5. Lecture 5

5.1. Quillen’s localization theorem and Bloch’s formula. Our next topic is a sketch of Quillen’s proof of Bloch’s formula, which is also a brief discussion of aspects of Quillen’s remarkable paper [4]. For $K_2$, this formula was proved by Bloch in [2].

We remind the reader that in this paper Quillen introduces the “Quillen Q-construction” $\mathcal{Q}\mathcal{E}$ of an exact category $\mathcal{E}$ and defines the $K$-groups of $\mathcal{E}$ to be the homotopy groups of the classifying space of $\mathcal{Q}\mathcal{E}$. Of most importance to us are the abelian category $\mathcal{M}_X$ of coherent sheaves on a Noetherian scheme $\mathcal{O}_X$ and the exact subcategory $\mathcal{P}_X \subset \mathcal{M}_X$, yielding

$$K_*(X) = \pi_*(B\mathcal{Q}\mathcal{P}_X), \quad K'_*(X) = \pi_*(B\mathcal{Q}\mathcal{M}_X).$$

A key ingredient in Quillen’s proof of Bloch’s formula is the localization sequence for $K'_*$, extending known localization sequences for low $K$-groups. Quillen formulates his results in an abstract, categorical setting.

**Theorem 5.1.** (Localization Theorem of Quillen, cf. [4]) Let $\mathcal{A}$ be an abelian category and $\mathcal{B} \subset \mathcal{A}$ a Serre subcategory with quotient category $\mathcal{A}/\mathcal{B}$. Then there is a long exact sequence of Quillen $K$-groups

$$\cdots \rightarrow K_1(\mathcal{A}) \rightarrow K_1(\mathcal{A}/\mathcal{B}) \rightarrow K_0(\mathcal{B}) \rightarrow K_0(\mathcal{A}) \rightarrow K_0(\mathcal{A}/\mathcal{B}) \rightarrow 0.$$

In conjunction with Quillen’s “devissage theorem”, this localization theorem implies the following:

**Theorem 5.2.** Consider $X \in (\text{Sch}/k)$ and let $\mathcal{M}_r(X)$ denote the Serre subcategory of the category $\mathcal{M}_X$ consisting of coherent sheaves whose support has codimension $\geq r$. Then there is a natural long exact sequence

$$\cdots \rightarrow \coprod_{x \in X_r} K_{i+1}(x) \rightarrow K_i(\mathcal{M}_{r+1}(X)) \rightarrow K_i(\mathcal{M}_r(X)) \rightarrow \coprod_{x \in X_r} K_i(x) \rightarrow \cdots$$

Here, $X_r$ denotes the set of points of $X$ of codimension $r$.

Correspondingly, there is a spectral sequence

$$E_1^{p,q}(X) = \coprod_{x \in X_p} K_{-p-q}(x) \Rightarrow K'_{-n}(X)$$

relating the $K$-theory of the residue fields of points of $X$ to the $K'$-theory of $X$.

**Proof.** Quillen’s devissage theorem tells us that $K_i(\mathcal{M}_r(X)/\mathcal{M}_{r+1}(X))$ is naturally isomorphic to $\coprod_{x \in X_r} K_i(x)$, where $\mathcal{M}_r(X)/\mathcal{M}_{r+1}(X)$ is the natural exact couple. The asserted exact sequences patch together to give an exact couple, with the indexing of the spectral sequence determined by this exact couple.

**Definition 5.3.** The Gersten complex for $K'_n$ is the complex

$$0 \rightarrow K'_n X \rightarrow \coprod_{x \in X_0} K_0(x) \rightarrow \coprod_{x \in X_1} K_{-1}(x) \rightarrow \cdots \rightarrow \coprod_{x \in X_n} K_0(x) \rightarrow 0$$

determined by the exact sequences of Theorem 5.2.

Essentially by inspection, we have the following theorem concerning the relationship of the spectral sequence of Theorem 5.2 and the exactness of the Gersten complex.
Proposition 5.4. Let $X \in (\text{Sch}/k)$. Then the following conditions are equivalent:

1.) For every $r \geq 0$, the inclusion $\mathcal{M}_{r+1}(X) \rightarrow \mathcal{M}_r(X)$ induces the zero map on $K$-groups.

2.) In the spectral sequence of Theorem 5.2, for all $q, E_2^{p,q} = 0$ for $p > 0$ and the edge homomorphism $K_{-q}^n X \rightarrow E_2^{0,q} X$ is an isomorphism.

3.) The Gersten complex for $X$ is exact.

Here is Quillen’s theorem establishing the validity of Bloch’s formula.

Theorem 5.5. (Bloch’s formula by Quillen [4]) Let $X \in \text{Sch}(k)$ be regular. Then there is a canonical isomorphism

$$H^q(X, K_q) \simeq CH^q(X)$$

where $K_q$ is the sheaf on $X$ (for the Zariski topology) associated to the presheaf $U \mapsto \mathcal{K}(U)$.

Proof. Granted the above analysis of the Quillen spectral sequence, there are two additional ingredients in the proof.

The first is Quillen’s theorem that the Gersten resolution is exact for $\text{Spec } O_X$, whenever $X \in \text{Sch}(k)$ and $x \in X$ is a regular point. This tells us that the Gersten complex for $K_n(X)$ becomes a resolution of $K_n(X)$ by flasque sheaves

$$0 \rightarrow K_n X \rightarrow \prod_{x \in X_0} i_x k(x) \rightarrow \prod_{x \in X_1} i_x k_{n-1}(x) \rightarrow \cdots$$

Consequently, the $E_2$-term of the Quillen spectral sequence has the form

$$E_2^{p,q}(X) = H^{p-q}(X, K_{-q}) \Rightarrow K_{-p-q}(X).$$

The second is Quillen’s determination of the last differential in the Gersten complex

$$\prod_{x \in X_{n-1}} K_1 k(x) \rightarrow \prod_{x \in X_n} K_0 k(x) = Z^{q}(X).$$

Quillen concludes that the image of this map is precisely the codimension $q$ cycles rationally equivalent to 0. □

5.2. Derived categories. In order to formulate motivic cohomology, we need to introduce the language of derived categories. Let $\mathcal{A}$ be an abelian category (e.g., the category of modules over a fixed ring) and consider the category of chain complexes $CH^\bullet(\mathcal{A})$. We shall index our chain complexes so that the differential has degree $+1$. We assume that $\mathcal{A}$ has enough injectives and projectives, so that we can construct the usual derived functors of left exact and right exact functors from $\mathcal{A}$ to another abelian category $\mathcal{B}$. For example, if $F : \mathcal{A} \rightarrow \mathcal{B}$ is right exact, then we define $L_i F(A)$ to be the $i$-th homology of the chain complex $F(P_\bullet)$ obtained by applying $F$ to a projective resolution $P_\bullet \rightarrow A$ of $A$; similarly, if $G : \mathcal{A} \rightarrow \mathcal{B}$ is left exact, then $R^i G(A) = H^i(\mathcal{I}^\bullet)$ where $A \rightarrow I^\bullet$ is an injective resolution of $A$.

The usual verification that these derived functors are well defined up to canonical isomorphism actually proves a bit more. Namely, rather take the homology of the complexes $F(P_\bullet)$, $G(I^\bullet)$, we consider these complexes themselves and observe that they are independent up to quasi-isomorphism of the choice of resolutions. Recall, that a map $C^\bullet \rightarrow D^\bullet$ is a quasi-isomorphism if it induces an isomorphism on homology; only in special cases is a complex $C^\bullet$ quasi-isomorphic to its homology $H^\bullet(C^\bullet)$ viewed as a complex with trivial differential.
We define the derived category $D(A)$ of $A$ as the category obtained from the category of $CH^*(A)$ of chain complexes of $A$ by inverting quasi-isomorphisms. Of course, some care must be taken to insure that such a localization of $CH^*(A)$ is well defined. Let $Hot(CH^*(A))$ denote the homotopy category of chain complexes of $A$. maps from the chain complex $C^\bullet$ to the chain complex $D^\bullet$ in $\mathcal{H}(CH^*(A))$ are chain homotopy equivalence classes of chain maps. Since chain homotopic maps induce the same map on homology, we see that $D(A)$ can also be defined as the category obtained from $Hot(CH^*(A))$ by inverting quasi-isomorphisms.

The derived category $D(A)$ of the abelian category $CH^*(A)$ is a triangulated category. Namely, we have a shift operator $[−][n]$ defined by

$$(A^\bullet[n])^j \equiv A^{n+j}.$$ 

This indexing is very confusing (as would be any other); we can view $A^\bullet[n]$ as $A^\bullet$ shifted “down” or “to the left”. We also have distinguished triangles

$$A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow A^\bullet[1]$$

defined to be those “triangles” quasi-isomorphic to short exact sequences $0 \rightarrow A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow 0$ of chain complexes.

This notation enables us to express $Ext$-groups quite neatly as

$$Ext^i_A(A, B) = H^i(Hom_A(P_\bullet, B)) = Hom_{D(A)}(A[-i], B)$$

$$= Hom_{D(A)}(A, B[i]) = H^i(Hom_A(A, P^\bullet)).$$

5.3. Bloch’s Higher Chow Groups. From our point of view, motivic cohomology should be a “cohomology theory” which bears a relationship to $K_\ast(X)$ analogous to the role Chow groups $CH^\ast(X)$ bear to $K_0(X)$ (and analogous to the relationship of $H^\ast_{sing}(T)$ to $K_{top}(T)$). In particular, motivic cohomology will be doubly indexed.

We now discuss a relatively naive construction by Spencer Bloch of “higher Chow groups” which satisfies this criterion. We shall then consider a more sophisticated version of motivic cohomology due to Suslin and Voevodsky.

We work over a field $k$ and define $\Delta^n$ to be $\text{Spec} k[x_0, \ldots, x_n]/(\sum_i x_i - 1)$, the algebraic $n$-simplex. As in topology, we have face maps $\partial_i : \Delta^{n-1} \rightarrow \Delta^n$ (sending the coordinate function $x_i \in k[\Delta^n]$ to 0) and degeneracy maps $\sigma_j : \Delta^{n+1} \rightarrow \Delta^n$ (sending the coordinate function $x_j \in k[\Delta^n]$ to $x_j + x_{j+1} \in k[\Delta^{n+1}]$). More generally, a composition of face maps determines a face $F \simeq \Delta^i \rightarrow \Delta^n$. Of course, $\Delta^n \simeq A^n$.

Bloch’s idea is to construct a chain complex for each $q$ which in degree $n$ would be the codimension $q$-cycles on $X \times \Delta^n$. In particular, the 0-th homology of this chain complex should be the usual Chow group $CH^q(X)$ of codimension $q$ cycles on $X$ modulo rational equivalence. This can not be done in a completely straightforward manner, since one has no good way in general to restrict a general cycle on $X \times \Delta^n$ via a face map $\partial_i$ to $X \times \Delta^{n-1}$. Thus, Bloch only considers codimension $q$ cycles on $X \times \Delta^n$ which restrict properly to all faces (i.e., to codimension $q$ cycles on $X \times F$).

Definition 5.6. Let $X$ be a variety over a field $k$. For each $p \geq 0$, we define a complex $z_p(X, *)$ which in degree $n$ is the free abelian group on the integral closed subvarieties $Z \subset X \times \Delta^n$ with the property that for every face $F \subset \Delta^n$

$$dim_k(Z \cap (X \times F)) \leq dim_k(F) + p.$$
The differential of \( z_p(X, \ast) \) is the alternating sum of the maps induced by restricting cycles to codimension 1 faces. Define the higher Chow cohomology groups by

\[
CH_p(X, n) = H_n(z_p(X, \ast)), \quad n, p \geq 0.
\]

If \( X \) is locally equi-dimensional over \( k \) (e.g., \( X \) is smooth), let \( z^q(X, n) \) be the free abelian group on the integral closed subvarieties \( Z \subset X \times \Delta^n \) with the property that for every face \( F \subset \Delta^n \)

\[
codim_{X \times F}(Z \cap (X \times F)) \geq q.
\]

Define the higher Chow cohomology groups by

\[
CH^q(X, n) = H_n(z^q(X, \ast)), \quad n, q \geq 0,
\]

where the differential of \( z^q(X, \ast) \) is defined exactly as for \( z_p(X, \ast) \).

Bloch, with the aid of Marc Levine, has proved many remarkable properties of these higher Chow groups.

**Theorem 5.7.** Let \( X \) be a quasi-projective variety over a field. Bloch’s higher Chow groups satisfy the following properties:

- \( CH_p(\ast, \ast) \) is covariantly functorial with respect to proper maps.
- \( CH^q(\ast, \ast) \) is contravariantly functorial on \( \text{Sm}_k \), the category of smooth quasi-projective varieties over \( k \).
- \( CH_p(X, 0) = CH_p(X) \), the Chow group of \( p \)-cycles modulo rational equivalence.
- (Homotopy invariance) \( \pi^* : CH_p(X, \ast) \xrightarrow{\sim} CH_{p+1}(X \times k^1) \).
- (Localization) Let \( i : Y \rightarrow X \) be a closed subvariety with \( j : U = X - Y \subset X \) the complement of \( Y \). Then there is a distinguished triangle

\[
z_p(Y, \ast) \xrightarrow{\sim} z_p(X, \ast) \xrightarrow{j^*} z_p(U, \ast) \rightarrow z_p(Y, \ast)[1]
\]

- (Projective bundle formula) Let \( E \) be a rank \( n \) vector bundle over \( X \). Then \( CH^\ast(P(E), \ast) \) is a free \( CH^\ast(X, \ast) \)-module on generators \( 1, \zeta, \ldots, \zeta^{n-1} \in CH^1(P(E), 0) \).
- For \( X \) smooth, \( K_i(X) \otimes \mathbb{Q} \simeq \oplus_q CH^q(X, i) \otimes \mathbb{Q} \) for every \( i \geq 0 \). Moreover, for any \( q \geq 0 \),

\[
(K_i(X) \otimes \mathbb{Q})^{(q)} \simeq CH^q(X, i) \otimes \mathbb{Q}.
\]

- If \( F \) is a field, the \( K^M_n(F) \simeq CH^n(Spec F, n) \).

The most difficult of these properties, and perhaps the most important, is localization. The proof requires a very subtle technique of moving cycles. Observe that \( z_p(X, \ast) \rightarrow z_p(U, \ast) \) is not surjective because the conditions of proper intersection on an element of \( z_p(U, n) \) (i.e., a cycle on \( U \times \Delta^n \)) might not continue to hold for the closure of that cycle in \( X \times \Delta^n \).

### 5.4. Beilinson’s Conjectures.

We give below a list of conjectures due to Beilinson which relate motivic cohomology and K-theory. Bloch’s higher Chow groups go some way toward providing a theory which satisfies these conjectures. Namely, Beilinson conjectures the existence of complexes of sheaves \( \Gamma_{Zar}(r) \) whose cohomology (in the Zariski topology) \( H^p(X, \Gamma_{Zar}(r)) \) one could call “motivic cohomology”. If we set

\[
H^p(X, \Gamma_{Zar}(r)) = CH^r(X, 2r - p),
\]
then many of the cohomological conjectures Beilinson makes for his conjectured complexes are satisfied by Bloch’s higher Chow groups $CH^\bullet(X, *)$.

**Conjecture 5.8. (Beilinson [1])** Let $X$ be a smooth variety over a field $k$. Then there should exist complexes of sheaves $\Gamma_{\text{Zar}}(r)$ of abelian groups on $X$ with the Zariski topology, well defined in $D(AbSh(X_{\text{Zar}}))$, functorial in $X$, and equipped with a graded product, which satisfy the following properties:

1. $\Gamma_{\text{Zar}}(1) = \mathbb{Z}$; $\Gamma_{\text{Zar}}(1) \simeq G_m[-1]$.
2. $H^{2n}(X, \Gamma_{\text{zar}}(n)) = CH^n(X)$.
3. $H^*(\text{Spec } k, \Gamma_{\text{Zar}}(i)) = K_i^M k$, Milnor $K$-theory.
4. (Motivic spectral sequence) There is a spectral sequence of the form
   $$E_2^{p,q} = H^{p-q}(X, \Gamma_{\text{Zar}}(q)) \Rightarrow K_{-p-q}(X)$$
   which degenerates after tensoring with $\mathbb{Q}$. Moreover, for each prime $\ell$, there is a mod-$\ell$ version of this spectral sequence
   $$E_2^{p,q} = H^{p-q}(X, \Gamma_{\text{Zar}}(q)) \otimes^L \mathbb{Z}/\ell \Rightarrow K_{-p-q}(X, \mathbb{Z}/\ell)$$
5. $\text{gr}^r(K_j(X) \otimes \mathbb{Q} \simeq H^{2r-j}(X_{\text{Zar}}, \Gamma_{\text{Zar}}(r)) \otimes \mathbb{Q}$.
6. (Beilinson-Lichtenbaum Conjecture) $\Gamma_{\text{Zar}} \otimes^L \mathbb{Z}/\ell \simeq \tau_{\leq r} R\pi_* (\mu^\otimes \otimes^L)$ in the derived category $D(AbSh(X_{\text{Zar}}))$ provided that $\ell$ is invertible in $O_X$, where $\pi : X_{et} \to X_{\text{Zar}}$ is the change of topology morphism.
7. (Vanishing Conjecture) $\Gamma_{\text{Zar}}(r)$ is acyclic outside $[1, r]$ for $r \geq 1$.

These conjectures require considerable explanation, of course. Essentially, Beilinson conjectures that algebraic $K$-theory can be computed using a spectral sequence of Atiyah-Hirzebruch type (4) using “motivic complexes” $\Gamma_{\text{Zar}}(r)$ whose cohomology plays the role of singular cohomology in the Atiyah-Hirzebruch spectral sequence for topological $K$-theory. I have indexed the spectral sequence as Beilinson suggests, but we could equally index it in the Atiyah-Hirzebruch way and write (by simply re-indexing)

$$E_2^{p,q} = H^{p-q}(X, \Gamma_{\text{Zar}}(-q/2)) \Rightarrow K_{-p-q}(X).$$

where $\Gamma_{\text{Zar}}(-q/2) = 0$ if $-q$ is not an even non-positive integer and $\Gamma_{\text{Zar}}(-q/2) = \Gamma_{\text{Zar}}(i)$ is $-q = 2i \geq 0$.

(1) and (2) just “normalize” our complexes, assuring us that they extend usual Chow groups and what is known in codimensions 0 and 1. Note that (1) and (2) are compatible in the sense that

$$H^2(X, \Gamma_{\text{Zar}}(1)) = H^2(X, O_X[-1]) = H^1(X, O_X^*) = Pic(X).$$

(3) asserts that for a field $k$, the $n$-th cohomology of $\Gamma_{\text{Zar}}(n)$ – the part of highest weight with respect to the action of Adams operations – should be Milnor $K$-theory. This has been verified for Bloch’s higher Chow groups by Suslin-Nesterenko and Totaro.

The (integral) spectral sequence of (4) has been established thanks to the work of many authors. This spectral sequence “collapses” at the $E_2$-level when tensored with $\mathbb{Q}$, so that $E_2 \otimes \mathbb{Q} = E_{\infty} \otimes \mathbb{Q}$. (5) asserts that this collapsing can be verified by using Adams operations, interpreted using the $\gamma$-filtration.

The vanishing conjecture of (7) is the most problematic, and there is no consensus on whether it is likely to be valid. However, (6) incorporates the mod-$\ell$ version of the vanishing conjecture and has apparently been proved by Rost and Voevodsky.
(6) asserts that if we consider the complexes $\Gamma_{Zar}^r$ modulo $\ell$ (in the sense of the derived category), then the result has cohomology closely related to etale cohomology with $\mu_\ell^{\otimes r}$ coefficients, where $\mu_\ell$ is the etale sheaf of $\ell$-th roots of unity (isomorphic to $\mathbb{Z}/\ell$ if all $\ell$-th roots of unity are in $k$). If the terms in the mod-$\ell$ spectral sequence were simply etale cohomology, then we would get etale $K$-theory which would violate the vanishing conjectured in (7) (and which would imply periodicity in low degrees which we know to be false). So Beilinson conjectures that the terms modulo $\ell$ should be the cohomology of complexes which involve a truncation.

More precisely, $R \pi_* F$ is a complex of sheaves for the Zariski topology (given by applying $\pi_*$ to an injective resolution $F \to I^\cdot$ of etale sheaves) with the property that $H^*_Zar(X, R \pi_* F) = H^*_et(X, F)$. Now, the $n$-th truncation of $R \pi_* F$, $\tau_{\leq n} R \pi_* F$, is the truncation of this complex of sheaves in such a way that its cohomology sheaves are the same as those of $R \pi_* F$ in degrees $\leq n$ and are 0 in degrees greater than $n$. (We do this by retaining coboundaries in degree $n+1$ and setting all higher degrees equal to 0.)

If $X = Spec k$, then $H^p(Spec k, \tau_{\leq n} R \pi_* \mu_\ell^{\otimes n})$ equals $H^p_{et}(Spec k, \mu_\ell^{\otimes n})$ for $p \leq n$ and is 0 otherwise. For a positive dimensional variety, this truncation has a somewhat mystifying effect on cohomology.

It is worth emphasizing that one of the most important aspects of Beilinson’s conjectures is its explicit nature: Beilinson conjectures precise values for algebraic $K$-groups, rather than the conjectures which preceded Beilinson which required the degree to be large or certain torsion to be ignored. Such a precise conjecture should be much more amenable to proof.

References


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