6. Lecture 6

6.1. **cdh and Nisnevich topologies.** These are Grothendieck topologies which play an important role in Suslin-Voevodsky's approach to not only motivic cohomology, but also to Morel-Voevodsky's formulation of $\mathbb{A}^1$-homotopy theory.

Earlier, when considering the algebraic singular complex and the relationship between its cohomology and that of etale cohomology, we utilized two Grothendieck topologies of Voevodsky (the h and the qfh topologies). We now consider Grothendieck topologies which are "completely decomposed"; in other words, each covering $\{U_i \to X\}$ in these topologies has the property that for every point $x \in X$ there is some u in some U_i mapping to x such that the induced map on residue fields $k(x) \to k(u)$ is an isomorphism.

The completely decomposed version of of the h-topology is the *cdh*-topology whereas the completely decomposed version of the etale topology is the Nisnevich topology (considered many years earlier by Nisnevich [5]).

Definition 6.1. A *Nisnevich covering* $\{U_i \to X\}$ of a scheme X is an etale covering with the property that every for every point $x \in X$ there exists some $u_x \in U_i$ mapping to x inducing an isomorphism $k(x) \to k(u_x)$ of residue fields. Such coverings determine the *Nisnevich topology* on the category of Sch/k of schemes of finite type over k.

The *cdh*-topology is the minimal Grothendieck topology of Sch/k for which Nisnevich coverings are coverings in this topology as are pairs $\{i_Y : Y \to X, \pi : X' \to X\}$ consisting of a closed immersion i_Y and a proper map π such that π restricted to $\pi^{-1}(X - X)$ is an isomorphism.

Examples of *cdh*-coverings which are not Nisnevich coverings include the closed immersion $X_{red} \to X$, the closed covering of a reducible variety by its irreducible components, and abstract blow-ups (see below).

It is instructive to observe that the "localization" of a scheme X at a point $x \in X$ is the spectrum of the local ring $\mathcal{O}_{X,x}$ if we consider the Zariski topology, is the spectrum of the *henselization* of $\mathcal{O}_{X,x}$ if we consider the Nisnevich topology, and is the the spectrum the *strict henselization* of $\mathcal{O}_{X,x}$ if we consider the etale topology.

The role of the Nisnevich topology can be seen in the following fundamental result of Voevodsky, a result whose analogue for the Zariski topology is not valid.

Theorem 6.2. [10, 3.1.3] *For any $X \in (Sch/k)$, denote by $L(X) : (SmCor/k) \to Ab$ be the representable presheaf with transfers which sends Y to the free abelian group of finite correspondences from $Y \to X$. Then if $\{U_i \to X\}$ is a Nisnevich cover and if $U = \coprod_i U_i$, then the complex*

$$\cdots \to L(U \times_X U \times_X U) \to L(U \times_X U) \to L(X)$$

is an exact sequence of Nisnevich sheaves with transfer.

Proof. The essence of the proof is that any finite surjective map $\operatorname{Spec} A \to \operatorname{Spec} R$ with R henselian admits a splitting, so that the above complex of sheaves splits locally. The local splittings give local homotopy contracting maps. □

Observe that the above argument does not apply if we replace the Nisnevich topology by the Zariski topology.

One can identify the fundamental role of the Nisnevich coverings in a somewhat different manner. Namely, if $Y \to X$ is a regular closed embedding of smooth varieties of dimension $e, d = e + c$, then this embedding is locally in the Nisnevich topology isomorphic to the linear embedding of affine spaces $(1,0) : \mathbb{A}^e \subset \mathbb{A}^e \times \mathbb{A}^c = \mathbb{A}^d$. This is proved by observing that any point of Y admits a Zariski open neighborhood $U \subset X$ together with an etale map $U \to \mathbb{A}^d$ such that the $U \cap Y$ maps to $\mathbb{A}^e \subset \mathbb{A}^d$.

Another special property of the Nisnevich topology is described in the following fundamental result of Voevodsky [9, 5.3]: if F is a presheaf with transfers, then $U \mapsto H^i_{Nis}(U, F_{Nis})$ admits a natural structure of a presheaf with transfers.

The *cdh*-topology is particularly useful for incorporating blow-ups: an *abstract blowup* of a scheme X consists of a nowhere dense closed immersion $i_Y : Y \to X$ together with a proper surjective map $\pi : X' \to X$ with the property that $p^{-1}(X - Y)_{red} \to (X - Y)_{red}$ is an isomorphism. In particular, any abstract blow-up yields a *cdh*-covering. Voevodsky has another theorem [9, 5.19] asserting that for any abelian Nisnevich sheaf one obtains a long exact sequence associated to the abstract blow-up of a smooth scheme along a smooth closed subscheme.

Finally, one can often replace Nisnevich cohomology by Zariski cohomology thanks to a difficult theorem of Voevodsky [9, 5.7] asserting that if $\mathcal{F}$ is a homotopy invariant presheaf with transfers, then $H^i_{Zar}(X, F_{Zar}) \to H^i_{Nis}(X, F_{Nis})$ is an isomorphism. This leads to [9, 5.11] asserting that if $\mathcal{F}$ is a presheaf with transfers whose associated sheaf F_{Nis} vanishes, then $h_i(F(- \times \Delta^\bullet))_{Zar} = 0$.

6.2. **Motivic cohomology.** Given an algebraic variety, we define its Voevodsky *motive* as follows, inspired by the previous discussion of Suslin homology.

Definition 6.3. Let X be a quasi-projective variety over k and let

$$c_{equi}(X, 0) : (Sm/k) \to Ab$$

be the presheaf taking a smooth variety U to $c_{equi}(X, 0)(U)$, the abelian group of finite correspondences from U to X. We define the motive of X, $M(X)$ by

$$M(X) \equiv C_*(c_{equi}(X, 0)) : (Sm/k)^{op} \to C_*(Ab)$$

(where $C_*(c_{equi}(X, 0))$ sends a smooth variety U to the normalized chain complex of the simplicial abelian group $c_{equi}(X, 0)(U \times \Delta^\bullet)$).

Remark 6.4. $c_{equi}(X, 0)$ is actually a sheaf for the qfh topology; in particular, it is a presheaf with transfers.

Remark 6.5. If X is normal, then the Suslin complex $Sus_*(X)$ equals the evaluation of $M(X)$ on $\operatorname{Spec} k$. Thus, the homology of $Sus_*(X)$, which we studied earlier and called the "algebraic singular complex of X", can be seen as the homological counter-point of motivic cohomology as defined below.

Definition 6.6. For a given $q > 0$, define F_q to be the sum of the q natural embeddings

$$c_{equi}((\mathbb{A}^1 - \{0\})^{q-1},) \to c_{equi}((\mathbb{A}^1 - \{0\})^q, 0)$$

and define

$$\mathbb{Z}(q) = C_*(c_{equi}((\mathbb{A}^1 - \{0\})^q, 0)/F_q)[-q].$$

(we view $\mathbb{Z}(q)$ as the motive of the "q-fold smash product of $\mathbb{G}_m$").

If X is a smooth quasi-projective variety over k, then we define the motivic cohomology of X by

$$H^p(X, \mathbb{Z}(q)) \equiv H^p_{Zar}(X, \mathbb{Z}(q)).$$

For X not necessarily smooth, we consider the sheafification of $\mathbb{Z}(q)$ in the *cdh*-topology (to incorporate resolution of singularities) and define motivic cohomology by

$$H^p(X, \mathbb{Z}(q)) \equiv H^p_{cdh}(X, \mathbb{Z}(q)_{cdh}).$$

Suslin and Voevodsky verify that the presheaves $c_{equi}(X, 0)$ satisfy numerous good formal properties. These apply more generally to the qfh-sheaves

$$z_{equi}(X, r) : (Sm/k) \to Ab$$

sending a smooth variety U to the free abelian group on the integral closed subschemes of $U \times X$ which are dominant with r-dimensional fibres onto some component of U.

property **Proposition 6.7.** [7, 4.37] *Let X be a smooth variety over k and let $\{U, V \subset X\}$ be a Zariski open covering. Then the short exact sequence of sheaves*

$$0 \to c_{equi}(U \cap V, 0) \to c_{equi}(U, 0) \oplus c_{equi}(V, 0) \to c_{equi}(X, 0) \to 0$$

is exact in the Nisnevich topology (as well as the cdh-topology) , thereby determining a Mayber-Vietoris long exact sequence in Zariski sheaf cohomology.

If $Y \subset X$ is a closed subvariety of the smooth variety X, then for any $r \geq 0$ there is an exact sequence of sheaves in the cdh-topology

$$0 \to z_{equi}(Y, r)_{cdh} \to z_{equi}(X, r)_{cdh} \to z_{equi}(X - Y)_{cdh} \to 0$$

thereby determining a localization long exact sequence in cdh sheaf cohomology.

Finally, if X is a quasi-projective variety over k and $Z \subset X$ is a closed subvariety and if $f : X' \to X$ is a proper morphism whose restriction $f^{-1}(X - Z) \to X - Z$ is an isomorphism, then there are short exact sequences of sheaves in the cdh topology for any $r \geq 0$

$$0 \to c_{equi}(f^{-1}(Z), 0)_{cdh} \to c_{equi}(X', 0)_{cdh} \oplus c_{equi}(Z, 0)_{cdh} \to c_{equi}(X, 0) \to 0$$

$$0 \to z_{equi}(f^{-1}(Z), r)_{cdh} \to z_{equi}(X', r)_{cdh} \oplus z_{equi}(Z, r)_{cdh} \to z_{equi}(X, r) \to 0$$

thereby giving blow-up long exact sequences in cdh sheaf cohomology.

When combined with the following fundamental theorem of Voevodsky, we conclude localization, Mayer-Vietoris, and blow-up exact sequences for motivic cohomology.

sheaf-acyclic **Theorem 6.8.** *Let P be a presheaf with transfers over a field of characteristic 0 with the property that $P_{cdh} = 0$. Then the complex of sheaves in the Zariski topology, $C_*(P)_{Zar}$, is acyclic.*

Proposition 6.7 and Theorem 6.8 enable us to prove various basic properties of $H^p(X, \mathbb{Z}(n))$. In particular, we have the following alternate formulations of motivic cohomology of a smooth scheme. The first asserted isomorphism follows from Zariski descent for $\mathbb{Z}(q)$ together with Theorem 6.8, whereas the second is a consequence of localization for $\mathbb{P}^{q-1} \subset \mathbb{P}^q$ with open complement $\mathbb{A}^q$.

Proposition 6.9. *If X is a smooth quasi-projective variety over k, then*

$$H^p(X, \mathbb{Z}(q)) \simeq h_{-p}(\mathbb{Z}(q)(X)) \simeq h_{-p}(C_*(z_{equi}(A^q, 0)(X)[-2q]).$$

As mentioned previously, these motivic cohomology groups agree with Bloch's higher Chow groups for smooth varieties, and therefore the Suslin-Voevodsky complexes satisfy most of the conjectures of Beilinson. Indeed, the key property of localization for Bloch's higher Chow groups is best proved using motivic cohomology. This comparison is achieved by means of a moving lemma argument of Suslin's given in [6], showing that cycles in Bloch's $z^q(X,n) \subset Z^q(X \times \Delta^n)$ can be moved to be equidimensional over X provided that $n \geq q$.

6.3. **Multi-relative K-theory and the Bloch-Lichtenbaum exact couple.** If $f : S \to T$ is a map of topological spaces, then there is a standard construction of a mapping fibration $\tilde{f} : \tilde{S} = S \times_{T\times\{0\}} T^I \to T$ together with a natural homotopy equivalence $S \to \tilde{S}$ over T. Granted a base point $t_0 \in T$, the *homotopy fibre* of f is given by

$$fib(f) \;=\; \tilde{f}^{-1}(t_0).$$

Similarly, consider f as a pointed map $f : (S, s_0) \to (T, t_0)$; then the *homotopy cofibre* of f is the reduced mapping cone

$$cofib(f) = cyl(f)/(S \times \{0\} \cup \{s\} \times I)$$

where the mapping cylinder is defined as $cyl(f) = S \times I \cup_{S\times\{1\}} T$.

If $i : Y \subset X$ is a closed immersion of schemes, then we define the *relative K-theory* $K_*(X,Y)$ to be the homotopy groups of the homotopy fibre of the map of K-theory spectra $\mathcal{K}(X) \to \mathcal{K}(Y)$ (where we define the 0-th space of the K-theory spectrum $\mathcal{K}(X)$ to be the geometric realization of the nerve of the Q-construction applied to the exact category $\mathcal{P}_X$ of coherent, locally free $\mathcal{O}_X$-modules).

For a non-negative integer n, an n-cube of pointed spaces (or spectra) is a functor from the category of subsets of $\{0,\ldots,n-1\}$ to pointed spaces. If $Y_{\bullet,\ldots,\bullet}$ is an n-cube of pointed spaces, then we inductively define $fib(Y_\bullet)$ as the homotopy fibre of the structure map induced by d_{n-1}

$$fib(Y_\bullet) \equiv fib(d_{n-1} : fib(Y_{\bullet,\ldots,\bullet,1}) \to fib(Y_{\bullet,\ldots,\bullet,0})),$$

thereby generalizing the usual homotopy fibre construction (obtained by taking $n = 1$). Similarly, we inductively define the homotopy cofibre $cof(Y_{\bullet,\ldots,\bullet})$ of the n-cube of pointed spaces as the homotopy cofibre

$$cofib(Y_\bullet) \equiv cofib(d_{n-1} : cofib(Y_{\bullet,\ldots,\bullet,1}) \to cofib(Y_{\bullet,\ldots,\bullet,0})),$$

If $X_\bullet$ is a simplicial space, then we define the associated n-cube (for any $n \geq 0$

$$Y_{\bullet,\bullet,\ldots,\bullet} = cube_n(X_\bullet)$$

by setting $Y_{i_0,i_1,\ldots,i_{n-1}} = X_{i_0+i_1+\cdots+i_{n-1}}$ and defining the maps of the cube in a natural way in terms of the face maps of $X_\bullet$.

Definition 6.10. Let X be a scheme and $Y_0,\ldots,Y_{n-1}$ be n closed subschemes. Define a n-cube of spectra $\mathcal{K}(X;\underline{Y})$ by associating to the subset $S \subset \{0,1,\ldots,n-1\}$ the spectrum $\mathcal{K}(\cap_{i\notin S} Y_i)$ and define the mulit-relative K-theory spectrum as

$$\mathcal{K}(X;Y_0,\ldots,Y_{n-1}) \equiv fib(\mathcal{K}(X;\underline{Y})).$$

We mention two important examples. Let Δ^n be the (algebraic) n-simplex, let $\partial(\Delta^n)$ denote the collection of the $n+1$-faces of Δ^n, and let $\Lambda(\Delta^n)$ denote the collection of all faces of Δ^n except the one given by $t = 0$. Then

$$\mathcal{K}(\Delta^n, \partial(\Delta^n)) = fib(cube_{n+1}(\mathcal{K}(\Delta^\bullet)),$$

$$\mathcal{K}(\Delta^n\; Lambda(\Delta^n) = fib(\{cube_{n+1}(\mathcal{K}(\Delta^\bullet)\}_1),$$

where $\{cube_{n+1}X_\bullet\}_1$ is the n-cube obtained by restricting to those subsets of $\{0, 1, \dots, n\}$ which contain 1.

We need to add one more complication to multi-relative K-theory. Namely, we consider the K-theory of a scheme X with supports in a family $\mathcal{C}$ of closed subschemes. Perhaps the best way to define this (following [8]) is to define K in terms of the category of complexes of vector bundles and to define K-theory with supports in terms of those complexes which are acyclic outside the union of the subschemes in $\mathcal{C}$. In this way, we get a functor $\mathcal{K}^{\mathcal{C}}(-)$ from schemes to spectra.

The following difficult theorem of Bloch and Lichtenbaum involves very intricate moving lemma arguments.

bl-exact

Theorem 6.11. [1] *Let F be a field and let*

$$K_0^{\mathcal{C}^q}(\Delta^n, \partial(\Delta^n)) = \pi_0(\mathcal{K}^{\mathcal{C}^q}(\Delta^n, \partial(\Delta^n)))$$

denote the 0-th multi-relative K-group of $Spec\, F$ with supports $\mathcal{C}^q$ consisting of those closed subschemes of Δ^n which meet each face of Δ^n in codimension $\geq q$. Define $K_0^{\mathcal{C}^q}(\Delta^n, \Lambda(\Delta^n))$ similarly. Then there is a natural long exact sequence (for each $n > 0$)

$$\cdots \to K_0^{\mathcal{C}^{q+1}}(\Delta^n, \partial(\Delta^n)) \xrightarrow{i} K_0^{\mathcal{C}^q}(\Delta^n, \partial(\Delta^n)) \xrightarrow{j} K_0^{\mathcal{C}^q}(\Delta^n, \Lambda(\Delta^n))$$

$$\xrightarrow{k} K_0^{\mathcal{C}^q}(\Delta^{n-1}, \partial(\Delta^{n-1})) \xrightarrow{i} K_0^{\mathcal{C}^{q-1}}(\Delta^{n-1}, \partial(\Delta^{n-1})) \to \cdots$$

Moreover, $K_0^{\mathcal{C}^q}(\Delta^n, \Lambda(\Delta^n))$ can be naturally identified with the n-th term of the complex $z^q(Spec\, F, *)_{norm}$, the normalization of the Bloch's complex which computes higher Chow groups $CH^q(Spec\, F, *)$. Under this identification, the composition $j \circ k : z^q(Spec\, F, n)_{norm} \to z^q(Spec\, F, n-1)_{norm}$ is the differential of Bloch's complex.

Patching together the exact sequences of Theorem 6.11, we obtain an exact couple and thus a spectral sequence, the motivic spectral sequence for X the spectrum of a field.

Theorem 6.12. [1] *Let F be a field. Then there is an exact couple (homologically indexed)*

$$D^1 = \bigoplus_{p,q} K_0^{\mathcal{C}^q}(\Delta^{p+q}, \partial(\Delta^{p+q})) \xrightarrow{i} \bigoplus_{p,q} K_0^{\mathcal{C}^q}(\Delta^{p+q}, \partial(\Delta^{p+q})) = D^1$$

$$E^1 = \bigoplus_{p,q} z^q(Spec\, F, p+q)_{norm}.$$

The associated spectral sequence has the form

$$E^2_{p,q} = CH^q(Spec\, F, p+q) \Rightarrow K_{p+q}(F).$$

A key step to globalization of this spectral sequence is the following theorem of Friedlander-Suslin. The proof of this result involves the identification of low dimensional homotopy groups of the homotopy fibres of the n-cube associated to a simplicial spectrum in terms of the homotopy groups (shifted by n) of the homotopy cofibre of this n-cubes which in turn are reinterpreted in terms of the homotopy groups (shifted by 1) of the total space of the simplicial spectrum.

Theorem 6.13. *The first derived exact couple of the Bloch-Lichtenbaum exact couple is given by*

$$D^2_{p,q} = K^{\mathcal{C}^{q+1}}_{p+q}(\Delta^\bullet) \equiv \pi_{p+q}(\mathcal{K}^{\mathcal{C}^{q+1}}(\Delta^\bullet)).$$

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