INTERACTION PURSUIT IN HIGH-DIMENSIONAL MULTI-RESPONSE REGRESSION VIA DISTANCE CORRELATION*

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Feature interactions can contribute to a large proportion of variation in many prediction models. In the era of big data, the coexistence of high dimensionality in both responses and covariates poses unprecedented challenges in identifying important interactions. In this paper, we suggest a two-stage interaction identification method, called the interaction pursuit via distance correlation (IPDC), in the setting of high-dimensional multi-response interaction models that exploits feature screening applied to transformed variables with distance correlation followed by feature selection. Such a procedure is computationally efficient, generally applicable beyond the heredity assumption, and effective even when the number of responses diverges with the sample size. Under mild regularity conditions, we show that this method enjoys nice theoretical properties including the sure screening property, support union recovery, and oracle inequalities in prediction and estimation for both interactions and main effects. The advantages of our method are supported by several simulation studies and real data analysis.

1. Introduction. Recent years have seen a surge of interests on interaction identification in the high-dimensional setting by many researchers. For instance, Hall and Xue [15] proposed a recursive approach to identify important interactions among covariates, where all $p$ covariates are first ranked by the generalized correlation and then only the top $p^{1/2}$ ones are retained to construct pairwise interactions of order $O(p)$ for further screening and selection of both interactions and main effects. A forward selection

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based screening procedure was introduced in [16] for identifying interactions in a greedy fashion under the heredity assumption. Such an assumption in the strong sense requires that an interaction between two covariates should be included in the model only if both main effects are important, while the weak version relaxes such a constraint to the presence of at least one main effect being important. Regularization methods have also been used for interaction selection under the heredity assumption. See, for example, [34], [8], and [2]. Under the inverse modeling framework, [19] proposed a new method, called the sliced inverse regression for interaction detection (SIRI), which can detect pairwise interactions among covariates without the heredity assumption. The theoretical development in [19] relies primarily on the joint normality assumption on the covariates. The innovated interaction screening procedure was introduced in [13] for high-dimensional nonlinear classification with no heredity assumption.

Although the aforementioned methods can perform well in many scenarios, they may have two potential limitations. First, those approaches assume mainly interaction models with a single response, while the coexistence of multiple responses becomes increasingly common in the big data era. Second, those developments are usually built upon the strong or weak heredity assumption, or the normality assumption, which may not be satisfied in certain real applications.

To enable broader applications in practice, in this paper we consider the following high-dimensional multi-response interaction model

\[ \mathbf{y} = \alpha + \mathbf{B}_x^T \mathbf{x} + \mathbf{B}_z^T \mathbf{z} + \mathbf{w}, \]

where \( \mathbf{y} = (Y_1, \cdots, Y_q)^T \) is a \( q \)-dimensional vector of responses, \( \mathbf{x} = (X_1, \cdots, X_p)^T \) is a \( p \)-dimensional vector of covariates, \( \mathbf{z} \) is a \( p(p-1)/2 \)-dimensional vector of all pairwise interactions between covariates \( X_j \)'s, \( \alpha = (\alpha_1, \cdots, \alpha_q)^T \) is a \( q \)-dimensional vector of intercepts, \( \mathbf{B}_x \in \mathbb{R}^{p \times q} \) and \( \mathbf{B}_z \in \mathbb{R}^{\lfloor p(p-1)/2 \rfloor \times q} \) are regression coefficient matrices for the main effects and interactions, respectively, and \( \mathbf{w} = (W_1, \cdots, W_q)^T \) is a \( q \)-dimensional vector of random errors with mean zero and being independent of \( \mathbf{x} \). Each response in this model is allowed to have its own regression coefficients, and to simplify the presentation, the covariate vector \( \mathbf{x} \) is assumed to be centered with mean zero. Commonly encountered is the setting of high dimensionality in both responses and covariates, where the numbers of responses and covariates, \( q \) and \( p \), can diverge with the sample size. It is of practical importance to consider sparse models in which the rows of the coefficient matrices \( \mathbf{B}_x \) and \( \mathbf{B}_z \) are sparse with only a fraction of nonzeros. We aim at identifying the important interactions and main effects, which have nonzero regression
coefficients, that contribute to the responses.

Interaction identification in the multi-response interaction model (1) with large \( p \) and \( q \) is intrinsically challenging. The difficulties include the high dimensionality in responses, the high computational cost caused by the existence of a large number of interactions among covariates, and the technical challenges associated with the complex model structure. The idea of variable screening can speed up the computation. Yet, under model setting (1) most existing variable screening methods based on the marginal correlation may no longer work. To appreciate this, let us consider a specific case of model (1) with only one response

\[
Y = \alpha + \sum_{j=1}^{p} \beta_j X_j + \sum_{k=1}^{p-1} \sum_{\ell=k+1}^{p} \gamma_{k\ell} X_k X_\ell + W,
\]

where all the notation is the same as therein with \( B_x = (\beta_j)_{1 \leq j \leq p} \) and \( B_z = (\gamma_{k\ell})_{1 \leq k \leq p-1, k+1 \leq \ell \leq p} \). For simplicity, assume that the covariates \( X_1, \ldots, X_p \) are independent of each other. Then under the above model setting (2), it is easy to see that

\[
E(Y|X_j) = \alpha + \beta_j X_j.
\]

This representation shows that if some covariate \( X_j \) is an unimportant main effect with \( \beta_j = 0 \), then the conditional mean of \( Y \) given \( X_j \) is free of \( X_j \), regardless of whether \( X_j \) contributes to interactions or not. When such a covariate \( X_j \) indeed appears in an important interaction, variable screening methods based on the marginal correlations of \( Y \) and \( X_k \)'s are not capable of detecting \( X_j \) if the heredity assumption fails to hold. As a consequence, there is an important need for new proposals of interaction screening. When the covariates are correlated, the conditional mean (3) may depend on \( X_j \) indirectly through correlations with other covariates when \( \beta_j = 0 \). Such a relationship can, however, be still weak if the correlations between \( X_j \) and other covariates are weak.

To address the aforementioned challenges, we suggest a new two-stage approach to interaction identification, named the interaction pursuit via distance correlation (IPDC), exploiting the idea of interaction screening and selection. In the screening step, we first transform the responses and covariates and then perform variable screening based on the transformed responses and covariates. Such a transformation enhances the dependence of responses on covariates that contribute to important interactions or main effects. The novelty of our interaction screening method is that it aims at recovering variables that contribute to important interactions instead of finding these interactions directly, which reduces the computational cost substantially from a
factor of \( O(p^2) \) to \( O(p) \). To take advantage of the correlation structure among multiple responses, we build our marginal utility function using the distance correlation proposed in [31]. After the screening step, we conduct interaction selection by constructing pairwise interactions with the retained variables from the first step, and applying the group regularization method to further select important interactions and main effects for the multi-response model in the reduced feature space.

The main contributions of this paper are twofold. First, the suggested IPDC method provides a computationally efficient approach to interaction screening and selection in ultra-high dimensional interaction models. Such a procedure accommodates the model setting with a diverging number of responses, and is generally applicable without the requirement of the heredity assumption. Second, our procedure is theoretically justified to be capable of retaining all covariates that contribute to important interactions or main effects with asymptotic probability one, the so-called sure screening property [11, 26], in the screening step. In the selection step, it is also shown to enjoy nice sampling properties for both interactions and main effects such as the support union recovery and oracle inequalities in prediction and estimation. In particular, there are two key messages that are delivered in this paper: a separate screening step for interactions can significantly enhance the screening performance if one aims at finding important interactions, and screening interaction variables can be more effective and efficient than screening interactions directly due to the noise accumulation. The former message is elaborated more with a numerical example presented in Section ?? of the Supplementary Material.

The rest of the paper is organized as follows. Section 2 introduces the new interaction screening approach and studies its theoretical properties. We illustrate the advantages of the proposed procedure using several simulation studies in Section 3 and a real data example in Section 4. Section 5 discusses some possible extensions of our method. The proofs of main results are relegated to the Appendix. Additional proofs of main results and technical details as well as additional numerical studies are provided in the Supplementary Material.


2.1. Motivation of the new method. To facilitate the presentation, we call \( X_kX_\ell \) an important interaction if the corresponding row of \( \mathbf{B}_2 \) is nonzero, and \( X_k \) an active interaction variable if there exists some \( 1 \leq \ell \neq k \leq p \) such that \( X_kX_\ell \) is an important interaction. Denote by \( \mathcal{I} \) the set of all important interactions. Similarly, \( X_j \) is referred to as an important main effect if its
associated row of \( B_X \) is nonzero. It is of crucial importance to identify both the set \( A \) of all active interaction variables and the set \( M \) of all important main effects.

Before presenting our main ideas, let us revisit the specific example (2) discussed in the Introduction. A phenomenon mentioned there is that variable screening methods using the marginal correlations between the response and covariates can fail to detect active interaction variables that have no main effects. We now consider the square transformation for the response. Some standard calculations (see Section ?? of the Supplementary Material) yield

\[
E(Y^2 | X_j) = \left[ \beta_j^2 + \sum_{k=1}^{j-1} \gamma_{kj}^2 E(X_k^2) + \sum_{\ell=j+1}^{p} \gamma_{j\ell}^2 E(X_{\ell}^2) \right] X_j^2 \\
+ 2 \left[ \beta_j \alpha + \sum_{k=1}^{j-1} \beta_k \gamma_{kj} E(X_k^2) + \sum_{\ell=j+1}^{p} \beta_{\ell} \gamma_{j\ell} E(X_{\ell}^2) \right] X_j + C_j,
\]

where \( C_j \) is some constant that is free of \( X_j \). This shows that the conditional mean \( E(Y^2 | X_j) \) is linear in \( X_j^2 \) as long as \( X_j \) is an active interaction variable, that is, \( \gamma_{kj} \) or \( \gamma_{j\ell} \neq 0 \) for some \( k \) or \( \ell \), regardless of whether it is also an important main effect or not. In fact, we can see from the above representation that the coefficient of \( X_j^2 \) reflects the cumulative contribution of covariate \( X_j \) to response \( Y \) as both an interaction variable and a main effect.

Motivated by the above example, we consider the approach of screening interaction variables via some marginal utility function for the transformed variables \( Y^2 \) and \( X_j^2 \), with the square transformation applied to both the response and covariates. Such an idea has been exploited in [12] for interaction screening in the setting of single-response interaction models. To rank the relative importance of features, they calculated the Pearson correlations between \( Y^2 \) and \( X_j^2 \). This idea is, however, no longer applicable when there are multiple responses, since the Pearson correlation is not well defined for the pair of \( q \)-vector \( y \) of responses with \( q > 1 \) and covariate \( X_j \). A naive strategy is to screen the interaction variables for each response \( Y_k \) with \( 1 \leq k \leq q \) using the approach of [12]. Such a naive procedure can suffer from several potential drawbacks. First, it may be inefficient and can result in undesirable results since the correlation structure among the responses \( Y_1, \cdots, Y_q \) is completely ignored. Second, when \( q \) is large it may retain too many interaction variables in total, which can in turn cause difficulty in model interpretation and high computational cost when further selecting active interaction variables.

To address the afore-discussed issues, we propose to construct the marginal utility function exploiting the distance correlation introduced in [31]. More
specifically, we identify the set of all active interaction variables $\mathcal{A}$ by ranking the distance correlations between the squared covariates $X^2_j$ and the squared response vector $y \circ y$, where $\circ$ denotes the Hadamard (component-wise) product of two vectors. The distance correlation
\[
dcorr(u, v) = \frac{d\text{cov}(u, v)}{\sqrt{d\text{cov}(u, u)d\text{cov}(v, v)}}
\]
is well defined for any two random vectors $u \in \mathbb{R}^{d_u}$ and $v \in \mathbb{R}^{d_v}$ of arbitrary mixed dimensions, where the distance covariance between $u$ and $v$ is given by
\[
d\text{cov}^2(u, v) = \frac{1}{c_{d_u}c_{d_v}} \int_{\mathbb{R}^{d_u+d_v}} \frac{|\varphi_{u,v}(s, t) - \varphi_u(s)\varphi_v(t)|^2}{\|s\|^{d_u+1}\|t\|^{d_v+1}} dsdt.
\]
Here $c_m = \pi^{(m+1)/2}/\Gamma((m+1)/2)$ is the half area of the unit sphere $S^m \subset \mathbb{R}^{m+1}$, $\varphi_{u,v}(s, t)$, $\varphi_u(s)$, and $\varphi_v(t)$ are the characteristic functions of $(u, v)$, $u$, and $v$, respectively, and $\|\cdot\|$ denotes the Euclidean norm. Compared to the Pearson correlation, it also has the advantage that the distance correlation of two random vectors is zero if and only if they are independent. Moreover, the distance correlation of two univariate Gaussian random variables is a strictly increasing function of the absolute value of the Pearson correlation between them. See [31] for more properties and discussions of the distance correlation, and [18] for a fast algorithm for computing the distance correlation.

It is worth mentioning that [23] introduced a model-free feature screening procedure based on the distance correlations of the original response and covariates. Their method is applicable to the cases of multiple responses and grouped covariates. Yet we found that the use of distance correlations for the transformed response vector and covariates, $y \circ y$ and $X^2_j$, can result in improved performance in interaction variable screening. The specific example considered in Section 2.1 provides some intuitive explanation of this phenomenon. To further illustrate this point, we generated 200 data sets from the following simple interaction model
\[
Y = X_1X_2 + W,
\]
where the covariate vector $x = (X_1, \cdots, X_p)^T \sim N(0, \Sigma)$ with $p = 1000$ and $\Sigma = (\rho^{j-k})_{1 \leq j,k \leq p}$, $\rho$ ranging in $(-1, 1)$ measures the correlation level among covariates, and the random error $W \sim N(0, 1)$ is independent of $x$. As shown in Figure 1, the sample distance correlation between $X^2_1$ and $Y^2$ is much larger than that between $X_1$ and $Y$. For covariates having weak correlation with active interaction variables $X_1$ and $X_2$, such as $X_{10}$ and $X_{1000}$, the square transformation does not increase their distance correlations with
2.2. Interaction screening. Suppose we have a sample \((y_i, x_i)_{i=1}^n\) of \(n\) independent and identically distributed (i.i.d.) observations from \((y, x)\) in the multi-response interaction model (1). For each \(1 \leq k \leq p\), denote by \(\text{dcorr}(X_k^2, y \circ y)\) the distance correlation between the squared covariate \(X_k^2\) and squared response vector \(y \circ y\). The idea of the screening step of our IPDC procedure is to rank the importance of the interaction variables \(X_k\) using the sample version of distance correlations \(\text{dcorr}(X_k^2, y \circ y)\). Similarly, we conduct screening of main effects based on the sample version of distance correlations \(\text{dcorr}(X_j, y)\) between covariates \(X_j\) and response vector \(y\).

For notational simplicity, we write \(X_k^* = X_k^2, \tilde{y} = y / \sqrt{q}\), and \(y^* = \tilde{y} \circ \tilde{y} = \)
\( y \circ y / q \). Define two population quantities
\[
\omega^*_k = \frac{\text{dcov}^2(X^*_k, y^*)}{\sqrt{\text{dcov}^2(X^*_k, X^*_k)}} \quad \text{and} \quad \omega_j = \frac{\text{dcov}^2(X_j, \tilde{y})}{\sqrt{\text{dcov}^2(X_j, X_j)}}
\]
with \( 1 \leq k, j \leq p \) for interaction variables and main effects, respectively. Denote by \( \hat{\omega}^*_k \) and \( \hat{\omega}_j \) the empirical versions of \( \omega^*_k \) and \( \omega_j \), respectively, constructed by plugging in the corresponding sample distance covariances based on the sample \((y_i, x_i)_{i=1}^n\). According to [31], the sample distance covariance between any two random vectors \( u \) and \( v \) based on a sample \((u_i, v_i)_{i=1}^n\) is given by
\[
\hat{\text{dcov}}^2(u, v) = \hat{S}_1 + \hat{S}_2 - 2\hat{S}_3,
\]
where the three quantities are defined as \( \hat{S}_1 = n^{-2} \sum_{i,j=1}^n \|u_i - u_j\| \|v_i - v_j\| \), \( \hat{S}_2 = \lceil n^{-2} \sum_{i,j=1}^n \|u_i - u_j\| \rceil \lceil n^{-2} \sum_{i,j=1}^n \|v_i - v_j\| \rceil \), and \( \hat{S}_3 = n^{-3} \sum_{i,j,k=1}^n \|u_i - u_k\| \|v_j - v_k\| \). In view of
\[
d\text{corr}^2(X^*_k, y \circ y) = d\text{corr}^2(X^*_k, y^*) = \omega^*_k / \{\text{dcov}^2(y^*, y^*)\}^{1/2}
\]
and
\[
d\text{corr}^2(X_j, y) = d\text{corr}^2(X_j, \tilde{y}) = \omega_j / \{\text{dcov}^2(\tilde{y}, \tilde{y})\}^{1/2},
\]
the procedure of screening the interaction variables and main effects via distance correlations \( d\text{corr}(X^*_k, y \circ y) \) and \( d\text{corr}(X_j, y) \) suggested above is equivalent to that of thresholding the quantities \( \omega^*_k \)'s and \( \omega_j \)'s, respectively.

More specifically, in the screening step of IPDC we estimate the sets of important main effects \( \mathcal{M} \) and active interaction variables \( \mathcal{A} \) as
\[
\hat{\mathcal{M}} = \{1 \leq j \leq p : \hat{\omega}_j \geq \tau_1\} \quad \text{and} \quad \hat{\mathcal{A}} = \{1 \leq k \leq p : \hat{\omega}^*_k \geq \tau_2\},
\]
where \( \tau_1 \) and \( \tau_2 \) are some positive thresholds. With the set \( \hat{\mathcal{A}} \) of retained interaction variables, we construct a set of pairwise interactions
\[
\hat{\mathcal{I}} = \{(k, l) : 1 \leq k < l \leq p \text{ and } k, l \in \hat{\mathcal{A}}\}.
\]
This gives a new interaction screening procedure. It is worth mentioning that the set of constructed interactions \( \hat{\mathcal{I}} \) tends to overestimate the set of all important interactions \( \mathcal{I} \) since the goal of the first step of IPDC is screening interaction variables. Such an issue can be addressed in the selection step of IPDC investigated in Section ??.
2.3. Sure screening property. We now study the sampling properties of the newly proposed interaction screening procedure. Some mild regularity conditions are needed for our analysis.

**Condition 1.** Both $\text{dcov}(X_k, X_k)$ and $\text{dcov}(X_k^2, X_k^2)$ are bounded away from zero uniformly in $k$.

**Condition 2.** There exists some constant $c_0 > 0$ such that $E\{\exp(c_0 X_k^2)\}$ and $E\{\exp(c_0 \|y\|/\sqrt{q})\}$ are uniformly bounded.

**Condition 3.** There exist some constants $c_1, c_2 > 0$ and $0 \leq \kappa_1, \kappa_2 < 1/2$ such that $\min_{j \in M} \omega_j \geq 3c_1 n^{-\kappa_1}$ and $\min_{k \in A} \omega_k \geq 3c_2 n^{-\kappa_2}$.

Condition 1 is a basic assumption requiring that the distance variances of covariates $X_k$ and squared covariates $X_k^2$ are at least of a constant order. Conditions 2 and 3 are analogous to the regularity conditions in [23]. In particular, Condition 2 controls the tail behavior of the covariates and responses, which facilitates the derivation of deviation probability bounds. Condition 3 also shares the same spirit as Condition 3 in [11], and can be understood as an assumption on the minimum signal strength in the feature screening setting. Smaller constants $\kappa_1$ and $\kappa_2$ correspond to stronger marginal signal strength for active interaction variables and important main effects, respectively. With these regularity conditions, we establish the sure screening property of IPDC in the following theorem.

**Theorem 1.** Under Conditions 1–2 with $\log p = o(n^{\eta_0})$ for $\eta_0 = \min\{(1 - 2\kappa_1)/3, (1 - 2\kappa_2)/5\}$, there exists some positive constant $C$ such that

\[
P\left(\max_{1 \leq j \leq p} |\hat{\omega}_j - \omega_j| \geq c_1 n^{-\kappa_1}\right) \leq O\left(\exp\left\{ - C n^{(1 - 2\kappa_1)/6} \right\}\right),
\]

\[
P\left(\max_{1 \leq k \leq p} |\hat{\omega}_k^* - \omega_k^*| \geq c_2 n^{-\kappa_2}\right) \leq O\left(\exp\left\{ - C n^{(1 - 2\kappa_2)/10} \right\}\right).
\]

Assume in addition that Condition 3 holds and set $\tau_1 = 2c_1 n^{-\kappa_1}$ and $\tau_2 = 2c_2 n^{-\kappa_2}$. Then we have

\[
P\left(\mathcal{M} \subset \hat{\mathcal{M}} \text{ and } \mathcal{I} \subset \hat{\mathcal{I}}\right) = 1 - O\left(\exp\left\{ - C n^{\eta_0/2} \right\}\right).
\]

Theorem 1 reveals that the IPDC enjoys the sure screening property that all active interaction variables and all important main effects can be retained in the reduced model with high probability. In particular, we see that it can handle ultra-high dimensionality with log $p = o(n^{\eta_0})$. A comparison of the
deviation probability bounds in (8) and (9) shows that interaction screening is generally more challenging and thus needs more restrictive constraint on dimensionality $p$ than main effect screening; see the probability bound (15) and its main effect counterpart for more details. It is also seen that when the marginal signal strength for interactions and main effects becomes stronger, the sure screening property of IPDC holds for higher dimensionality $p$.

For any feature screening procedure, it is of practical importance to control the dimensionality of the reduced feature space, since feature selection usually follows the screening for further selection of important features in such a space. We next investigate such an aspect for IPDC. Let $s_1$ and $s_2$ be the cardinalities of sets of all important main effects $\mathcal{M}$ and all active interaction variables $\mathcal{A}$, respectively. With the thresholds $\tau_1 = 2c_1n^{-\kappa_1}$ and $\tau_2 = 2c_2n^{-\kappa_2}$ specified in Theorem 1, we introduce two sets of unimportant main effects and inactive interaction variables

\begin{equation}
\mathcal{M}_1 = \{ j \in \mathcal{M}^c : \omega_j \geq c_1 n^{-\kappa_1} \} \quad \text{and} \quad \mathcal{A}_1 = \{ k \in \mathcal{A}^c : \omega_k^* \geq c_2 n^{-\kappa_2} \}
\end{equation}

that are of significant marginal effects. Denote by $s_3$ and $s_4$ the cardinalities of these two sets $\mathcal{M}_1$ and $\mathcal{A}_1$, respectively. Larger values of $s_3$ and $s_4$ indicate more difficulty in the problem of interaction and main effect screening in the high-dimensional multi-response interaction model (1).

**Theorem 2.** Assume that all the conditions of Theorem 1 hold and set $\tau_1 = 2c_1n^{-\kappa_1}$ and $\tau_2 = 2c_2n^{-\kappa_2}$. Then we have

\begin{equation}
P\left( |\hat{\mathcal{M}}| \leq s_1 + s_3 \quad \text{and} \quad |\hat{\mathcal{I}}| \leq (s_2 + s_4)(s_2 + s_4 - 1)/2 \right) = 1 - O\left\{ \exp\left( -Cn^{\eta_0/2} \right) \right\}
\end{equation}

for some positive constant $C$.

Theorem 2 quantifies how the size of the reduced model for interactions and main effects is related to the thresholding parameters $\tau_1$ and $\tau_2$, and the cardinalities of the two sets $\mathcal{M}_1$ and $\mathcal{A}_1$. In particular, we see that when $s_i = O(n^{\delta_i})$ with some constants $\delta_i \geq 0$ for $1 \leq i \leq 4$, the total number of retained interactions and main effects in the reduced feature space can be controlled as $O(n^\delta)$ with $\delta = \max\{\delta_1 \lor \delta_3, 2(\delta_2 \lor \delta_4)\}$, where $\lor$ denotes the maximum of two values. In contrast, the dimensionality $p$ is allowed to grow nonpolynomially with sample size $n$ in the rate of $\log p = o(n^{\eta_0})$ with $\eta_0 = \min\{(1 - 2\kappa_1)/3, (1 - 2\kappa_2)/5\}$. The reduced model size can fall below the sample size and be a smaller order of $n$ when both $\max\{\delta_1, \delta_3\} < 1$ and $\max\{\delta_2, \delta_4\} < 1/2$ are satisfied. The post-screening interaction selection
and its sampling properties are further investigated in Section ?? of the Supplementary Material.

3. Simulation studies. We illustrate the finite-sample performance of our method using several simulation examples. Two sets of models are considered for the single-response case and the multi-response case, respectively. This section evaluates the screening performance, while the post-screening selection performance is investigated in Section ?? of the Supplementary Material.

3.1. Screening in single-response models. We begin with the following four high-dimensional single-response interaction models:

Model 1: $Y = 2X_1 + 2X_2 + X_1X_2 + W$,
Model 2: $Y = 2X_1 + 3X_1X_2 + 3X_1X_3 + W$,
Model 3: $Y = 3X_1X_2 + 3X_1X_3 + W$,
Model 4: $Y = 3I(X_{12} \geq 0) + 2X_{22} + 3X_1X_2 + W$,

where all the notation is the same as in (1) and $I(\cdot)$ denotes the indicator function. The covariate vector $x = (X_1, \ldots, X_p)^T$ is sampled from the distribution $N(0, \Sigma)$ with covariance matrix $\Sigma = (\rho^{j-k})_{1 \leq j,k \leq p}$ for $\rho \in (-1, 1)$, and the error term $W \sim N(0, 1)$ is generated independently of $x$ to form an i.i.d. sample of size $n = 200$. For each of the four models, we further consider three different settings with $(p, \rho) = (2000, 0.5), (5000, 0.5), \text{ and } (2000, 0.1)$, respectively. In particular, Models 2 and 3 are adapted from simulation scenarios 2.2 and 2.3 in Jiang and Liu [19], whereas Model 4 is adapted from simulation example 2.b of Li et al. [23] and accounts for model misspecification since without any prior information, our working model treats $X_{12}$ as a linear predictor instead of $I(X_{12} \geq 0)$. We see that Model 1 satisfies the strong heredity assumption and Model 2 obeys the weak heredity assumption, while Models 3 and 4 violate the heredity assumption since none of the active interaction variables are important main effects.

We compare the interaction and main effect screening performance of the IPDC with the SIS [11], DCSIS [23], SIRI [19], IP [12], and iFORT and iFORM [16]. Like IPDC, SIRI and IP were developed for screening interaction variables and main effects separately. In particular, SIRI is an iterative procedure, while all others are non-iterative ones. For a fair comparison, we adopt the initial screening step described in Section 2.3 of Jiang and Liu [19] to implement SIRI in a non-iterative fashion, and keep the top ranked covariates. Since the SIS is originally designed only for main effect screening and the original DCSIS screens variables without the distinction between
main effects and interaction variables, we construct pairwise interactions based on the covariates recruited by SIS and DCSIS, and refer to the resulting procedures as SIS2 and DCSIS2, respectively, to distinguish them from the original ones. It is worth mentioning that the SIS2 shares a similar spirit to the TS-SIS procedure proposed in [22], where the difference is that the latter constructs pairwise interactions between the main effects retained by SIS and all \( p \) covariates. Following the suggestions of Fan and Lv [11] and Li et al. [23], we keep the top \( \lfloor n/(\log n) \rfloor \) variables after ranking for each screening procedure. We examine the screening performance by the proportions of important main effects, important interactions, and all of them being retained by each screening procedure over 100 replications.

Table 1 reports the screening results of different methods. In Model 1, all screening methods are able to retain almost all important main effects and interactions across all three settings. The IPDC outperforms SIS2, DCSIS2, SIRI, iFORT, and iFORM in Models 2–4 over all three settings. It is seen that SIS2 can barely identify important interactions for those three models. The advantage of IPDC over SIS2, DCSIS2, and SIRI is most pronounced when the heredity assumption is violated as in Models 3 and 4. We also

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<th>Model 1</th>
<th>Model 2</th>
<th>Model 3</th>
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<td>iFORT</td>
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<tr>
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</tr>
<tr>
<td>IP</td>
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<tr>
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<table>
<thead>
<tr>
<th>Setting 1: ((p, \rho) = (2000, 0.5))</th>
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<tbody>
<tr>
<td>SIS2</td>
</tr>
<tr>
<td>iFORT</td>
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<tr>
<td>iFORM</td>
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<tr>
<td>DCSIS2</td>
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<tr>
<td>SIRI</td>
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<tr>
<td>IP</td>
</tr>
<tr>
<td>IPDC</td>
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<table>
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<tr>
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<td>iFORT</td>
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<td>DCSIS2</td>
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<tr>
<td>SIRI</td>
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<td>IPDC</td>
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</table>

<table>
<thead>
<tr>
<th>Setting 3: ((p, \rho) = (2000, 0.1))</th>
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<tr>
<td>iFORT</td>
</tr>
<tr>
<td>iFORM</td>
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<tr>
<td>DCSIS2</td>
</tr>
<tr>
<td>SIRI</td>
</tr>
<tr>
<td>IP</td>
</tr>
<tr>
<td>IPDC</td>
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Table 1 Proportions of important main effects, important interactions, and all of them retained by different screening methods. For SIS2, DCSIS2, and SIRI, interactions are constructed using the top \( \lfloor n/(\log n) \rfloor \) covariates ranked by their marginal utilities with the response.
observe significant improvement of IPDC over IP in many of those model settings. When the dimensionality increases from 2000 to 5000 (settings 1 and 2), the problem of interaction and main effect screening becomes more challenging as indicated by the drop of the screening probabilities. Compared to others, IPDC consistently performs well.

It is interesting to observe that in view of settings 1 and 3, the interaction screening performance can be improved in the presence of a higher level of correlation among covariates. One possible explanation is that high correlation among covariates may increase the dependence of the response on the interaction variables and thus benefit interaction screening. For instance, in Model 2 due to the correlation between the interaction variable $X_2$ (or $X_3$) and main effect $X_1$, the response $Y$ depends on $X_2$ (or $X_3$) not only directly through the interaction $X_1X_2$ (or $X_1X_3$) but also indirectly through the main effect $X_1$. Therefore, in this case high correlation among covariates can boost the performance of interaction screening. Similar phenomenon has been documented for DCSIS in the literature; see, for example, Models 1.b and 1.c in Table 2 of Li et al. [23].

3.2. Screening in multi-response model. We next consider the setting of interaction model with multiple responses and specifically Model 5 with $q = 10$ responses:

$$Y_1 = \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_1 X_2 + W_1, \quad Y_2 = \beta_4 X_1 + \beta_5 X_2 + \beta_6 X_1 X_3 + W_2,$$

$$Y_3 = \beta_7 X_1 + \beta_8 X_2 + \beta_9 X_6 X_7 + W_3, \quad Y_4 = \beta_{10} X_1 + \beta_{11} X_2 + \beta_{12} X_8 X_9 + W_4,$$

$$Y_5 = \beta_{13} X_6 X_7 + \beta_{14} X_8 X_9 + W_5, \quad Y_6 = \beta_{15} X_1 + \beta_{16} X_2 + \beta_{17} X_1 X_2 + W_6,$$

$$Y_7 = \beta_{18} X_1 + \beta_{19} X_2 + \beta_{20} X_1 X_3 + W_7, \quad Y_8 = \beta_{21} X_1 + \beta_{22} X_2 + \beta_{23} X_6 X_7 + W_8,$$

$$Y_9 = \beta_{24} X_1 + \beta_{25} X_2 + \beta_{26} X_8 X_9 + W_9, \quad Y_{10} = \beta_{27} X_6 X_7 + \beta_{28} X_8 X_9 + W_{10},$$

where all the notation and setup are the same as in Section 3.1, the covariate vector $x = (X_1, \ldots, X_p)^T$ is sampled from distribution $N(0, \Sigma)$ with covariance matrix $\Sigma = (0.5^{(|j-k|)})_{1 \leq j,k \leq p}$, and the error vector $w = (W_1, \ldots, W_q)^T \sim N(0, I_q)$ is independent of $x$. The nonzero regression coefficients $\beta_k$ with $1 \leq k \leq 28$ for all important main effects and interactions are generated independently as $\beta_k = (-1)^U \text{Uniform}(1, 2)$, where $U$ is a Bernoulli random variable with success probability 0.5 and $\text{Uniform}(1, 2)$ is the uniform distribution on $[1, 2]$. For simplicity, we consider only the setting of $(n, p, \rho) = (100, 1000, 0.5)$. In Model 5, covariates $X_1$ and $X_2$ are both active interaction variables and important main effects, whereas covariates $X_3$ and $X_j$ with $6 \leq j \leq 9$ are active interaction variables only.

To simplify the presentation, we examine only the proportions of active interaction variables and important main effects retained by different screening
procedures. A direct application of SIS to each response $Y_k$ with $1 \leq k \leq q$ results in $q$ marginal correlations for each covariate $X_j$ with $1 \leq j \leq p$. We thus consider two modifications of SIS to deal with multi-response data. Specifically, we exploit two new marginal measures, $\max_{1 \leq k \leq q} |\hat{\text{corr}}(Y_k, X_j)|$ and $\sum_{k=1}^{q} |\hat{\text{corr}}(Y_k, X_j)|$, to quantify the importance of covariates $X_j$, where $\hat{\text{corr}}$ denotes the sample correlation. We refer to these two methods as SIS.max and SIS.sum, respectively. The SIRI and IP are not included for comparison in this model since both methods were not designed for multi-response models, while the DCSIS is still applicable since the distance correlation is well defined in such a scenario.

Since feature screening is more challenging in multi-response models, we implement IPDC in a slightly different fashion than in single-response models. Recall that in Section 3.1, IPDC screens interaction variables and main effects separately, and keeps the top $\lfloor n/(\log n) \rfloor$ of each type of variables. For Model 5, we take a union of these two sets of variables and regard an active interaction variable or important main effect as being retained if such a variable belongs to the union, which can contain up to $2\lfloor n/(\log n) \rfloor$ variables. Consequently we construct pairwise interactions of all variables in the union. To ensure fair comparison, we keep the top $2\lfloor n/(\log n) \rfloor$ variables for the other screening methods SIS.max, SIS.sum, and DCSIS.

Table 2 summarizes the screening results under Model 5. We see that all methods perform well in recovering variables $X_1$, $X_2$, and $X_3$. Yet only IPDC is able to retain active interaction variables $X_6, \cdots, X_9$ with large probability.

### Table 2

<table>
<thead>
<tr>
<th>Method</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X_6$</th>
<th>$X_7$</th>
<th>$X_8$</th>
<th>$X_9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SIS.max</td>
<td>1.00 (0.00)</td>
<td>1.00 (0.00)</td>
<td>0.98 (0.01)</td>
<td>0.12 (0.03)</td>
<td>0.18 (0.04)</td>
<td>0.13 (0.03)</td>
<td>0.08 (0.03)</td>
</tr>
<tr>
<td>SIS.sum</td>
<td>1.00 (0.00)</td>
<td>1.00 (0.00)</td>
<td>0.99 (0.01)</td>
<td>0.17 (0.04)</td>
<td>0.17 (0.04)</td>
<td>0.17 (0.04)</td>
<td>0.17 (0.04)</td>
</tr>
<tr>
<td>DCSIS</td>
<td>1.00 (0.00)</td>
<td>1.00 (0.00)</td>
<td>1.00 (0.00)</td>
<td>0.61 (0.05)</td>
<td>0.57 (0.05)</td>
<td>0.72 (0.05)</td>
<td>0.68 (0.05)</td>
</tr>
<tr>
<td>IPDC</td>
<td>1.00 (0.00)</td>
<td>1.00 (0.00)</td>
<td>0.99 (0.01)</td>
<td>0.91 (0.03)</td>
<td>0.90 (0.03)</td>
<td>0.95 (0.02)</td>
<td>0.90 (0.03)</td>
</tr>
</tbody>
</table>

3.3. **Screening in multi-response model with discrete covariates.** We now turn to the scenario of multi-response interaction model with mixed covariate types and specifically Model 6 with $q = 50$ responses and $(n, p) =$
\begin{align*}
Y_1 &= \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4 + \beta_5 X_1 X_2 + \beta_6 X_3 X_4 + W_1, \\
Y_2 &= \beta_7 X_1 + \beta_8 X_2 + \beta_9 X_3 + \beta_{10} X_4 + \beta_{11} X_1 X_3 + \beta_{12} X_4 X_5 + W_2, \\
Y_3 &= \beta_{13} X_1 + \beta_{14} X_2 + \beta_{15} X_3 + \beta_{16} X_4 + \beta_{17} X_4 X_5 + \beta_{18} X_9 X_{13} + W_3, \\
Y_4 &= \beta_{19} X_1 + \beta_{20} X_2 + \beta_{21} X_3 + \beta_{22} X_4 + \beta_{23} X_9 X_{12} + \beta_{24} X_{12} X_{13} + W_4, \\
Y_5 &= \beta_{25} X_9 X_{12} + \beta_{26} X_9 X_{13} + \beta_{27} X_{12} X_{13} + W_5,
\end{align*}

and the remaining nine groups of five responses are defined in a similar way to how \(Y_6, \cdots, Y_{10}\) were defined in Model 5 in Section 3.2, that is, repeating the support of each response but with regression coefficients \(\beta_k\) generated independently from the same distribution as in Model 5. There are several key differences with Model 5. We consider higher response dimensionality \(q = 50\), higher population collinearity level \(\rho = 0.8\), and larger true model sizes for the responses. The covariates \(X_1, \cdots, X_p\) are sampled similarly as in Model 5, but the even numbered covariates are further discretized. More specifically, each even numbered covariate is assigned values 0, 1, or 2 if the original continuous covariate takes values below 0, between 0 and 1.5, or above 1.5, respectively, and then centered with mean zero. These discrete covariates are included in the model because in real applications some covariates can also be discrete. For instance, the covariates in the single nucleotide polymorphism (SNP) data are typically coded to take values 0, 1, and 2. In addition, the random errors \(W_1, \cdots, W_q\) are sampled independently from the \(t\)-distribution with 5 degrees of freedom. Thus Model 6 involves both non-Gaussian design matrix with mixed covariate types and non-Gaussian error vector.

We list in Table 3 the screening performance of all the methods as in Section 3.2. Note that the standard errors are omitted in this table to save space. Comparing these results to those in Table 2, we see that the problem of interaction screening becomes more difficult in this model. This result is reasonable since the scenario of Model 6 is more challenging than that of Model 5. Nevertheless IPDC still improves over other methods in retaining active interaction variables \(X_9, X_{12},\) and \(X_{13}\).

4. Real data analysis. We further evaluate the performance of our proposed procedure on a multivariate yeast cell-cycle data set from Spellman et al. [30], which can be accessed in the R package “spls.” This data set has been studied in Chun and Keles [9] and Chen and Huang [7]. Our goal is to predict how much mRNA is produced by 542 genes related to the yeast cell’s replication process. For each gene, the binding levels of 106 transcription factors (TFs) are recorded. The binding levels of the TFs play a role in
Table 3
Proportions of important main effects and active interaction variables retained by different screening methods.

<table>
<thead>
<tr>
<th>Method</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X_4$</th>
<th>$X_5$</th>
<th>$X_{12}$</th>
<th>$X_{13}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SIS.max</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>0.65</td>
<td>0.44</td>
</tr>
<tr>
<td>SIS.sum</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>0.76</td>
<td>0.45</td>
</tr>
<tr>
<td>DCSIS</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>0.93</td>
<td>0.28</td>
</tr>
<tr>
<td>IPDC</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>0.92</td>
<td>0.67</td>
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</table>

Table 4
Means and standard errors (in parentheses) of prediction error as well as numbers of selected main effects and interactions for each method in yeast cell-cycle data.

<table>
<thead>
<tr>
<th>Method</th>
<th>PE ($\times 10^{-3}$)</th>
<th>Model Size</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Main</td>
<td>Interaction</td>
</tr>
<tr>
<td>SIS.max-GLasso</td>
<td>224.05 (1.20)</td>
<td>73.73 (7.96)</td>
</tr>
<tr>
<td>SIS.sum-GLasso</td>
<td>223.42 (1.17)</td>
<td>50.76 (3.97)</td>
</tr>
<tr>
<td>DCSIS-GLasso</td>
<td>223.93 (1.16)</td>
<td>63.67 (7.47)</td>
</tr>
<tr>
<td>IPDC-GLasso</td>
<td>220.44 (1.14)</td>
<td>113.78 (9.74)</td>
</tr>
<tr>
<td>SIS.max-GLasso-Lasso</td>
<td>226.66 (1.45)</td>
<td>47.56 (3.66)</td>
</tr>
<tr>
<td>SIS.sum-GLasso-Lasso</td>
<td>225.07 (1.48)</td>
<td>50.76 (3.97)</td>
</tr>
<tr>
<td>DCSIS-GLasso-Lasso</td>
<td>226.40 (1.43)</td>
<td>47.12 (3.92)</td>
</tr>
<tr>
<td>IPDC-GLasso-Lasso</td>
<td>222.43 (1.39)</td>
<td>56.08 (3.33)</td>
</tr>
</tbody>
</table>

determining which genes are expressed and help detail the process behind eukaryotic cell-cycles. Messenger RNA is collected for two cell-cycles for a total of eighteen time points. Thus this data set has sample size $n = 542$, number of covariates $p = 106$, and number of response $q = 18$, with all variables being continuous.

Considering the relatively large sample size, we use 30% of the data as training and the rest as testing, and repeat such random splitting for 100 times. We follow the same screening and selection procedures as in the simulation study for the setting of multiple responses in Section 3. Similarly we take a union of the set of retained interaction variables and the set of retained main effects. For fair comparison, we keep $2[n/(\log n)] = 62$ variables in the screening procedures of SIS.max, SIS.sum, and DCSIS, and use those variables to construct pairwise interactions for the selection step.

Table 4 presents the results on the prediction error and selected model size. Paired $t$-tests of prediction errors on the 100 splits of IPDC-GLasso against SIS.max-GLasso, SIS.sum-GLasso, and DCSIS-GLasso result in $p$-values $2.86 \times 10^{-11}$, $1.70 \times 10^{-13}$, and $1.15 \times 10^{-14}$, respectively. Moreover,
paired $t$-tests of prediction errors on the 100 splits of IPDC-GLasso-Lasso against SIS.max-GLasso-Lasso, SIS.sum-GLasso-Lasso, and DCSIS-GLasso-Lasso give $p$-values $2.92 \times 10^{-4}$, $2.30 \times 10^{-2}$, and $9.73 \times 10^{-5}$, respectively. These results show significant improvement of our method over existing ones.

5. Discussions. We have investigated the problem of interaction identification in the setting where the numbers of responses and covariates can both be large. Our suggested two-stage procedure IPDC provides a scalable approach with the idea of interaction screening and selection. It exploits the joint information among all the responses by using the distance correlation in the screening step and the regularized multi-response regression in the selection step. One key ingredient is the use of the square transformation to responses and covariates for effective interaction screening. The established sure screening and model selection properties enable its broad applicability beyond the heredity assumption.

Although we have focused our attention on the square transformation of the responses and covariates due to its simplicity and the motivation discussed in Section 2.1, it is possible that other functions can also work for the idea of IPDC. It would be interesting to investigate and characterize what class of functions is optimal for the purpose of interaction screening.

Like all independence screening methods using the marginal utilities including the SIS and DCSIS, our feature screening approach may fail to identify some important interactions or main effects that are marginally weakly related to the responses. One possible remedy is to exploit the idea of the iterative SIS proposed in [11] which has been shown to be capable of ameliorating the SIS. Recently, Zhong and Zhu [36] also introduced an iterative DCSIS procedure and demonstrated that it can improve the finite-sample performance of the DCSIS. The theoretical properties of these iterative feature screening approaches are, however, less well understood. It would be interesting to develop an effective iterative IPDC procedure for further improving on the IPDC and investigate its sampling properties. For more flexible modeling, it is also of practical importance to extend the idea of IPDC to high-dimensional multi-response interaction models in the more general settings of the generalized linear models, nonparametric models, and survival models, as well as other single-index models and multi-index models. These possible extensions are beyond the scope of the current paper and will be interesting topics for future research.
APPENDIX A: PROOFS OF MAIN RESULTS

We provide the main steps of the proof of Theorem 1 and the proof of Theorem 2 in this appendix. Some intermediate steps of the proof of Theorem 1 and additional technical details are included in the Supplementary Material. Hereafter we denote by $\tilde{C}_i$ with $i \geq 0$ some generic positive constants whose values may vary from line to line.

A.1. Proof of Theorem 1. The proof of Theorem 1 consists of two parts. The first part establishes the exponential probability bounds for $\hat{\omega}_j - \omega_j$ and $\hat{\omega}_k^* - \omega_k^*$, and the second part proves the sure screening property.

Part 1. We first prove inequalities (8) and (9), which give the exponential probability bounds for $\hat{\omega}_j - \omega_j$ and $\hat{\omega}_k^* - \omega_k^*$, respectively. Since the proofs of (8) and (9) are similar, here we focus on (9) to save space. Recall that

$$\omega_k^* = \frac{\text{dco}v^2(X_k^*, y^*)}{\sqrt{\text{dco}v^2(X_k^*, X_k^*)}}$$

and

$$\hat{\omega}_k^* = \frac{\hat{\text{dco}v}^2(X_k^*, y^*)}{\sqrt{\hat{\text{dco}v}^2(X_k^*, X_k^*)}}.$$

The key idea of the proof is to show that for any positive constant $\tilde{C}$, there exist some positive constants $\tilde{C}_1, \cdots, \tilde{C}_4$ such that

$$P \left( \max_{1 \leq k \leq p} |\hat{\text{dco}v}^2(X_k^*, y^*) - \text{dco}v^2(X_k^*, y^*)| \geq \tilde{C} n^{-\kappa_2} \right) \leq p \tilde{C}_1 \exp\{-\tilde{C}_2 n^{(1-2\kappa_2)/5} \} + \tilde{C}_3 \exp\{-\tilde{C}_4 n^{(1-2\kappa_2)/10} \},$$

(13)

and

$$P \left( \max_{1 \leq k \leq p} |\hat{\text{dco}v}^2(X_k^*, X_k^*) - \text{dco}v^2(X_k^*, X_k^*)| \geq \tilde{C} n^{-\kappa_2} \right) \leq p \tilde{C}_1 \exp\{-\tilde{C}_2 n^{(1-2\kappa_2)/5} \}.$$

(14)

for all $n$ sufficiently large. Once these two probability bounds are obtained, it follows from Conditions 1–2 and Lemmas ??–?? and ?? that

$$P \left( \max_{1 \leq k \leq p} |\hat{\omega}_k^* - \omega_k^*| \geq c_2 n^{-\kappa_2} \right) \leq O \left( p \exp\{-C_1 n^{(1-2\kappa_2)/5} \} \right) + \exp\{-C_2 n^{(1-2\kappa_2)/10} \} \leq O \left( \exp\{ -C n^{(1-2\kappa_2)/10} \} \right),$$

(15)

where $C_1, C_2$, and $C$ are some positive constants, and the last inequality follows from the condition that $\log p = o(n^{\eta_0})$ with $\eta_0 = \min\{(1-2\kappa_1)/3, (1-2\kappa_2)/5\}$.

It thus remains to prove (13) and (14). Again we concentrate on (13) since (14) can be shown using similar arguments. Define $\phi(X_{1k}, X_{2k}) = |X_{1k} - X_{2k}|$.
and \( \psi(y_1^*, y_2^*) = ||y_1^* - y_2^*|| \). According to [31], we have
\[
dcov^2(X_k^*, y^*) = T_{k1} + T_{k2} - 2T_{k3} \quad \text{and} \quad \tilde{\text{dcov}}^2(X_k^*, y^*) = \tilde{T}_{k1} + \tilde{T}_{k2} - 2\tilde{T}_{k3},
\]
where \( T_{k1} = E[\phi(X_{1k}^*, X_{2k}^*) \psi(y_1^*, y_2^*)], T_{k2} = E[\phi(X_{1k}^*, X_{2k}^*)] E[\psi(y_1^*, y_2^*)], \)
\( T_{k3} = E[\phi(X_{1k}^*, X_{2k}^*) \psi(y_1^*, y_2^*)], \) and
\[
\tilde{T}_{k1} = n^{-2} \sum_{i,j=1}^n \phi(X_{ik}^*, X_{jk}^*) \psi(y_i^*, y_j^*),
\]
\[
\tilde{T}_{k2} = [n^{-2} \sum_{i,j=1}^n \phi(X_{ik}^*, X_{jk}^*)] [n^{-2} \sum_{i,j=1}^n \psi(y_i^*, y_j^*)],
\]
\[
\tilde{T}_{k3} = n^{-3} \sum_{i=1}^n \sum_{j,l=1}^n \phi(X_{ik}^*, X_{jk}^*) \psi(y_i^*, y_l^*).
\]

It follows from the triangle inequality that
\[
\max_{1 \leq k \leq p} |\tilde{\text{dcov}}^2(X_k^*, y^*) - \text{dcov}^2(X_k^*, y^*)| \leq \max_{1 \leq k \leq p} |\tilde{T}_{k1} - T_{k1}|
+ \max_{1 \leq k \leq p} |\tilde{T}_{k2} - T_{k2}| + 2 \max_{1 \leq k \leq p} |\tilde{T}_{k3} - T_{k3}|.
\]

To establish the probability bound for the term \( \max_{1 \leq k \leq p} |\tilde{T}_{k1} - T_{k1}| \), it is sufficient to bound each term on the right hand side above. To enhance the readability, we proceed with three main steps.

**Step 1.** We start with the first term \( \max_{1 \leq k \leq p} |\tilde{T}_{k1} - T_{k1}| \). An application of the Cauchy-Schwarz inequality gives
\[
T_{k1} \leq \{E[\phi^2(X_{1k}^*, X_{2k}^*)] E[\psi^2(y_1^*, y_2^*)]\}^{1/2}.
\]

It follows from the triangle inequality that
\[
\phi(X_{1k}^*, X_{2k}^*) = |X_{1k}^* - X_{2k}^*| \leq |X_{1k}^*| + |X_{2k}^*| = X_{1k}^2 + X_{2k}^2
\]
and
\[
\psi(y_1^*, y_2^*) = ||y_1^* - y_2^*|| \leq ||y_1^*|| + ||y_2^*|| \leq ||\tilde{y}_1||^2 + ||\tilde{y}_2||^2,
\]
in view of \( y_1^* = \tilde{y}_1 \circ \tilde{y}_1 \) and the fact that \( ||a \circ a|| \leq ||a||^2 \) for any \( a \in \mathbb{R}^q \). By (17), we have \( E[\phi^2(X_{1k}^*, X_{2k}^*)] \leq E\{2(X_{1k}^4 + X_{2k}^4)\} = 4E(X_{1k}^4) \). Similarly, it holds that \( E[\psi^2(y_1^*, y_2^*)] \leq 4E(||\tilde{y}_1||^4) \). Combining these results leads to
0 \leq T_{k_1} \leq 4 \left\{ E(X_{1k}^4)E(||\tilde{y}_1||^4) \right\}^{1/2}. \text{ By Condition 2, } E(X_{1k}^4) \text{ and } E(||\tilde{y}_1||^4) \text{ are uniformly bounded by some positive constant for all } 1 \leq k \leq p. \text{ Thus for any positive constant } \tilde{C}, \ |T_{k_1}/n| < \tilde{C}n^{-\kappa_2}/8 \text{ holds uniformly for all } 1 \leq k \leq p \text{ when } n \text{ is sufficiently large.}

Let \( \hat{T}_{k_1} = n(n-1)^{-1}T_{k_1} = \{n(n-1)^{-1}\sum_{i \neq j} \phi(X_{ik}^*, X_{jk}^*)\psi(y_i^*, y_j^*) \}. \text{ Then we have}

\[ |\hat{T}_{k_1} - T_{k_1}| \leq n^{-1}(n-1)|\hat{T}_{k_1} - T_{k_1}| + |T_{k_1}/n| \leq |\hat{T}_{k_1} - T_{k_1}| + \tilde{C}n^{-\kappa_2}/8 \]

for all \( 1 \leq k \leq p \), which entails

(19) \( P(\max_{1 \leq k \leq p} |\hat{T}_{k_1} - T_{k_1}| \geq \tilde{C}n^{-\kappa_2}/4) \leq P(\max_{1 \leq k \leq p} |\hat{T}_{k_1} - T_{k_1}| \geq \tilde{C}n^{-\kappa_2}/8) \)

for sufficiently large \( n \). Thus it is sufficient to bound \( \hat{T}_{k_1} - T_{k_1} \).

Since \( X_{ik}^* \) and \( y_i^* \) are generally unbounded, we apply the technique of truncation in the technical analysis. Define

\[
\hat{T}_{k_{1,1}} = \{n(n-1)^{-1}\sum_{i \neq j} \phi(X_{ik}^*, X_{jk}^*)\psi(y_i^*, y_j^*)\}^\mathbb{I}\{\phi(X_{ik}^*, X_{jk}^*) \leq M_1\}
\]

\[
\hat{T}_{k_{1,2}} = \{n(n-1)^{-1}\sum_{i \neq j} \phi(X_{ik}^*, X_{jk}^*)\psi(y_i^*, y_j^*)\}^\mathbb{I}\{\phi(X_{ik}^*, X_{jk}^*) > M_2\}
\]

\[
\hat{T}_{k_{1,3}} = \{n(n-1)^{-1}\sum_{i \neq j} \phi(X_{ik}^*, X_{jk}^*)\psi(y_i^*, y_j^*)\}^\mathbb{I}\{\phi(X_{ik}^*, X_{jk}^*) > M_1\}
\]

where \( \mathbb{I}\{\cdot\} \) denotes the indicator function and the thresholds \( M_1, M_2 > 0 \) will be specified later. Then we have \( \hat{T}_{k_1} = \hat{T}_{k_{1,1}} + \hat{T}_{k_{1,2}} + \hat{T}_{k_{1,3}} \). Consequently, we can rewrite \( T_{k_1} \) as \( T_{k_1} = T_{k_{1,1}} + T_{k_{1,2}} + T_{k_{1,3}} \) with

\[
T_{k_{1,1}} = E[\phi(X_{1k}, X_{2k})\psi(y_1^*, y_2^*)^\mathbb{I}\{\phi(X_{1k}, X_{2k}) \leq M_1\}^\mathbb{I}\{\psi(y_1^*, y_2^*) \leq M_2\}],
\]

\[
T_{k_{1,2}} = E[\phi(X_{1k}, X_{2k})\psi(y_1^*, y_2^*)^\mathbb{I}\{\phi(X_{1k}, X_{2k}) \leq M_1\}^\mathbb{I}\{\psi(y_1^*, y_2^*) > M_2\}],
\]

\[
T_{k_{1,3}} = E[\phi(X_{1k}, X_{2k})\psi(y_1^*, y_2^*)^\mathbb{I}\{\phi(X_{1k}, X_{2k}) > M_1\}].
\]

Clearly, \( \hat{T}_{k_{1,1}}, \hat{T}_{k_{1,2}}, \) and \( \hat{T}_{k_{1,3}} \) are unbiased estimators of \( T_{k_{1,1}}, T_{k_{1,2}}, \) and \( T_{k_{1,3}} \), respectively. Therefore, it follows from Bonferroni’s inequality that

(20) \( P(\max_{1 \leq k \leq p} |\hat{T}_{k_1} - T_{k_1}| \geq \tilde{C}n^{-\kappa_2}/8) \)

\[
\leq \sum_{j=1}^3 P(\max_{1 \leq k \leq p} |\hat{T}_{k_{1,j}} - T_{k_{1,j}}| \geq \tilde{C}n^{-\kappa_2}/24). \]
In what follows, we will provide details on deriving an exponential tail probability bound for each term on the right hand side above.

**Step 1.1.** We first consider \( \hat{T}_{k1,1} - T_{k1,1} \). For any \( \delta > 0 \), by Markov’s inequality we have

\[
(21) \quad P(\hat{T}_{k1,1} - T_{k1,1} \geq \delta) \leq \exp(-t\delta) \exp(-tT_{k1,1}) E[\exp(t\hat{T}_{k1,1})]
\]

for \( t > 0 \). Let \( h(X_{1k}^*, y_1^*; X_{1k}^*, y_2^*) = \phi(X_{1k}^*, X_{2k}^*) \psi(y_1^*, y_2^*) \{ \phi(X_{1k}^*, X_{2k}^*) \leq M_1 \} \{ \psi(y_1^*, y_2^*) \leq M_2 \} \) be the kernel of the \( U \)-statistic \( \hat{T}_{k1,1} \) and define

\[
(22) \quad W(X_{1k}^*, y_1^*; \cdots; X_{nk}^*, y_n^*) = m^{-1} \left\{ h(X_{1k}^*, y_1^*; X_{1k}^*, y_2^*) + h(X_{3k}^*, y_3^*; X_{4k}^*, y_4^*) + \cdots + h(X_{2m-1,k}^*, y_{2m-1}^*; X_{2m,k}^*, y_{2m}^*) \right\},
\]

where \( m = \lfloor n/2 \rfloor \) is the integer part of \( n/2 \). According to the theory of \( U \)-statistics [29, Section 5.1.6], any \( U \)-statistic can be expressed as an average of averages of i.i.d. random variables. This representation gives

\[
\hat{T}_{k1,1} = (n!)^{-1} \sum_{n!} W(X_{i_1k}^*, y_{i_1}^*; \cdots; X_{i_nk}^*, y_{i_n}^*),
\]

where \( \sum_{n!} \) represents the summation over all possible permutations \((i_1, \cdots, i_n)\) of \((1, \cdots, n)\). An application of Jensen’s inequality yields that for any \( t > 0 \),

\[
E[\exp(t\hat{T}_{k1,1})] = E\left\{ \exp\left( (n!)^{-1} \sum_{n!} tW(X_{i_1k}^*, y_{i_1}^*; \cdots; X_{i_nk}^*, y_{i_n}^*) \right) \right\}
\leq E\left\{ (n!)^{-1} \sum_{n!} \exp\left( tW(X_{i_1k}^*, y_{i_1}^*; \cdots; X_{i_nk}^*, y_{i_n}^*) \right) \right\}
= E\{ \exp[tW(X_{1k}^*, y_1^*; \cdots; X_{nk}^*, y_n^*)] \}
= E^m \{ \exp[tm^{-1}h(X_{1k}^*, y_1^*; X_{2k}^*, y_2^*)] \},
\]

where the last equality follows from (22). The above inequality together with (21) leads to

\[
P(\hat{T}_{k1,1} - T_{k1,1} \geq \delta) \leq \exp(-t\delta) E^m \left\{ e^{tm^{-1}[h(X_{1k}^*, y_1^*; X_{2k}^*, y_2^*) - T_{k1,1}]} \right\}.
\]

Note that \( E[h(X_{1k}^*, y_1^*; X_{2k}^*, y_2^*) - T_{k1,1}] = 0 \) and

\[
-T_{k1,1} \leq h(X_{1k}^*, y_1^*; X_{2k}^*, y_2^*) - T_{k1,1} \leq M_1 M_2 - T_{k1,1}.
\]
Hence it follows from Lemma ?? that
\[ P(\hat{T}_{k,1} - T_{k,1} \geq \delta) \leq \exp[-t\delta + t^2 M_1^2 M_2^2/(8m)] \]
for any \( t > 0 \). Minimizing the right hand side above with respect to \( t \) gives 
\[ P(\hat{T}_{k,1} - T_{k,1} \geq \delta) \leq \exp(-2m\delta^2/M_1^2 M_2^2) \]
for any \( \delta > 0 \). Similarly, we can show that 
\[ P(\hat{T}_{k,1} - T_{k,1} \leq -\delta) \leq \exp(-2m\delta^2/M_1^2 M_2^2) \]
Therefore, it holds that 
\[ P(\left|\hat{T}_{k,1} - T_{k,1}\right| \geq \delta) \leq 2\exp(-2m\delta^2/M_1^2 M_2^2) \]
Recall that \( m = \lfloor n/2 \rfloor \). If we set \( M_1 = n^{\xi_1} \) and \( M_2 = n^{\xi_2} \) with some positive constants \( \xi_1 \) and \( \xi_2 \), then for \( \delta = \tilde{C} n^{-\kappa_2/24} \) with any positive constant \( \tilde{C} \), there exists some positive constant \( \tilde{C}_1 \) such that when \( n \) is sufficiently large, 
\[ P(\left|\hat{T}_{k,1} - T_{k,1}\right| \geq \tilde{C} n^{-\kappa_2/24}) \leq 2\exp(-\tilde{C}^2 \tilde{C}_1 n^{1-2\kappa_2-2\xi_1-2\xi_2}) \]
for all \( 1 \leq k \leq p \). This along with Bonferroni’s inequality entails 
\[ P(\max_{1 \leq k \leq p} |\hat{T}_{k,1} - T_{k,1}| \geq \tilde{C} n^{-\kappa_2/24}) \leq 2p\exp(-\tilde{C}^2 \tilde{C}_1 n^{1-2\kappa_2-2\xi_1-2\xi_2}). \]

**Step 1.2.** We next deal with \( \hat{T}_{k,1,2} - T_{k,1,2} \). Note that 
\[ 0 \leq T_{k,1,2} \leq M_1 E[\psi(y_1^1, y_2^2)\mathbb{I}\{\psi(y_1^1, y_2^2) > M_2\}] \]
for all \( 1 \leq k \leq p \). It follows from the Cauchy-Schwarz inequality that 
\[ E[\psi(y_1^1, y_2^2)\mathbb{I}\{\psi(y_1^1, y_2^2) > M_2\}] \leq (E[\psi^2(y_1^1, y_2^2)] E[\{\psi(y_1^1, y_2^2) > M_2\}])^{1/2}. \]
In view of (18), we see that 
\[ E[\psi^2(y_1^1, y_2^2)] \leq E[(\|\tilde{y}_1\|^2 + \|\tilde{y}_2\|^2)^2] \leq E[2(\|\tilde{y}_1\|^4 + \|\tilde{y}_2\|^4)] = 4E(\|\tilde{y}_1\|^4) \]
and the probability term in (24) is bounded from above by 
\[ P(\|\tilde{y}_1\|^2 + \|\tilde{y}_2\|^2 > M_2) \leq P(\|\tilde{y}_1\|^2 > M_2/2) + P(\|\tilde{y}_2\|^2 > M_2/2) \leq 2P(\|\tilde{y}_1\| > \sqrt{M_2/2}) \leq 2\exp(-c_0 \sqrt{M_2/2}) E[\exp(c_0 \|\tilde{y}_1\|)], \]
where \( c_0 \) is a positive constant given in Condition 2 and the last inequality follows from Markov’s inequality. Combining inequalities (24)–(26) and by Condition 2, we obtain 
\[ E[\psi(y_1^1, y_2^2)\mathbb{I}\{\psi(y_1^1, y_2^2) > M_2\}] \leq \tilde{C}_2 \exp(-2^{-1} c_0 \sqrt{M_2/2}) \]
and thus $0 \leq T_{k1,2} \leq \tilde{C}_2 M_1 \exp(-2^{-1}c_0 \sqrt{M_2/2})$, where $\tilde{C}_2$ is some positive constant. Recall that $M_1 = n^{\xi_1}$ and $M_2 = n^{\xi_2}$. Then for any positive constant $\tilde{C}$, it holds that

$$0 \leq T_{k1,2} \leq \tilde{C}_2 n^{\xi_1} \exp(-2^{-3/2}c_0 n^{\xi_2/2}) \leq \tilde{C} n^{-\xi_2}/48$$

for all $1 \leq k \leq p$ when $n$ is sufficiently large. This inequality gives

(28) \[ P(\max_{1 \leq k \leq p} |\hat{T}_{k1,2}^* - T_{k1,2}| \geq \tilde{C} n^{-\xi_2}/24) \leq P(\max_{1 \leq k \leq p} |\hat{T}_{k1,2}^*| \geq \tilde{C} n^{-\xi_2}/48) \]

for all $n$ sufficiently large.

Note that for all $1 \leq k \leq p$, $|\hat{T}_{k1,2}^*|$ is uniformly bounded from above by $M_1 \left[ n(n-1) \right]^{-1} \sum_{i \neq j} \psi(y_i^*, y_j^*) \| \psi(y_i^*, y_j^*) \| > M_2 \}$. Thus in view of (27), applying Markov’s inequality yields that for any $\delta > 0$,

$$P(\max_{1 \leq k \leq p} |\hat{T}_{k1,2}^*| \geq \delta/2) \leq P\left\{ M_1 \left[ n(n-1) \right]^{-1} \sum_{i \neq j} \psi(y_i^*, y_j^*) \| \psi(y_i^*, y_j^*) \| > M_2 \} \right\} \leq (\delta/2)^{-1} M_1 \exp(-2^{-1}c_0 \sqrt{M_2/2}).$$

Since $M_1 = n^{\xi_1}$ and $M_2 = n^{\xi_2}$, setting $\delta = \tilde{C} n^{-\xi_2}/24$ in the above inequality entails

$$P(\max_{1 \leq k \leq p} |\hat{T}_{k1,2}^*| \geq \tilde{C} n^{-\xi_2}/48) \leq 48 \tilde{C}^{-1} \tilde{C}_2 n^{\xi_2+\xi_1} \exp(-2^{-3/2}c_0 n^{\xi_2/2}).$$

Combining this inequality with (28) gives

(29) \[ P(\max_{1 \leq k \leq p} |\hat{T}_{k1,2}^* - T_{k1,2}| \geq \tilde{C} n^{-\xi_2}/24) \leq 48 \tilde{C}^{-1} \tilde{C}_2 n^{\xi_2+\xi_1} \exp(-2^{-3/2}c_0 n^{\xi_2/2}). \]

**Step 1.3.** We finally handle the term $\hat{T}_{k1,3}^* - T_{k1,3}$ and show that it satisfies

(30) \[ P(\max_{1 \leq k \leq p} |\hat{T}_{k1,3}^* - T_{k1,3}| \geq \tilde{C} n^{-\xi_2}/24) \leq 48p \tilde{C}^{-1} \tilde{C}_3 n^{\xi_2} \exp(-8^{-1}c_0 n^{\xi_1}) \]

with $\tilde{C}_3$ some positive constant in Section ?? of the Supplementary Material.
Combining the results in (20), (23), and (29)–(30) leads to
\[
P(\max_{1 \leq k \leq p} |\tilde{T}_{k1} - T_{k1}| \geq \tilde{C}n^{-\kappa_2}/8) \leq 2p \exp(-\tilde{C}^2 \tilde{C}_1 n^{1-2\kappa_2-2\xi_1-2\xi_2}) \\
+ 48p\tilde{C}^{-1} \tilde{C}_3 n^{\kappa_2} \exp(-8^{-1} c_0 n^{\xi_1}) \exp(-2^{3/2} c_0 n^{\xi_2/2}).
\]
Let \(\xi_1 = (1 - 2\kappa_2)/3 - 2\eta\) and \(\xi_2 = 3\eta\) with some \(0 < \eta < (1 - 2\kappa_2)/6\). Then we have
\[
P(\max_{1 \leq k \leq p} |\tilde{T}_{k1} - T_{k1}| \geq \tilde{C}n^{-\kappa_2}/8) \leq p\tilde{C}_1 \exp\{-\tilde{C}_2 n^{(1-2\kappa_2)/3-2\eta}\} \\
+ \tilde{C}_3 \exp\{-\tilde{C}_4 n^{3\eta/2}\},
\]
where \(\tilde{C}_1, \ldots, \tilde{C}_4\) are some positive constants. This inequality along with (19) yields
\[
P(\max_{1 \leq k \leq p} |\tilde{T}_{k1} - T_{k1}| \geq \tilde{C}n^{-\kappa_2}/4) \leq p\tilde{C}_1 \exp\{-\tilde{C}_2 n^{(1-2\kappa_2)/3-2\eta}\} \\
+ \tilde{C}_3 \exp\{-\tilde{C}_4 n^{3\eta/2}\}.
\]

**Step 2.** For the second term \(\max_{1 \leq k \leq p} |\tilde{T}_{k2} - T_{k2}|\), we show in Section ?? of the Supplementary Material that
\[
P(\max_{1 \leq k \leq p} |\tilde{T}_{k2} - T_{k2}| \geq \tilde{C}n^{-\kappa_2}/4) \leq \sum_{k=1}^{p} P(|\tilde{T}_{k2} - T_{k2}| \geq \tilde{C}n^{-\kappa_2}/4) \\
\leq p\tilde{C}_5 \exp\{-\tilde{C}_6 n^{(1-2\kappa_2)/5}\}
\]
holds, where \(\tilde{C}_5\) and \(\tilde{C}_6\) are some positive constants.

**Step 3.** We further prove that the third term \(\tilde{T}_{k3} - T_{k3}\) satisfies
\[
P(\max_{1 \leq k \leq p} |\tilde{T}_{k3} - T_{k3}| \geq \tilde{C}n^{-\kappa_2}/4) \leq p\tilde{C}_1 \exp\{-\tilde{C}_2 n^{(1-2\kappa_2)/3-2\eta}\} \\
+ \tilde{C}_3 \exp\{-\tilde{C}_4 n^{3\eta/2}\}
\]
with \(\tilde{C}_1, \ldots, \tilde{C}_4\) some positive constants in Section ?? of the Supplementary Material.

Combining inequalities (16) and (32)–(34) and setting \(\eta = (1 - 2\kappa_2)/15\) entail
\[
P\left\{ \max_{1 \leq k \leq p} |\text{dco}v^2(X_k^*, y^*) - \text{dco}v^2(X_k^*, y^*)| \geq \tilde{C}n^{-\kappa_2} \right\} \\
\leq p\tilde{C}_1 \exp\{-\tilde{C}_2 n^{(1-2\kappa_2)/5}\} + \tilde{C}_3 \exp\{-\tilde{C}_4 n^{(1-2\kappa_2)/10}\}
\]
with $\tilde{C}_1, \ldots, \tilde{C}_4$ some positive constants, which completes the proof for the first part of Theorem 1.

**Part 2.** We now proceed to prove the second part of Theorem 1. The main idea is to build the probability bounds for two events $\{M \subset \hat{M}\}$ and $\{I \subset \hat{I}\}$. We first bound $P(M \subset \hat{M})$. Define an event $\Omega_1 = \{\max_{j \in M} |\hat{\omega}_j - \omega_j| < c_1 n^{-\kappa_1}\}$. Then by Condition 3, conditional on the event $\Omega_1$ we have $\hat{\omega}_j \geq 2c_1 n^{-\kappa_1}$ for all $j \in M$, which gives

\begin{equation}
P(M \subset \hat{M}) \geq P(\Omega_1) = 1 - P(\Omega_1^c) = 1 - P(\max_{j \in M} |\hat{\omega}_j - \omega_j| \geq c_1 n^{-\kappa_1}).
\end{equation}

Following similar arguments as for proving (15), it can be shown that there exist some positive constants $\tilde{C}_5$ and $\tilde{C}_6$ such that

\begin{equation}
P(M \subset \hat{M}) \geq 1 - O\left(s_1 \exp\left\{-\tilde{C}_5 n^{(1-2\kappa_1)/3}\right\} + \exp\left\{-\tilde{C}_6 n^{(1-2\kappa_1)/6}\right\}\right),
\end{equation}

where $s_1$ is the cardinality of $M$. This inequality together with (36) yields

\begin{equation}
P(M \subset \hat{M}) \geq 1 - O\left(s_1 \exp\left\{-\tilde{C}_5 n^{(1-2\kappa_1)/3}\right\} + \exp\left\{-\tilde{C}_6 n^{(1-2\kappa_1)/6}\right\}\right).
\end{equation}

We next bound $P(I \subset \hat{I})$. Note that $P(I \subset \hat{I}) \geq P(A \subset \hat{A})$ since conditional on the event $\{A \subset \hat{A}\}$ it holds that $\{I \subset \hat{I}\}$. Define an event $\Omega_2 = \{\max_{k \in A} |\hat{\omega}_k^* - \omega_k^*| < c_2 n^{-\kappa_2}\}$. Then by Condition 3, we have $\hat{\omega}_k \geq 2c_2 n^{-\kappa_2}$ for all $k \in A$ conditional on the event $\Omega_2$, which leads to $P(A \subset \hat{A}) \geq P(\Omega_2)$. Combining these results yields

\begin{equation}
P(I \subset \hat{I}) \geq P(\Omega_2) = 1 - P(\Omega_2^c) = 1 - P(\max_{k \in A} |\hat{\omega}_k - \omega_k| \geq c_2 n^{-\kappa_2}).
\end{equation}

Using similar arguments as for proving (15) shows that there exist some positive constants $\tilde{C}_7$ and $\tilde{C}_8$ such that

\begin{equation}
P(\max_{k \in A} |\hat{\omega}_k - \omega_k| \geq c_2 n^{-\kappa_2}) = O\left(s_2 \exp\left\{-\tilde{C}_7 n^{(1-2\kappa_2)/5}\right\} + \exp\left\{-\tilde{C}_8 n^{(1-2\kappa_2)/10}\right\}\right),
\end{equation}
where \( s_2 \) is the cardinality of \( \mathcal{A} \). This together with (38) entails
\[
\Pr(\mathcal{I} \subset \widehat{\mathcal{I}}) \geq 1 - O\left( s_2 \exp\{-\tilde{C}_7 n^{(1-2\kappa_2)/5}\} + \exp\{-\tilde{C}_8 n^{(1-2\kappa_2)/10}\} \right).
\]

Finally combining (37) and (39), we obtain
\[
\Pr(\mathcal{M} \subset \widehat{\mathcal{M}} \text{ and } \mathcal{I} \subset \widehat{\mathcal{I}}) \geq \Pr(\mathcal{M} \subset \widehat{\mathcal{M}}) + \Pr(\mathcal{I} \subset \widehat{\mathcal{I}}) - 1
\]
\[
\geq 1 - O\left( s_1 \exp\{-\tilde{C}_5 n^{(1-2\kappa_1)/3}\} + \exp\{-\tilde{C}_6 n^{(1-2\kappa_1)/6}\} \right)
\]
\[
- O\left( s_2 \exp\{-\tilde{C}_7 n^{(1-2\kappa_2)/5}\} + \exp\{-\tilde{C}_8 n^{(1-2\kappa_2)/10}\} \right)
\]
\[
\geq 1 - O\left( \exp\{- C n^{\eta_0/2}\} \right),
\]
where \( C \) is some positive constant, and the last inequality follows from the facts \( s_1, s_2 \leq p \) and the condition that \( \log p = o(n^{\eta_0}) \) with \( \eta_0 = \min\{(1-2\kappa_1)/3, (1-2\kappa_2)/5\} \). This concludes the proof for the second part of Theorem 1.

A.2. Proof of Theorem 2. Define an event \( \Omega_3 = \{ \max_{1 \leq j \leq p} |\tilde{\omega}_j - \omega_j| < c_1 n^{-\kappa_1} \} \). For any \( j \in \mathcal{M}^c \), if \( \omega_j < c_1 n^{-\kappa_1} \) and \( |\tilde{\omega}_j - \omega_j| < c_1 n^{-\kappa_1} \), then we have \( \tilde{\omega}_j < 2c_1 n^{-\kappa_1} \). Thus conditional on the event \( \Omega_3 \), the cardinality of \( \{ j : \tilde{\omega}_j \geq 2c_1 n^{-\kappa_1} \text{ and } j \in \mathcal{M}^c \} \) cannot exceed that of \( \{ j : \omega_j \geq c_1 n^{-\kappa_1} \text{ and } j \in \mathcal{M}^c \} \). This entails that the cardinality of \( \{ j : \tilde{\omega}_j \geq 2c_1 n^{-\kappa_1} \} \) is no larger than that of \( \{ j : \omega_j \geq c_1 n^{-\kappa_1} \text{ and } j \in \mathcal{M}^c \} \cup \mathcal{M} \), which is in turn bounded from above by \( |\mathcal{M}| + s_3 \). Thus it follows from (8) in Theorem 1 that
\[
\Pr(|\widehat{\mathcal{M}}| \leq |\mathcal{M}| + s_3) \geq \Pr(\Omega_3) = 1 - \Pr(\max_{1 \leq j \leq p} |\tilde{\omega}_j - \omega_j| \geq c_1 n^{-\kappa_1})
\]
\[
\geq 1 - O\left( \exp\{- C n^{(1-2\kappa_1)/6}\} \right).
\]
Similarly we can show that
\[
\Pr \left\{ |\widehat{\mathcal{I}}| \leq (|\mathcal{A}| + s_4)(|\mathcal{A}| + s_4 - 1)/2 \right\} \geq 1 - O\left( \exp\{- C n^{(1-2\kappa_2)/10}\} \right).
\]
Combining these two probability bounds yields
\[
\Pr \left\{ |\widehat{\mathcal{M}}| \leq |\mathcal{M}| + s_3 \text{ and } |\widehat{\mathcal{I}}| \leq (|\mathcal{A}| + s_4)(|\mathcal{A}| + s_4 - 1)/2 \right\}
\]
\[
\geq 1 - O\left( \exp\{- C n^{(1-2\kappa_1)/6}\} \right) - O\left( \exp\{- C n^{(1-2\kappa_2)/10}\} \right)
\]
\[
\geq 1 - O\left( \exp\{- C n^{\eta_0/2}\} \right),
\]
where \( C \) is some positive constant, and the last inequality follows from the condition that \( \log p = o(n^{\eta_0}) \) with \( \eta_0 = \min\{(1-2\kappa_1)/3, (1-2\kappa_2)/5\} \). This completes the proof of Theorem 2.
SUPPLEMENTARY MATERIAL

Supplementary material to “Interaction Pursuit in High-Dimensional Multi-Response Regression via Distance Correlation” (doi: 10.1214/00-AOSXXXXSUPP; .pdf). Due to space constraints, some intermediate steps of the proof of Theorem 1 and additional numerical studies and technical details are provided in the Supplementary Material [20].

REFERENCES


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