse measure (see [EpM, §3]), but technically harder in the general case [Bo3]. It uses an analog of the Gap Formula in the (non-commutative) group \( \text{Isom} (\mathbb{H}^3) \).

We can use this to locally parametrize near \( \rho \) the space of representations

\[
\rho : \pi_1 (S) \to \text{Isom} (\mathbb{H}^3)
\]

which are \textit{cusp-preserving}, in the sense that \( \rho' \) sends the elements of \( \pi_1 (S) \) corresponding to cusps of \( S \) to parabolic elements of \( \text{Isom} (\mathbb{H}^3) \). For this, assume in addition (and without loss of generality) that \( \lambda \) is \textit{maximal}, namely that it is maximal for inclusion among all geodesic laminations. This is equivalent to the property that its complement consists only of infinite triangles.

Deform the representation \( \rho \) to a nearby cusp-preserving representation

\[
\rho' : \pi_1 (S) \to \text{Isom} (\mathbb{H}^3).
\]

If \( \rho' \) is sufficiently close to \( \rho \), there is a pleated surface \( \tilde{f}' : \tilde{S} \to \mathbb{H}^3 \) which still has pleating locus \( \lambda \) but is now equivariant with respect to \( \rho' \), and this pleated surface is unique modulo pre-composition by the lift of an isotopy of \( S \). This pleated surface has hyperbolic metric

\[
m' \in \mathcal{T} (S) = \{ \text{hyperbolic metrics on } S \} / \text{isotopy}
\]

and bending cocycle

\[
\beta' \in \mathcal{H}_0 (l; \mathbb{R} / 2\pi \mathbb{Z}).
\]

**Proposition 23.** The map \( \rho' \mapsto (m', \beta') \) defines a diffeomorphism between a neighborhood of \( \rho \) in

\[
\mathcal{R} = \{ \text{cusp-preserving representations } \pi_1 (S) \to \text{Isom} (\mathbb{H}^3) \} / \text{Isom} (\mathbb{H}^3)
\]

and a neighborhood of \( (m, \beta) \) in

\[
\mathcal{T} (S) \times \mathcal{H}_0 (l; \mathbb{R} / 2\pi \mathbb{Z}).
\]

When \( \lambda \) is maximal, it can be shown that \( \mathcal{H}_0 (\lambda; \mathbb{R} / 2\pi \mathbb{Z}) \) consist of two copies of \( (\mathbb{R} / 2\pi \mathbb{Z})^{3g_\lambda (S) - c} \), where \( c \) is the number of cusps of \( S \).

The main point of Proposition 23 is that it provides a local parametrization of \( \mathcal{R} \) by \( \mathcal{T} (S) \times \mathcal{H}_0 (\lambda; \mathbb{R} / 2\pi \mathbb{Z}) \) which is very well adapted to the geodesic lamination \( \lambda \). We now discuss an application of this machinery to convex cores of hyperbolic 3-manifolds.
Variations of the geometry of convex cores

Let $M_0$ be a hyperbolic 3–manifold, and deform the metric of $M_0$ to $M$.

Under relatively mild hypotheses (for instance if $M_0$ is convex co-compact, or if $M_0$ is geometrically finite and we restrict to cusp-preserving deformations, or if we restrict to holomorphic deformations, etc ...), the convex core boundary $\partial C(M)$ has a natural topological identification with $\partial C(M_0)$, which is well-defined up to isotopy. If $S$ is a component of $\partial C(M_0)$, the geometry of the corresponding component of $\partial C(M)$ is then described by its hyperbolic metric $m \in \mathcal{T}(S)$ and its bending measured lamination $\beta \in \mathcal{ML}(S)$.

The set

$$\mathcal{D} = \{ \text{deformations of the hyperbolic metric of } M \}$$

suitably restricted as above so that the topology of $\partial C(M)$ does not change, is a differentiable manifold. Recall that the Teichmüller space $\mathcal{T}(S)$ is a differentiable manifold, and that the space $\mathcal{ML}(S)$ of measured geodesic lamination is a piecewise linear manifold. This leads to the following natural questions:

**Question 1.** Is the map $\mathcal{D} \to \mathcal{T}(S)$ defined by $M \mapsto m$ differentiable

**Question 2.** Is the map $\mathcal{D} \to \mathcal{ML}(S)$ defined by $M \mapsto \beta$ differentiable, in the appropriate sense?

To answer these questions, we take the problem backwards, mixing the differentiable and piecewise linear context.

If $U$ is open in $\mathbb{R}^m$, the tangent map of $\varphi : U \to \mathbb{R}^n$ is, if it exists, the map $T_x \varphi : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$T_x \varphi (v) = (\varphi \circ \alpha)'(0)$$

for every curve $\alpha : [0, \varepsilon[ \to U$ with $\alpha (0) = x$ and $\alpha' (0) = v$. The map $\varphi$ is *tangential* if it admits a tangent map everywhere. Typical examples of tangent maps include differentiable maps and piecewise linear maps, and these are exactly the examples which are relevant here.

This immediately leads to a notion of *tangential manifolds*, defined by atlases of charts where the changes of charts are all tangential. Typical examples are differentiable manifolds, piecewise linear manifolds or, as we will encounter later on, products of differentiable and piecewise linear manifolds. By an immediate translation of the corresponding definitions in the differentiable case, one can introduce notions of tangent vectors to a tangential manifold $X$ at the point $x$, of tangent spaces $T_x X$, of tangential maps $f : X \to Y$ between tangential manifolds, and of tangent maps $T_x f : T_x X \to T_{f(x)} Y$ of tangential maps. We already encountered these notions when discussing the piecewise linear structure of $\mathcal{ML}(S)$. In particular, the tangent space $T_x X$ is homeomorphic to $\mathbb{R}^n$, where $n$ is the dimension of $X$, but does not necessarily admit a natural vector space structure.

The key to computing tangent maps is the following very elementary lemma.

**Lemma 24.** If a local homeomorphism $\varphi$ is tangential at $x$ and if the tangent map $T_x \varphi$ is injective, then $\varphi^{-1}$ is tangential at $\varphi (x)$ and $T_{\varphi(x)} \varphi^{-1} = (T_x \varphi)^{-1}$. (The tangent map of a local homeomorphism is always surjective, by a degree argument.)
We want to show that the map $\mathcal{D} \to \mathcal{T}(S) \times \mathcal{ML}(S)$ defined by $M \mapsto (m, \beta)$ is tangential at $m_0$.

For this, it is convenient to consider the space

$$\mathcal{R} = \{\text{cusp-preserving representations } \pi_1(S) \to \text{Isom}(\mathbb{H}^3)\} / \text{Isom}(\mathbb{H}^3)$$

and the map $\mathcal{D} \to \mathcal{R}$ which to a hyperbolic structure $M$ associates $\rho \circ i$, where $\rho : \pi_1(S) \to \text{Isom}(\mathbb{H}^3)$ is the monodromy of $M$ and where $i : \pi_1(S) \to \pi_1(M)$ is the homomorphism induced by the inclusion map. When discussing Proposition 23, we also explained how, given a hyperbolic metric $m \in \mathcal{T}(S)$ and a measured geodesic lamination $\beta \in \mathcal{ML}(S)$, we can reconstruct a pleated surface $\tilde{f} : \tilde{S} \to \mathbb{H}^3$ which is equivariant with respect to a representation $\rho : \pi_1(S) \to \text{Isom}(\mathbb{H}^3)$, and which induces on $S$ the hyperbolic metric $m$ and the bending measured lamination $\beta$. This defines a map

$$\varphi : \mathcal{T}(S) \times \mathcal{ML}(S) \to \mathcal{R}.$$ 

In his study of complex projective structures on surfaces (see [KaT]), Thurston showed that $\varphi$ is a local homeomorphism. It is then immediate that the map $\mathcal{D} \to \mathcal{T}(S) \times \mathcal{ML}(S)$ we are interested in coincides near $M_0$ with the composition of the map $\mathcal{D} \to \mathcal{R}$ and of a local inverse $\varphi^{-1}$ of $\varphi$.

By Proposition 23, for instance, the space $\mathcal{R}$ is a differentiable manifold near $\rho_0$. Also, the map $\mathcal{D} \to \mathcal{R}$ is a differentiable map between differentiable manifolds, by construction. Therefore, to prove that the map $\mathcal{D} \to \mathcal{T}(S) \times \mathcal{ML}(S)$ is tangential at $M_0$, it suffices to show that the local inverse $\varphi^{-1}$ is tangential at $\rho_0$. By Lemma 24, we only need to show that $\varphi$ is tangential at the point $(m_0, \beta_0) \in \mathcal{ML}(S)$ corresponding to $M_0$, and that the tangent map $T_{(m_0, \beta_0)}\varphi$ is injective.

These properties are proved in [Bo4], by carefully analyzing the proof of Proposition 23. It follows that the map $\mathcal{D} \to \mathcal{T}(S) \times \mathcal{ML}(S)$ is tangential. Composing with the appropriate projections, we conclude that the maps $\mathcal{D} \to \mathcal{T}(S)$ and $\mathcal{D} \to \mathcal{ML}(S)$ are tangential at $M_0$. A close examination of the proof actually shows that the tangent map $T_{M_0}\mathcal{D} \to T_{m_0}\mathcal{T}(S)$ is linear (which makes sense since $\mathcal{D}$ and $\mathcal{T}(S)$ are differentiable manifolds), and varies continuously with $M_0$; in other words, the map is continuously differentiable in the usual sense.

Letting $S$ range over all components of $\partial C(M_0)$, this proves:

**Theorem 25.** [Bo4] The map $\mathcal{D} \to \mathcal{T}(\partial C(M_0))$, which associates to a deformation $M \in \mathcal{D}$ of the hyperbolic metric $M_0$ the hyperbolic metric $m \in \mathcal{T}(\partial C(M_0))$ of $\partial C(M) \equiv \partial C(M_0)$, is continuously differentiable.

**Theorem 26.** [Bo4] The map $\mathcal{D} \to \mathcal{ML}(\partial C(M_0))$, which associates to a deformation $M \in \mathcal{D}$ of the hyperbolic metric $M_0$ the bending measured lamination $\beta \in \mathcal{ML}(\partial C(M_0))$ of $\partial C(M) \equiv \partial C(M_0)$, is tangential.

Explicit examples show that the map $\mathcal{D} \to \mathcal{T}(S)$ is in general not $C^2$. This low differentiability can actually be attributed to the ‘corners’ of the piecewise linear structure of $\mathcal{ML}(S)$.

This machinery has a nice application to the function $\text{vol} : \mathcal{D} \to \mathbb{R}$ which associates to $M$ the volume of its convex core $C(M)$.

Consider the bending map $b : \mathcal{D} \to \mathcal{ML}(\partial C(M_0))$. We saw that $b$ is tangential. If $v \in T_{M_0}\mathcal{D}$ is a tangent vector of $\mathcal{D}$, we can therefore consider its image $T_{m_0}b(v) \in T_b(M_0)\mathcal{ML}(S)$ under the tangent map of $b$. By Theorem 20, this tangent vector $T_{M_0}b(v)$ can be interpreted as a geodesic lamination with a transverse
Hölder distribution, and consequently has a well-defined length \( l_{m_0}(T_{M_0}b(v)) \) for the hyperbolic metric \( m_0 \) of the boundary \( \partial C(M_0) \).

**Theorem 27.** [Bo5] The convex core volume function \( \text{vol} : D \to \mathbb{R} \) is tangential at \( M_0 \) and, for every tangent vector \( v \in T_{M_0}D \),

\[
T_{M_0}\text{vol}(v) = \frac{1}{2}l_{m_0}(T_{M_0}b(v)).
\]

This is the natural generalization of the well-known Schlaffi formula for the variation of volumes of polyhedra in hyperbolic spaces.

**Rotation angles, bending cocycles, and Thurston’s intersection form**

We conclude with a last application of the bending transverse cocycle of a pleated surface.

Let \( M \) be an oriented hyperbolic 3-manifold, and let \( \gamma \) be a closed geodesic of \( M \). If we start with a vector \( v_x \) orthogonal to \( \gamma \) at \( x \), we can extend the vector \( v_x \) to a parallel vector field \( v \) (orthogonal to \( \gamma \)) along \( \gamma \). However, as we go around \( \gamma \), there is no reason why \( v \) should return to the value \( v_x \) at \( x \). Let \( \text{rot}_M(\gamma) \in \mathbb{R}/2\pi\mathbb{Z} \) be the angle between the new value \( v'_x \) of \( v \) at \( x \) and the original vector \( v_x \), measured in the plane orthogonal to \( \gamma \) at \( x \). It is immediate that \( \text{rot}_M(\gamma) \) is independent of the choices of \( x, v_x \) and of an orientation for \( \gamma \).

By definition, \( \text{rot}_M(\gamma) \in \mathbb{R}/2\pi\mathbb{Z} \) is the rotation angle of the closed geodesic \( \gamma \).

![Fig. 25. Measuring the rotation angle of a closed geodesic](image)

This definition can be extended to the wider context of measured geodesic laminations. Suppose that we are given a finite type surface \( S \) and a cusp-preserving map \( f : S \to M \). A geodesic lamination \( \lambda \) in \( S \) is realizable if there exists a pleated surface \( f' : S \to M \), homotopic to \( f \) through a family of cusp-preserving maps, which admits a pleating locus containing \( \lambda \). In particular, \( f' \) sends each leaf of \( \lambda \) to a complete geodesic of \( M \).

Let \( \mathcal{O}_f \subset \mathcal{ML}(S) \) denote the set of those measured geodesic laminations whose underlying geodesic lamination is realizable. Under relatively mild hypotheses on \( f \) or \( M \), the subset \( \mathcal{O}_f \) is open in \( \mathcal{ML}(S) \). For instance, \( \mathcal{O}_f \) is equal to all of \( \mathcal{ML}(S) \) if the convex core \( C(M) \) is compact.

By re-interpreting the rotation angles of closed geodesics in the appropriate framework, one can show:

**Proposition 28.** [Bo3] There is a continuous function:

\[
\text{rot}_M : \mathcal{O}_f \to \mathbb{R}
\]
such that
\[ \tilde{\text{rot}}_M (\gamma) = \text{rot}_M (\gamma^*) \mod 2\pi \]
for every simple closed curve \( \gamma \in \mathcal{O}_f \subset \mathcal{MC}(S) \), where \( \gamma^* \) denotes the closed geodesic of \( M \) that is homotopic to \( f(\gamma) \), and such that \( \tilde{\text{rot}}_M (a\alpha) = a\tilde{\text{rot}}_M (\alpha) \) for every \( \alpha \in \mathcal{O}_f \), \( a \in \mathbb{R}^+ \).

Note that the last property obliges us to replace \( \mathbb{R}/2\pi \mathbb{Z} \) by \( \mathbb{R} \). The function \( \tilde{\text{rot}}_M : \mathcal{O}_f \to \mathbb{R} \) is not unique and depends on certain choices, as there are several ways to lift rotation angles from \( \mathbb{R}/2\pi \mathbb{Z} \) to \( \mathbb{R} \) in a continuous way. However, compare Theorem 30 below and the remark thereafter.

The same construction, which enabled us to define a rotation angle for transverse measures for a realizable geodesic lamination \( \lambda \), automatically extends to transverse Hölder distributions for \( \lambda \), and defines a linear map
\[ \tilde{\text{rot}}_M : \mathcal{H}(\lambda) \to \mathbb{R} \]
when the geodesic lamination \( \lambda \) stays away from the cusps of \( S \). Note that this is always the case when \( \lambda \) is the underlying geodesic lamination of a measured geodesic lamination, or when \( \lambda \) is the underlying geodesic lamination of a tangent vector to \( \mathcal{MC}(S) \) interpreted as a geodesic lamination with a transverse Hölder distribution.

As in the case of Theorem 22, the naturality of the construction gives in a quasi-automatic way the following differentiability result.

**Theorem 29.** The function \( \tilde{\text{rot}}_M : \mathcal{O}_f \to \mathbb{R} \) of Proposition 28 is tangential at each \( \alpha \in \mathcal{O}_f \), and the tangent map
\[ T_\alpha \tilde{\text{rot}}_M : T_\alpha \mathcal{MC}(S) \to \mathbb{R} \]
just associates to \( v \in T_\alpha \mathcal{MC}(S) \) the rotation angle \( \tilde{\text{rot}}_M (v) \), if we interpret the tangent vector \( v \) as a geodesic lamination with a transverse Hölder distribution.

When the geodesic lamination \( \lambda \) has leaves converging towards a cusp of \( S \), one can still define a rotation angle \( \tilde{\text{rot}}_M (\alpha) \) for a transverse cocycle \( \alpha \in \mathcal{H}(\lambda) \) provided that \( \alpha \) satisfies an additional condition: for every cusp \( c \) of \( S \), the weights assigned by \( \alpha \) to the finitely many half-leaves converging to \( c \) must add up to 0.

The same construction as above, subject to the same choices, then defines a linear function
\[ \tilde{\text{rot}}_M : \mathcal{H}_0 (\lambda) \to \mathbb{R} \]
where \( \mathcal{H}_0 (\lambda) \) denotes the vector space of all transverse cocycles for \( \lambda \) which satisfy the above cusp condition.

It turns out that these rotation angles are related to the bending cocycle of a pleated surface through the Thurston intersection form, which has an algebro-topological flavor.

Intuitively, if \( \alpha \) and \( \beta \) are two transverse cocycles for the same geodesic lamination \( \lambda \), their Thurston intersection number \( \tau (\alpha, \beta) \in \mathbb{R} \) is the algebraic intersection number of \( \lambda \) with multiplicity \( \alpha \) and of \( \lambda \) with multiplicity \( \beta \).

In general, to compute an algebraic intersection number between two 1-dimensional objects \( a \) and \( b \) on the surface \( S \), one needs an orientation for \( a \) and \( b \), after perturbing \( b \) to some \( b' \) in transverse position with respect to \( a \), the algebraic intersection
of $a$ and $b$ is then defined by adding up the signs of the intersection points of $a$ and $b'$, taking into account whatever multiplicities may have been assigned to $a$ and $b$.

In our case, we do not have any orientation assigned to the geodesic lamination $\lambda$. However, we can proceed as follows. Choose a small $C^1$ perturbation from $\lambda$ to a lamination $\lambda'$, so that the leaves of $\lambda'$ are transverse to the leaves of $\lambda$. The key property is that at each $x \in \lambda \cap \lambda'$, the leaf $l'$ of $\lambda'$ passing through $x$ makes a very small angle with the leaf $l$ of $\lambda$ passing through $x$. If we choose an orientation for the leaf $l$ at $x$, this uniquely determines an orientation for $l'$ at $x$, namely the one for which their oriented tangent vectors make a small angle; this assigns a sign to the intersection of $l$ and $l'$ at $x$. If we reverse the orientation of $l$, the orientation of $l'$ also gets reversed and the sign of the intersection at $x$ is unchanged. This assigns a well-defined sign $\pm 1$ to each intersection point $x \in \lambda \cap \lambda'$. Then, if $\alpha$ and $\beta$ are two transverse cocycles for $\lambda$, the Thurston intersection number $\tau (\alpha, \beta) \in \mathbb{R}$ is defined by taking the ‘sum’ of the signs of the intersection points of $\lambda$ with multiplicity $\alpha$ and $\lambda'$ with multiplicity $\beta$ since $\lambda \cap \lambda'$ may be uncountable, one needs to be a little careful when computing this sum, by grouping these intersection points in finitely many boxes in $S$. The correct framework to formalize this, and to make sure that the output does not depend on the choice of the perturbation of $\lambda$ to $\lambda'$, is to lift the situation to the orientation covering of the geodesic lamination $\lambda$; see [PeH, §3.2] or [Bo3, §3], for instance.

Actually, when $\lambda$ has leaves converging towards a cusp of $S$, one needs an additional condition for $\alpha$ and $\beta$ for $\tau (\alpha, \beta)$ to be well-defined: They need to satisfy the *cusp condition* that, for every cusp $c$ of $S$, the weights assigned by $\alpha$ (resp. $\beta$) to the finitely many half-leaves converging to $c$ add up to 0.

If $\mathcal{H}_0 (\lambda)$ denotes the vector space of all transverse cocycles for $\lambda$ which satisfy the above cusp condition, this defines the *Thurston intersection form*

$$\tau : \mathcal{H}_0 (\lambda) \times \mathcal{H}_0 (\lambda) \to \mathbb{R}.$$ 

This is an antisymmetric bilinear form. An easy exercise shows that $\tau$ is non-degenerate if and only if every end of $S - \lambda$ that is not a whole cusp of $S$ has an odd number of spikes. In particular, $\tau$ is non-degenerate when $\lambda$ is maximal.
Theorem 30. [Bo3] Let $f : S \to M$ be a pleated surface with bending locus $\lambda$ and with bending transverse cocycle $\beta \in \mathcal{H}_\lambda (\lambda; \mathbb{R}/2\pi \mathbb{Z})$. Then, for every choice of a rotation angle function
\[ \tilde{\text{rot}}_M : \mathcal{H}_\lambda (\lambda) \to \mathbb{R} \]
there is a lift of $\beta$ to an $\mathbb{R}$-valued transverse cocycle $\tilde{\beta} \in \mathcal{H}_\lambda (\lambda)$ such that
\[ \tilde{\text{rot}}_M (\alpha) = \tau (\alpha, \beta) \]
for every $\alpha \in \mathcal{H}_\lambda (\lambda)$.

Conversely, choosing a lift $\tilde{\beta} \in \mathcal{H}_\lambda (\lambda)$ for the bending transverse cocycle $\beta \in \mathcal{H}_\lambda (\lambda; \mathbb{R}/2\pi \mathbb{Z})$ defines a rotation angle function $\tilde{\text{rot}}_M : \mathcal{H}_\lambda (\lambda) \to \mathbb{R}$ by the property that $\tilde{\text{rot}}_M (\alpha) = \tau (\alpha, \beta)$. Making such a choice at the pleated surface realizing a measured geodesic lamination $\alpha \in \mathcal{O}_f$ in general determines in a unique way a function $\tilde{\text{rot}}_M : \mathcal{O}_f \to \mathbb{R}$ as in Proposition 28.

References


[Bo7] F. Bonahon, Closed curves on surfaces, manuscript in progress.


GEODESIC LAMINATIONS ON SURFACES


Index

bending locus .......................... 27  measure ................................... 12
bending measured lamination .... 27 measured geodesic lamination .... 19
bending transverse cocycle .... 29 piecewise linear manifold .... 19
Birman-Series Theorem ............. 10 piecewise linear map .......... 19
complete geodesic ...................... 3 piecewise linear submanifold .... 19
convex core ............................. 25 pleated surface ................. 27
convex hull .............................. 26 pleating locus ................. 27
convex subset ........................... 25 Radon measure ................. 12
cusp condition ........................... 35 realizable geodesic lamination .... 33
cusp-preserving map ................... 28 regular leaf .................... 5
cusp-preserving representation ... 30 rotation angle ............... 33
distribution .............................. 13 simple geodesic ................ 3
essential curve ......................... 18 sub-lamination ............... 7
Gap Formula ............................. 16 support of a (Hölder) distribution .... 17
generic arc ............................... 19 Support Lemma ............... 17
goedesic lamination .................... 3 suspension ................ 5
Hölder continuous map ................ 13 tangent map ................. 31
Hölder distribution ..................... 14 tangent vector to a piecewise
Hölder exponent ....................... 13 linear manifold .............. 21
hyperbolic metric ...................... 3 Schläfli formula ............... 33
interval exchange map ............... 5 tangential manifold ........... 31
isolated leaf .............................. 8 tangential map ............... 31
leaf of a geodesic lamination ...... 3 Thurston intersection form .... 35
length function ......................... 21 transverse cocycle .......... 15
limit set ................................. 26 transverse Hölder distribution .... 14
Lipschitz continuous map .......... 13 transverse measure .......... 11
maximal geodesic lamination ....... 30 Zeghib Theorem ............... 8

Department of Mathematics, University of Southern California, Los Angeles CA 90089-1113, U.S.A.
E-mail address: fbonahon@math.usc.edu