7. **$A^1$-Homotopy Theory**

### 7.1. Closed model categories

We begin with Quillen’s generalization of the derived category. Recall that if $\mathcal{A}$ is an abelian category and if $C_\bullet(\mathcal{A})$ denotes the abelian category of chain complexes of $\mathcal{A}$ (say bounded above), then the derived category $C_\bullet(\mathcal{A}) \to D_-(\mathcal{A})$ is universal among functors from $C_\bullet(\mathcal{A})$ to additive categories which invert quasi-isomorphisms. Quillen extends this construction (so that it is applicable in homotopy theory) by associating to a closed model category $\mathcal{C}$ a functor $\mathcal{C} \to \mathcal{H}o(\mathcal{C})$ which is universal among functors to categories in which weak equivalences are mapped to isomorphisms.

**Example 7.1.** Our basic example is one of considerable interest. Let $\mathcal{S}$ denote the category of simplicial sets and define a weak equivalence $f : S_\bullet \to T_\bullet$ to be a map of simplicial sets whose geometric realization is a homotopy equivalence. This is equivalent to saying that $f$ is a disjoint union (indexed by $\pi_0(S_\bullet) = \pi_0(T_\bullet)$) of maps of connected simplicial sets each of which induces an isomorphism on homotopy groups for any choice of base points. This is part of a natural structure of a closed model category on $\mathcal{S}$.

Let $\mathcal{T}op$ denote the category of topological spaces and define a weak equivalence $g : V \to W$ to be a continuous map which induces a homotopy equivalence on associated singular complexes. This is equivalent to the assertion that $g$ is a disjoint union of maps of connected spaces (indexed by $\pi_0(V) = \pi_0(W)$) each of which induces an isomorphism on homotopy groups for any choice of base points. This is part of a natural structure of a closed model category on $\mathcal{T}op$.

Then the singular and geometric realization functors induce an equivalence of categories

$$\mathcal{H}o(\mathcal{T}op) \sim \mathcal{H}o(\mathcal{S}).$$

This is called the homotopy category.

There is more to this example which is reflected in the general context of closed model categories. Namely, we can consider the full subcategory $\mathcal{K} \subset \mathcal{S}$ of “Kan complexes”, simplicial sets which have a certain lifting property (i.e., are “fibrant”) and we can consider the full subcategory $\mathcal{C}W \subset \mathcal{T}op$ of C.W. complexes. A map in either $\mathcal{K}$ or in $\mathcal{C}W$ which is a weak equivalence is actually a homotopy equivalence. Moreover, we have equivalences of categories

$$\mathcal{H}o(\mathcal{C}W) \sim \mathcal{H}o(\mathcal{T}op) \sim \mathcal{H}o(\mathcal{S}) \sim \mathcal{H}o(\mathcal{K}).$$

Here is the abstract definition of a Quillen closed model category.

**Definition 7.2.** A closed model category $\mathcal{C}$ is a category equipped with three distinguished classes of morphisms called cofibrations, fibrations, and weak equivalences which satisfying the following:

- $\mathcal{C}$ is closed under finite limits and colimits.
- Given $X \overset{g}{\to} Y \overset{f}{\to} Z$, if any two of $f, g, f \circ g$ are weak equivalences then so is the third.
- Any retract of of cofibration (respectively, fibration; resp., weak equivalence) is again a cofibration (resp., fibration; resp. weak equivalence).
• Given a commutative diagram

\[ \begin{array}{ccc}
S & \xrightarrow{g} & X \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle p} \\
T & \xrightarrow{f} & Y 
\end{array} \]

(7.2.1)

where \( i \) is a cofibration and \( p \) is a fibration and at least one of \( i, p \) is a weak equivalence, then there exists some \( h : T \to X \) such that \( h \circ i = g, p \circ h = f \).

• Any map \( f : X \to Y \) in \( \mathcal{C} \) can be factored as

i.) \( f = p \circ i \), where \( p \) is a fibration and \( i \) is a trivial cofibration (i.e., \( i \) is both a cofibration and a weak equivalence).

ii.) \( f = q \circ j \), where \( q \) is a trivial fibration (i.e., \( q \) is both a fibration and a weak equivalence) and \( j \) is a cofibration.

**Remark 7.3.** Given a category \( \mathcal{C} \) and given two classes of morphisms which we view as cofibrations and weak equivalences (respectively, fibrations and weak equivalences; resp., cofibrations and fibrations), there exists at most one class of morphisms which constitute the class of fibrations (resp., cofibrations; resp., weak equivalences) for a closed model category structure on \( \mathcal{C} \) compatible with the given choices of the other two classes. In other words, in a closed model category, two out of three of the class of cofibrations, fibrations, and weak equivalences determines the third.

**Remark 7.4.** Given a closed model category \( \mathcal{C} \), Quillen introduces “cylinder objects” \( A \coprod A \xrightarrow{i} A \xrightarrow{p} A \) for objects \( A \in \mathcal{C} \), where \( i \) is a cofibration and \( p \) is a trivial fibration and the composition is the identity on each summand. Quillen also introduces “path objects” \( B \xrightarrow{s} B' \xrightarrow{q} B \times B \) with \( s \) a trivial cofibration and \( q \) a fibration and the composition equal to the diagonal. These objects are not uniquely defined, but permit Quillen to talk about left and right homotopy classes of maps in \( \mathcal{C} \). If \( X, Y \) are both fibrant and cofibrant, then left and right homotopy relating maps \( f : X \to Y \) coincide and determine an equivalence relation. We denote by \( \pi(X, Y) \) the set of homotopy classes of maps from the fibrant/cofibrant \( X \) to the fibrant/cofibrant \( Y \).

**Proposition 7.5.** If \( \mathcal{C} \) is a closed model category, \( f : X \to Y \) is a morphism of \( \mathcal{C} \) with \( X, Y \) fibrant and cofibrant. Then \( f \) is a weak equivalence if and only if it is an equivalence.

We now can state Quillen’s theorem asserting the existence of a good localization of a closed model category with respect to its class of weak equivalences. An object \( X \in \mathcal{C} \) is cofibrant if the unique map \( \emptyset \to X \) from the initial \( \emptyset \in \mathcal{C} \) to \( X \) is a cofibration; and object \( X \) is fibrant if the unique map \( X \to * \) from \( X \) to the final object \( * \in \mathcal{C} \) is a fibration. Using the axioms, we readily check that any map \( f : X \to Y \) lifts to a map \( \tilde{f} : QX \to QY \) of fibrant objects which map via weak equivalences to \( f : X \to Y \), and this in turn extends to a map \( Rf : RQX \to RQY \) of fibrant/cofibrant objects. There is no naturality (much less uniqueness) of the choice of \( Rf \) associated to \( f \).

**Theorem 7.6.** Let \( \mathcal{C} \) be a closed model category. Then there is a functor

\[ \gamma : \mathcal{C} \to \mathcal{Ho}(\mathcal{C}) \]
which is the identity on objects and whose set of morphisms from $X$ to $Y$ equals
the set of homotopy classes of morphisms from some fibrant/cofibrant replacement
of $X$ to some fibrant/cofibrant replacement of $Y$:

$$\text{Hom}_{\mathcal{H}_0(C)}(X, Y) = \pi(RQX, RQY).$$

If $F : C \to D$ is a functor with the property that $F$ sends weak equivalences to
isomorphisms, then there is a unique functor $RF : \mathcal{H}_0(C) \to D$ such that

$$F = R \circ \gamma : C \to \mathcal{H}_0(C) \to D.$$

7.2. $\mathbb{A}^1$ homotopy category. Let $k$ be a field and let $\text{Sm}/k$ denote the
category of quasi-projective varieties over $k$. Let $\text{PreShv}(\text{Sm}/k)$ denote the category
of contravariant functors from $\text{Sm}/k$ to (sets), and consider the natural embedding
$\text{Sm}/k \to \text{PreShv}(\text{Sm}/k)$ The great advantage of $\text{PreShv}(\text{Sm}/k)$ is that it
is closed under finite limits (e.g., quotients). Thus, if $X \to Y$ is a map in $\text{Sm}/k$
and if we denote by $h_X \to h_Y$ the induced map of representable presheaves (i.e.,
$\text{Hom}(-, X) \to \text{Hom}(-, Y)$), then we can consider $h_Y/h_X$ which is the colimit of
the diagram $\ast = h_{\text{Spec}k} \leftarrow h_X \to h_Y$.

Following Voevodsky, we denote by $\text{Spc}_k$ the category of sheaves of sets
$\text{Sm}/k$ for the Nisnevich topology. (Voevodsky thinks of $\text{Spc}_k$ as the category of “spaces" over
$k$.) Taking the associated sheaf gives us a functor from $\text{PreShv}(\text{Sm}/k) \to \text{Spc}_k$.
Voevodsky proposes to view the affine line $\mathbb{A}^1$ as the analogue of the interval in
ordinary topology, so that to build the homotopy category of $\text{Spc}_k$ we should
localize maps of the form $X \times \mathbb{A}^1 \to X$.

**Proposition 7.7.** There is a closed model category structure on $\text{Spc}_k$ whose cofi-
brations are monomorphisms (of Nisnevich sheaves on $\text{Sm}/k$).

The class of weak equivalences of this closed model category is the smallest class
of morphisms in $\text{Spc}_k$ containing all isomorphisms and closed under the following:

- $X \times \mathbb{A}^1 \to X$ is in the class
- If $X \to Y \to Z$ and if two of $f, g, g \circ f$ are in the class, then so is the third.
- The colimit of a filtered system of maps which are both cofibrations and
  weak equivalences (i.e., trivial cofibrations) is again a weak equivalence.
- The pushout of a weak equivalence along a cofibration is a weak equivalence.
- The pushout of a trivial cofibration along any map is a weak equivalence.

The resulting homotopy category is denoted $\text{Hot}_{\mathbb{A}^1, k}$.

As is almost always the case when considering a closed model category, one of
the two classes of cofibrations and fibrations is easily identified (in the case of $\text{Spc}_k$
it is the case of cofibrations) and the other class is determined formally by this class
of cofibrations and the chosen class of weak equivalences.

The homotopy category $\text{Hot}_{\mathbb{A}^1, k}$ has another description which is useful not only
to establish the well-definedness of $\text{Hot}_{\mathbb{A}^1, k}$ but also to compare it with the derived
category of Nisnevich sheaves with transfers which we used to formulate motivic
cohomology, we state the following theorem.

**Theorem 7.8.** Let $\Delta^{op}\text{Spc}_k$ denote the category of simplicial objects of $\text{Spc}_k$
(i.e., simplicial sheaves in the Nisnevich topology). There is a closed model cate-
gory structure on $\Delta^{op}\text{Spc}_k$ whose cofibrations are monomorphisms and whose weak
equivalences are maps $f : F_\bullet \to G_\bullet$ with the property that for each point $x$ (in the
Nisnevich topology) the induced map of simplicial sets $f : (F_\bullet)_x \to (G_\bullet)_x$ is a
weak equivalence. The resulting homotopy category $\text{Ho}(\Delta^{op}\text{Spc}_k)$ can be further localized by inverting all projections $C_*(\mathbb{A}^1) \times X_\ast \to X_\ast$, determining a category $\text{Ho}_{\mathbb{A}^1}(\Delta^{op}\text{Spc}_k)$.

Then there is an equivalence of categories
\[
\text{Hot}_{\mathbb{A}^1,k} \cong \text{Ho}_{\mathbb{A}^1}(\Delta^{op}\text{Spc}_k).
\]

Another advantage of the closed model category $\Delta^{op}\text{Spc}_k$ is that it is a simplicial closed model category in a sense made precise in the following definition. Essentially, this means that the category has representable function spaces.

**Definition 7.9.** A category $\mathcal{C}$ is a simplicial category if there is a mapping space functor $\text{Hom}_\mathcal{C} : \mathcal{C}^{op} \times \mathcal{C} \to \Delta^{op}(\text{sets})$ such that
- $\text{Hom}_\mathcal{C}(A, B)_0 = \text{Hom}_\mathcal{C}(A, B)$
- $\text{Hom}_\mathcal{C}$ has an associative left adjoint $\otimes : \Delta^{op}(\text{sets}) \to \mathcal{C}$
- $\text{Hom}_\mathcal{C}(-, B) : \mathcal{C}^{op} \to \Delta^{op}(\text{sets})$ has as left adjoint $\text{Hom}_\mathcal{C}(-, B) : \Delta^{op}(\text{sets}) \to \mathcal{C}^{op}$.

A category $\mathcal{C}$ which is both a closed model category and simplicial is a simplicial closed model category if it satisfies the following condition: if $j : A \to B$ is a cofibration and $q : X \to Y$ is a fibration, then
\[
\text{Hom}_\mathcal{C}(B, X) \to \text{Hom}_\mathcal{C}(A, X) \times_{\text{Hom}_\mathcal{C}(A, Y)} \text{Hom}_\mathcal{C}(B, Y)
\]
is a fibration of simplicial sets which is trivial if either $j$ or $q$ is trivial.

We can relate this to motivic cohomology using the following commutative square.

**Theorem 7.10.** The construction of the homotopy category $\text{Hot}_{\mathbb{A}^1,k}$ and the derived category $\text{cDM}_k$ used for the formulation of motivic cohomology are related by the following commutative diagram:

\[
\begin{array}{cccccc}
Sm/k & \to & \text{PreShv}(Sm/k) & \to & \Delta^{op}\text{Spc}_k & \to & \text{Hot}_{\mathbb{A}^1,k} \\
\downarrow_{h_-} & & \downarrow & & \downarrow & & \\
(\text{PreShv/tr}) & \to & D_-(\text{PreShv/tr}) & \to & D_-(\text{NisShv/tr}) & \to & \text{DM}_{\text{eff}}^k
\end{array}
\]

### 7.3. Stable $\mathbb{A}^1$ homotopy theory; $\text{SHot}_k$. Morel and Voevodsky essentially repeat the constructions in the previous subsection with spaces replaces by spectra. A key insight of Voevodsky is that one has to make invertible “suspending by each of two circles”, the “simplical circle $S^1$ and the “Tate circle” $S^1_t = \mathbb{A}^1 - \{0\}$.

One can expand the above commutative diagram on the right by stabilizing in top row with respect to $T = S^1 \land S^1_t$, $\text{Hot}_{\mathbb{A}^1,k} \to \text{SHot}_k$ and stabilizing in the bottom row by inverting the Tate twist $M \to M \otimes \mathbb{Z}(1)$ (where the tensor product is taken in the sense of the derived category of Nisnevich sheaves with transfer):

\[
\begin{array}{cccc}
\text{Hot}_{\mathbb{A}^1,k} & \to & \text{SHot}_{\mathbb{A}^1,k} \\
\downarrow & & \\
\text{DM}^\text{eff}_k & \to & \text{DM}_k
\end{array}
\]
Among other pleasing results is their proof that algebraic K-theory is representable by a T-spectrum.

References


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