4. Higher Algebraic K-theory

With the "model" of topological K-theory in mind, we proceed to investigate the formulation of the algebraic K-theory of a variety in positive degrees. The reader should be forewarned of an indexing confusion: if \( X \) is a quasi-projective complex algebraic variety, then there is a natural map

\[
K_i(X) \to K_{top}(X^{an}).
\]

The explanation of the change of index is that these higher algebraic \( K \)-groups were originally defined for affine schemes \( X = \text{Spec} R \), so that the contravariance of \( K \)-theory with respect to maps of schemes is seen as covariance of \( K \)-theory with respect to maps of rings.

The strategy is to formulate a suitable space \( K(X) \) with the property that \( \pi_0(K(X)) = K_0(X) \); we then define \( K_i(X) = \pi_i(K(X)) \). Indeed, the space \( K(X) \) which is constructed is an infinite loop space, enabling the definition of \( K_i(X, \mathbb{Z}/n) \equiv \pi_i(K(X, \mathbb{Z}/n)) \) for every \( i \geq 0, n > 1 \). Many procedures have been developed to produce such spaces, some of which give different answers. However, \( K \)-theorists accept Quillen’s constructions (of which we shall mention three, which give the same \( K \)-groups whenever each is defined) as the “correct” ones, for they produce \( K \)-groups which are well behaved and somewhat computable.

The construction of these spaces (or spectra) is quite interesting, but we shall be content to try to convey some sense of homotopy-theoretic group completion which plays a key role and to merely sketch various ideas involved in Quillen’s constructions. Before we discuss these definitions (and some of the formal properties of \( K_*(X) \)), we briefly mention the “lower \( K \)-groups” \( K_1(-), K_2(-) \) since a criterion for a good definition of the space \( K(X) \) is that \( \pi_1(K(X)) \) should give \( K_1, \pi_2(K(X)) \) should give \( K_2 \).

**Definition 4.1.** Let \( R \) be a ring (assumed associative, as always and with unit). We define \( K_1(R) \) by the formula

\[
K_1(R) \equiv \text{GL}(R)/[\text{GL}(R), \text{GL}(R)],
\]

where \( \text{GL}(R) = \lim_{\to n} \text{GL}(n, R) \) and where \([\text{GL}(r), \text{GL}(R)]\) is the commutator subgroup of the group \( \text{GL}(R) \). Thus, \( K_1(R) \) is the maximal abelian quotient of \( \text{GL}(R) \),

\[
K_1(R) = H_1(\text{GL}(R), \mathbb{Z}).
\]

The commutator subgroup \([\text{GL}(R), \text{GL}(R)]\) equals the subgroup \( E(R) \subset \text{GL}(R) \) defined as the subgroup generated by elementary matrices \( E_{i,j}(r), r \in R, i \neq j \) (i.e., matrices which differ by the identity matrix by having \( r \) in the \((i,j)\) position). This group is readily seen to be perfect (i.e., \( E(R) = [E(r), E(R)] \)); indeed, it is an elementary exercise to verify that \( E(n, R) = E(R) \cap \text{GL}(n, R) \) is perfect for \( n \geq 3 \).

**Proposition 4.2.** If \( R \) is a commutative ring, then the determinant map

\[
det : K_1(R) \to R^x
\]

from \( K_1(R) \) to the multiplicative group of units of \( R \) provides a natural splitting of \( R^x = \text{GL}(1, R) \to \text{GL}(R) \to K_1(R) \). Thus, we can write

\[
K_1(R) = R^x \times \text{SL}(R)
\]

where \( \text{SL}(R) = \ker \{ \text{det} \} \).

If \( R \) is a field or more generally a local ring, then \( SK_1(R) = 0 \). Moreover, \( SK_1(O_F) = 0 \) for the ring of integers \( O_F \) in a number field \( F \).
The consideration of $K_1(-)$ played an important role in the “Congruent subgroup problem” by Bass, Milnor, and Tate: for a ring of integers $O$, is it the case that all subgroups of finite index in $GL(n, O)$ are given as the kernels of maps $GL(n, O) \to GL(n, O/I)$ for some ideal $I \subset O$? The answer is yes if the number field $F$ admits a real embedding, and no otherwise.

For a group ring $\mathbb{Z}[G]$ of a discrete group $G$, the Whitehead group

$$Wh_1(G) = K_1(\mathbb{Z}[G])/ < \pm g, g \in G>$$

is an important topological invariant of a connected cell complex with fundamental group $G$.

One can think of $K_0(R)$ as the “stable group” of projective modules “modulo trivial projective modules” and of $K_1(R)$ of the stabilized group of automorphisms of the trivial projective module modulo “trivial automorphisms” (i.e., the elementary matrices up to isomorphism. This philosophy can be extended to the definition of $K_2$, but has not been extended to $K_i, i > 2$. Namely, $K_2(R)$ can be viewed as the relations among the trivial automorphisms (i.e., elementary matrices) modulo those relations which hold universally.

**Definition 4.3.** Let $St(R)$, the Steinberg group of $R$, denote the group generated by elements $X_{i,j}(r), i \neq j, r \in R$ subject to the following commutator relations:

$$[X_{i,j}(r), X_{k,\ell}(s)] = \begin{cases} 1 & \text{if } j \neq k, i \neq \ell \\ X_{i,\ell}(rs) & \text{if } j = k, i \neq \ell \\ X_{k,j}(-rs) & \text{if } j \neq k, i = \ell \end{cases}$$

We define $K_2(R)$ to be the kernel of the map $St(R) \to E(R)$, given by sending $X_{i,j}(r)$ to the elementary matrix $E_{i,j}(r)$, so that we have a short exact sequence

$$1 \to K_2(R) \to St(R) \to E(R) \to 1.$$ 

**Proposition 4.4.** The short exact sequence

$$1 \to K_2(R) \to St(R) \to E(R) \to 1$$

is the universal central extension of the perfect group $E(R)$. Thus, $K_2(R) = H_2(E(R), \mathbb{Z})$, the Schur multiplier of $E(r)$.

**Proof.** Once can show that a universal central extension of a group $E$ exists if and only $E$ is perfect. In this case, a group $S$ mapping onto $E$ is the universal central extension if and only if $S$ is also perfect and $H_2(S, \mathbb{Z}) = 0$. $\square$

**Example 4.5.** If $R$ is a field, then $K_1(F) = F^\times$, the non-zero elements of the field viewed as an abelian group under multiplication. By a theorem of Matsumoto, $K_2(F)$ is characterized as the target of the “universal Steinberg symbol”. Namely, $K_2(F)$ is isomorphic to the free abelian group with generators “Steinberg symbols” $\{a, b\}, a, b \in F^\times$ and relations

- i. $\{ac, b\} = \{a, b\} \{c, b\}$,
- ii. $\{a, bd\} = \{a, b\} \{a, d\}$,
- iii. $\{a, 1-a\} = 1, a \neq 1 \neq 1-a$. (Steinberg relation)

Observe that for $a \in F^\times$, $-a = \frac{1+a}{1-a+a} \cdot a$, so that

$$\{a, -a\} = \{a, 1-a\} \{a, 1-a^{-1}\}^{-1} = \{a, 1-a^{-1}\}^{-1} = \{a^{-1}, 1-a^{-1}\} = 1.$$
Then we conclude the skew symmetry of these symbols:
\[\{a, b\} = \{a - b, a\} = \{a, -b\} = \{a, -ab\} = \{ab, -ab\} = 1.\]

Milnor used this presentation of \(K_2(F)\) as the starting point of his definition of the Milnor \(K\)-theory of a field \(F\).

**Definition 4.6.** Let \(F\) be a field with multiplicative group of units \(F^\times\). The Milnor \(K\)-group \(K_n^\text{Milnor}(F)\) is defined to be the \(n\)-th graded piece of the graded ring defined as the tensor algebra \(\bigoplus_{i \geq 0} (F^\times)^{\otimes n}\) modulo the ideal generated by the hyperbolic space \(W\) \(\langle\rangle\) consisting of elements \(\{a, 1-a\}, a \neq 1\neq 1 - a\) of homogeneous degree 2.

In particular, \(K_1(F) = K_1^\text{Milnor}(F), K_2(F) = K_2^\text{Milnor}(F)\) for any field \(F\), and \(K_n^\text{Milnor}(F)\) is a quotient of \(\Lambda^n(F^\times)\). For \(F\) an infinite field, Suslin proved that there are natural maps
\[K_n^\text{Milnor}(F) \to K_n(F) \to K_n^\text{Milnor}(F)\]
whose composition is \((-1)^{n-1}(n-1)!\). This immediately implies, for example, that the higher \(K\)-groups of a field of high transcendence degree are very large.

**Remark 4.7.** It is difficult to even briefly mention \(K_2\) of fields without also mentioning the deep and import theorem of Mekurjev and Suslin: for any field \(F\) and positive integer \(n\),
\[K_2(F)/nK_2(F) \simeq _n\text{Br}(F)\]
where \(_n\text{Br}(F)\) denotes the subgroup of the Brauer group of \(F\) consisting of elements which are \(n\)-torsion.

The most famous success of \(K\)-theory in recent years is the following theorem of Voevodsky, extending foundational work of Milnor.

**Theorem 4.8.** Let \(F\) be a field of characteristic \(\neq 2\). Let \(W(F)\) denote the Witt ring of \(F\), the quotient of the Grothendieck group of symmetric inner product spaces modulo the ideal generated by the hyperbolic space \(\langle 1 \rangle \oplus \langle -1 \rangle\) and let \(I = \ker (W(F) \to \mathbb{Z}/2)\) be given by sending a symmetric inner product space to its rank (modulo 2). Then the map
\[K_n^\text{Milnor}(F)/2 \cdot K_n^\text{Milnor}(F) \to I^n/I^{n+1}, \quad \{a_1, \ldots, a_n\} \mapsto \prod_{i=1}^n((a_i) - 1)\]
is an isomorphism for all \(n \geq 0\). Here, \(\langle a \rangle\) is the 1 dimensional symmetric inner product space with inner product \((-,-)_a\) defined by \((c,d)_a = acd\).

We next turn to the construction of Quillen \(K\)-theory spaces, the most readily accessible of which is that of the Quillen plus construction for a ring \(R\) (e.g., an affine algebraic variety).

**Proposition 4.9.** Let \(R\) be a ring (associative, with unit). There is a unique homotopy class of maps
\[i : BGL(R) \to BGL(R)^+\]
of connected spaces (of the homotopy type of C.W. complexes) with the following properties:
- \(i_\# : \pi_1(BGL(R)) \to \pi_1(BGL(R)^+)\) is the abelianization map \(GL(R) \to GL(R)/[GL(R), GL(R)] = K_1(R)\).
• For any local coefficient system on $BGL(R)^+$ (i.e., any $K_1(R)$-module $M$), $i^*: H^*(BGL(R)^+, M) \to H^*(BGL(R), i^*M)$ is an isomorphism.

Proof. The uniqueness up to homotopy of such a map follows from obstruction theory. For example, the uniqueness is equivalent to the uniqueness of a homology equivalence from the covering space of $BGL(R)$ associated to the subgroup $[GL(R), GL(R)] \subset \pi_1(BGL(R))$ to a simply connected space which is a homology equivalence.

The existence of this map depends upon the fact that $[GL(R), GL(R)] = E(R)$ is a perfect normal subgroup of $GL(R)$. One adds 2-cells to $BGL(R)$ to kill $E(R) \subset \pi_1(BGL(R))$ and then using the fact that $E(R)$ is perfect and normal one verifies that one can add 3-cells to kill the resulting extra homology in dimension 2 in such a way that the resulting space has the same homology as $BGL(R)$.

\[\square\]

Remark 4.10. By construction, $\pi_1(BGL(R)^+) = K_1(R)$. Moreover, one verifies directly that $\pi_2(BGL(R)^+) = H_2(BGL(R)^+)$ is the kernel of the universal central extension of $E(R)$. Namely, let $F$ denote the homotopy fibre of the acyclic map $BGL(R) \to BGL(R)^+$ and recall that the boundary map in the long exact homotopy sequence sends $\pi_2(BGL(R)^+) = H_2(BGL(R)^+)$ to the center of $E(R) = ker(\pi_1(BGL(R)) \to \pi_1(BGL(R)^+))$. Since $H_1(F) = H_2(F) = 0$, we conclude that $\pi_1(F)$ is perfect and thus $\pi_2(BGL(R)^+ \to \pi_1(F) \to E(R)$ is the universal central extension of $E(R)$.

Definition 4.11. Let $R$ be an associative ring with unit. We define the $K$-theory space of $R$ to be the space

$$K(R) \equiv K_0(R) \times BGL(R)^+.$$ 

Although we are now close to a full computation of $K_*(\mathbb{Z})$, finite fields and formal constructions applied to finite fields given us the only examples of rings for which the $K$-theory is completely known. Since Quillen had the following computation in mind when he introduced his plus construction, this indicates how difficult it is to compute $k$-groups. One should contrast this calculation with the computation $K_i^{M_{\text{fin}}}$.

Theorem 4.12. Let $\mathbb{F}_q$ be a finite field with $q$ elements. Then $BGL(\mathbb{F}_q)^+$ is homotopy equivalent to the homotopy fibre of the map $1 - \Psi^q : BU \to BU$. Consequently,

$$K_i(\mathbb{F}_q) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}/q^j - 1 & \text{if } i = 2j > 0 \\ 0 & \text{if } i = 2j - 1 > 0 \end{cases}$$

A “cheap” way to define the $K$-theory space of a quasi-projective variety is to define the $K$-theory as that of a Jouanolou resolution as exhibited in the following proposition. The fact that this agrees with more satisfying definitions is a consequence of a theorem of Quillen asserting that such an affine fibre bundle induces an isomorphism on $K$-groups.

Proposition 4.13. Let $X$ be a quasi-projective variety over a field. Then there exists an affine torsor $P \to X$ with $P$ itself affine.

Proof. Consider the affine variety $J_n$ of rank 1 projectors of $k^{n+1}$ (i.e., linear maps $p : k^{n+1} \to k^{n+1}$ of rank 1 satisfying $p^2 = p$). Then $J^n$ is the 0-locus inside the affine space of all $n+1 \times n+1$ matrices $A = (a_{i,j})$ of the polynomial equations in the

$$a_{i,j} = a_{j,i}.$$
coordinates \(x_{i,j}\) which impose the conditions that \(A^2 = A, \det(A) = 0, \det(I-A) = 0\). The natural map \(J^n \to \mathbb{P}^n\) sending a projector \(p\) to the line \(p(k^{n+1})\) has fibre over \(p\) the vector group \(\text{Hom}_k(\ker\{p\}, \text{im}\{p\})\). If \(X \subset \mathbb{P}^n\) is closed, then we can take \(P \to X\) to be the restriction of \(J^n \to \mathbb{P}^n\), \(P = J^n \times_{\mathbb{P}^n} X \to X\).

An \(H\)-space \(T\) is a pointed space of the homotopy type of a pointed C.W. complex equipped with a continuous pointed pairing \(\mu : T \times T \to T\) with the property that \(\mu(t_0, -), \mu(-, t_0) : T \to T\) are both homotopic as pointed maps to the identity. A homotopy associative \(H\)-space equipped with a homotopy inverse \(i : T \to T\) is called a group-like \(H\)-space. It is a useful observation that if an \(H\)-space \(T\) has the homotopy type of a C.W. complex and if \(\pi_0(T)\) is a group, then it is group-like.

**Proposition 4.14.** The \(K\)-theory space \(K(R)\) admits the structure of a group-like \(H\)-space. Moreover, if \(Y\) is any group-like \(H\)-space, and if \(f : BGL(R) \to Y\) is a map, then there exist a unique homotopy class of maps \(BGL(R)^+ \to Y\) whose composition with \(BGL(R) \to K(R)\) is homotopic to \(f\).

Let \(P\) denote the abelian monoid of isomorphism classes of finitely generated projective \(R\)-modules. Then the \(K\)-theory space \(K(R)\) together with its \(H\)-group structure can be characterized as follows: there is a natural homotopy class of maps of \(H\)-spaces

\[
\coprod_{[P] \in \mathcal{P}} BAut(P) \to K(R)
\]

which is a homotopy-theoretic group completion in the sense that it satisfies the following two properties.

- The induced map on connected components is the group completion map \(\mathcal{P} \to K_0(R)\).
- The induced map on homology can be identified with the localization map

\[
H_*(\coprod_{[P]} BAut(P)) \to \mathbb{Z}[K_0(R)] \otimes_{\mathbb{Z}[P]} H_*(\coprod_{[P]} BAut(P)).
\]

Such a homotopy-theoretic group completion of an \(H\)-space \(T\) is the universal map (in the pointed homotopy category) of \(H\)-spaces from \(T\) to a group-like \(H\)-space.

Our definition of the \(K\)-theory space is not satisfactory for several reasons. The \(H\)-space structure is not very natural, the extension to non-affine varieties is also unnatural, and we have as yet no way to define \(K_1(-, \mathbb{Z}/n)\). The accepted definition which escapes all of these difficulties involves the Quillen Q-construction. Quillen’s comparison for \(X\) affine of the Q-construction and Quillen plus construction is via a third construction, the \(S^{-1}S\)-construction of Quillen. This is presented in [8].

Recall that a symmetric monoidal category \(S\) is a (small) category with a unit object \(e \in S\) and a functor \(\Box : S \times S \to S\) which is associative and commutative up to coherent natural isomorphisms. For example, if we consider the category \(\mathcal{P}\) of finitely generated projective \(R\)-modules, then the direct sum \(\oplus : \mathcal{P} \times \mathcal{P} \to \mathcal{P}\) is associative but only commutative up to natural isomorphism. The symmetric monoidal category relevant for the \(K\)-theory of a ring \(R\) is the category \(\text{Iso}(\mathcal{P})\) whose objects are finitely generated projective \(R\)-modules and whose morphisms are isomorphisms between projective \(R\)-modules.

Quillen’s construction of \(S^{-1}S\) for a symmetric monoidal category \(S\) is appealing, modelling one way we would introduce inverses to form the group completion of an abelian monoid.
Definition 4.15. Let $S$ be a symmetric monoidal category. The category $S^{-1}S$ is the category whose objects are pairs $\{a, b\}$ of objects of $S$ and whose maps from $\{a, b\}$ to $\{c, d\}$ are equivalence classes of compositions of the following form:

$$\{a, b\} \xrightarrow{s} \{s \square a, s \square b\} \xrightarrow{(f, g)} \{c, d\}$$

where $s$ is some object of $S$, $f, g$ are morphisms in $S$. Another such composition

$$\{a, b\} \xrightarrow{s'} \{s' \square a, s' \square b\} \xrightarrow{(f', g')} \{c, d\}$$

is declared to be the same map in $S^{-1}S$ from $\{a, b\}$ to $\{c, d\}$ if and only if there exists some isomorphism $\theta : s \rightarrow s'$ such that $f = f' \circ (\theta \square a)$, $g = g' \circ (\theta \square b)$.

Heuristically, we view $\{a, b\} \in S^{-1}S$ as representing $a - b$, so that $\{s \square a, s \square b\}$ also represents $a - b$. Moreover, we are forcing morphisms in $S$ to be invertible in $S^{-1}S$. If we were to apply this construction to the natural numbers $\mathbb{N}$ viewed as a discrete category with addition as the operation, then we get $\mathbb{N}^{-1} \mathbb{N} = \mathbb{Z}$.

We briefly recall the construction of the “classifying space” of a (small) category $\mathcal{C}$. Namely, we associate to $\mathcal{C}$ its “nerve” $N\mathcal{C}$, the simplicial set whose set of 0-simplices is the set of objects of $\mathcal{C}$ and whose set of $n$-simplices for $n > 0$ is the set of sequences of maps $X_0 \rightarrow \cdots \rightarrow X_n$ in $\mathcal{C}$; face maps are given by either dropping an end object or composing adjacent maps; degeneracies are given by inserting an identity map. We then define $B\mathcal{C}$ to be the geometric realization of the nerve of $\mathcal{C}$,

$$B\mathcal{C} \equiv |N\mathcal{C}|.$$ 

Since the geometric realization functor $|-| : \{simplicial\ \text{sets}\} \rightarrow \{spaces\}$ takes values in C.W. complexes, $B\mathcal{C}$ is a C.W. complex.

An informative example is the category $\mathcal{C}(P)$ of simplices of a polyhedron (objects are simplices, maps are inclusions). Then $B\mathcal{C}(P)$ can be identified with the first barycentric subdivision of $P$.

Theorem 4.16. (Quillen) Let $S$ be a symmetric monoidal category with the property that for all $s, t \in S$ the map $s \square - : \text{Aut}(t) \rightarrow \text{Aut}(s \square t)$ is injective. Then the natural map $BS \rightarrow B(S^{-1}S)$ is a homotopy-theoretic group completion.

In particular, if $S$ denotes the category whose objects are finite dimensional projective $R$-modules and whose maps are isomorphisms (so that $BS = \prod_{[P]} B\text{Aut}(P)$), then $K(R)$ is homotopy equivalent to $B(S^{-1}S)$.

Quillen proves this theorem using ingenious techniques of recognizing when a functor between categories induces a homotopy equivalence on classifying spaces and when a triple $\mathcal{D} \rightarrow \mathcal{D} \rightarrow \mathcal{E}$ of categories determines a fibration sequence of spaces (i.e., yields a long exact sequence of homotopy groups). This leads us to Quillen’s $Q$-construction which applies to any small exact category (e.g., the category $\mathcal{P}(X)$ of algebraic vector bundles on a variety $X$, or the category $\mathcal{M}(X)$ of coherent sheaves on $X$).

Definition 4.17. An exact category $\mathcal{P}$ is an additive category equipped with a family of “exact sequences” $0 \rightarrow X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow 0$ in $\mathcal{P}$ which satisfies the following: there exists an embedding of $\mathcal{P}$ in an abelian category $\mathcal{A}$ such that

- A sequence $0 \rightarrow X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow 0$ is $\mathcal{P}$ is exact if and only if it is exact in $\mathcal{A}$.
- If $0 \rightarrow A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow 0$ is an exact sequence in $\mathcal{A}$ with $A_1, A_3 \in \mathcal{P}$, then this is an exact sequence of $\mathcal{P}$ (in particular, $A_2 \in \mathcal{P}$).
If a map \( i : X_1 \rightarrow X_2 \) in \( \mathcal{P} \) fits in an exact sequence of the form \( 0 \rightarrow X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow 0 \) in \( \mathcal{P} \) then we say that \( i \) is an admissible monomorphism and write it as \( X_1 \overset{i}{\rightarrow} X_2 \); if \( j : X_2 \rightarrow X_3 \) in \( \mathcal{P} \) fits in an exact sequence of the form \( 0 \rightarrow X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow 0 \) in \( \mathcal{P} \), then we say that \( j \) is an admissible epimorphism and write it as \( X_2 \overset{j}{\rightarrow} X_3 \).

If \( S \) is a symmetric monoidal category, then \( S \) is an exact category when provided with the class of split exact sequences (i.e., those sequences isomorphic to \( e \rightarrow X_1 \rightarrow X_1 \sqcup X_2 \rightarrow X_2 \rightarrow e \)).

**Definition 4.18.** Let \( \mathcal{P} \) is an exact category, we define \( Q\mathcal{P} \) to be the category whose objects are the same as the objects of \( \mathcal{P} \) and whose maps \( X \rightarrow Y \) are equivalence classes of diagrams \( X \leftarrow Z \rightarrow Y \). Two such diagram are equivalent (and thus determine the same morphism in \( Q\mathcal{P} \) provided that there is an isomorphism between the two diagrams which is the identity on both \( X \) and \( Y \) (i.e., an isomorphism \( f : Z \rightarrow Z' \) such that \( i = i' \circ f, j = j' \circ f \)).

If \( Y \overset{\sim}{\rightarrow} W \overset{\sim}{\rightarrow} S \) is another morphism in \( Q\mathcal{P} \) then the composition is defined to be the naturally induced \( X \overset{\sim}{\rightarrow} X \times_Y W \overset{\sim}{\rightarrow} S \).

**Remark 4.19.** We can identify morphisms \( X \rightarrow Y \) in \( Q\mathcal{P} \) with an isomorphism of \( Y_2/Y_1 \cong X \) where \( Y_1 \rightarrow Y_2 \rightarrow Y \) is an “admissible layer” of \( Y \).

**Theorem 4.20.** Let \( R \) be a ring and let \( \mathcal{P}(R) \) denote the exact category of finitely generated projective \( R \)-modules. Let \( S \) denotes the category whose objects are those of \( \mathcal{P}(R) \) and whose maps are the isomorphisms of \( \mathcal{P}(R) \). Then there is a natural homotopy equivalence

\[
B(S^{-1}S) \simeq \Omega BQ\mathcal{P}(R).
\]

In particular,

\[
K_i(R) \equiv \pi_i(K(R)) \simeq \pi B(S^{-1}S) \simeq \pi_{i+1}BQ\mathcal{P}(R).
\]

**Definition 4.21.** For any variety (or scheme) \( X \), define the \( K \)-theory space \( K(X) \) to be \( \Omega BQ(\mathcal{P}(X)) \), the loop space on the classifying space of the Quillen construction applied to the exact category of algebraic vector bundles over \( X \). Moreover, define \( K_i(X) \) by

\[
K_i(X) \equiv \pi_{i+1}K(X)
\]

and define \( K_i(X, \mathbb{Z}/n) \) by

\[
K_i(X, \mathbb{Z}/n) \equiv \pi_{i+1}(K(X), \mathbb{Z}/n), n > 0; \ K_0(X, \mathbb{Z}/n) = K_0(X) \otimes \mathbb{Z}/n).
\]

We conclude this section with the following theorem of Quillen, extending work of Bloch. This gives another hint of the close connection of algebraic \( K \)-theory and algebraic cycles.

**Theorem 4.22.** Let \( X \) be a smooth algebraic variety over a field. For any \( i \geq 0 \), let \( K_i \) denote the sheaf on \( X \) (for the Zariski topology) sending an open subset \( U \subset X \) to \( K_i(U) \). Then there is a natural isomorphism

\[
CH^i(X) = H^i(X, K_i)
\]

relating the Chow group of codimension \( i \) cycles on \( X \) and the Zariski cohomology of the sheaf \( K_i \). In particular, for \( i = 1 \), this becomes the familiar identification \( \text{Pic}(X) = H^1(X, \mathcal{O}_X) \).
References


Department of Mathematics, Northwestern University, Evanston, IL 60208
E-mail address: eric@math.nwu.edu