4. Lecture 4

4.1. Intersection product. Let us briefly return to the ring structure on \( CH^r(X) \) for a smooth variety \( X \). This ring structure arises from intersection of cycles, as described in the following theorem.

**Theorem 4.1.** Let \( X \) be a smooth quasi-projective variety of dimension \( d \). Then there exists a pairing

\[
CH_r(X) \otimes CH_s(X) \to CH_{d-r-s},
\]

with the property that if \( Z = [Y], Z' = [W] \) are irreducible cycles of dimension \( r, s \) respectively and if \( Y \cap W \) has dimension \( \leq d - r - s \), then \( Z \cdot Z' \) is a cycle which is a sum with positive coefficients indexed by the irreducible subvarieties of \( Y \cap W \) of dimension \( d - r - s \).

Thus, the intersection product determines an intersection pairing of the form

\[
CH^*(X) \otimes CH^t(X) \to CH^{*+t}(X).
\]

**Proof.** Classically, this was proved by showing the following geometric fact: given a codimension \( r \) cycle \( Z \) and a codimension \( s \) cycle \( W = \sum_j m_j R_j \) with \( r + s \leq d \), there is another codimension \( r \) cycle \( Z' = \sum_i n_i Y_i \) rationally equivalent to \( Z \) (i.e., determining the same element in \( CH^r(X) \)) such that \( Z' \) meets \( W \) “properly”; in other words, every component \( C_{i,j,k} \) of each \( Y_i \cap R_j \) has codimension \( r + s \). One then defines

\[
Z' \cdot W = \sum_{i,j,k} n_i m_j \cdot int(Y_i \cap R_j, C_{i,j,k}) C_{i,j,k}
\]

where \( int(Y_i \cap R_j, C_{i,j,k}) \) is a positive integer determined using local commutative algebra, the intersection multiplicity. Furthermore, one shows that if one chooses a \( Z'' \) rationally equivalent to both \( Z, Z' \) and also intersecting \( W \) properly, then \( Z' \cdot W \) is rationally equivalent to \( Z'' \cdot W \).

In [3], Blaine Lawson and I showed how one could generalize this, considering any finite dimensional family of codimension \( r \) cycles \( Z_\alpha \) and any finite dimensional family of codimension \( s \) cycles \( W_\beta \), moving simultaneously each \( Z_\alpha \) to a suitable \( Z'_\alpha \) so that each \( Z'_\alpha \) meets properly each \( W_\beta \). This is achieved by showing that the classical argument admits parametrizations by large dimensional parameter spaces. This classical argument consists of two steps, in order to move \( Z \) into better position with respect to \( W \). The first step involves a choice of a finite projection \( \pi_L : X \subset \mathbb{P}^N \) onto \( \mathbb{P}^n \), \( n = dim(X) \), which determines the projective cone \( C_L(Z) = \pi_L^*(\pi_L^*(Z)) \subset \mathbb{P}^N \). The first parametrized family is the family of “linear centers” \( L \) parametrizing these projections. The second parametrized family is the family of moves in \( \mathbb{P}^N \) which enables us to move the resulting projecting cone \( C_L(Z) \) in order that \( C_L(Z) \cdot X \) meets \( Y \) properly on \( X \). The variation in the family of projections \( \pi_L \) enables us to arrange that the residual cycle \( R_L(Z) = C_L(Z) \cdot X - Z \) has improved intersection with \( W \) that \( Z \).

A completely different proof is given by William Fulton and Robert MacPherson. (cf [4]). They use a powerful geometric technique discovered by MacPherson called deformation to the normal cone. For \( Y \subset X \) closed, the deformation space \( M_Y(X) \) is a variety mapping to \( \mathbb{P}^1 \) defined as the complement in the blow-up of \( X \times \mathbb{P}^1 \) along \( Y \times \infty \) of the blow-up of \( X \times \infty \) along \( Y \times \infty \). One readily verifies that \( Y \times \mathbb{P}^1 \subset M(X,Y) \) restricts above \( \infty \neq p \in \mathbb{P}^1 \) to the given embedding \( Y \subset X \); and above \( \infty \),
restricts to the inclusion of \( Y \) into the normal cone \( C_Y(X) = \text{Spec}(\oplus_{n \geq 0} T^n Y / T^n Y^*) \), where \( T_Y \subset O_X \) is the ideal sheaf defining \( Y \subset X \). When \( Y \subset X \) is a regular closed embedding, then this normal cone is a bundle, the normal bundle \( N_Y(X) \).

This enables a regular closed embedding (e.g., the diagonal \( \delta : X \to X \times X \) for \( X \) smooth) to be deformed into the embedding of the 0-section of the normal bundle \( N_{\delta(X)}(X \times X) \). One defines the intersection of \( Z, W \) as the intersection of \( \delta(X), Z \times W \) and thus one reduces the problem of defining intersection product to the special case of intersection of the 0-section of the normal bundle \( N_X(X \times X) \) with the normal cone \( N((Z \times W) \cap \delta(X))(Z \times W) \).

4.2. Chern classes and the Chern character. Grothendieck introduced many basic techniques which we now use as a matter of course when working with bundles. The following splitting principle is one such technique, a technique which enable one to frequently reduce constructions for arbitrary vector bundles to those which are a sum of line bundles.

**Proposition 4.2.** Let \( E \) be a rank \( r+1 \) vector bundle on a quasi-projective variety \( X \) and define \( p_1 : \mathbb{P}(E) = \text{Proj}(\text{Sym}_{O_X} E) \to X \) to be the projective bundle of lines in \( E \). Then \( p_i^* : K_i(X) \to K_{i+r}(\mathbb{P}(E)) \) is split injective and \( p_1^* = E \), \( E_1 \) is a direct sum of a rank \( r \) bundle and a line bundle.

Applying this construction to \( E_1 \) on \( \mathbb{P}(E) \), we obtain \( p_2 : \mathbb{P}(E_1) \to \mathbb{P}(E) \); proceeding inductively, we obtain

\[
p = p_r \circ \cdots \circ p_1 : \mathbb{P}(E) = \mathbb{P}(E_{r-1}) \to X
\]

with the property that \( p^* : K_0(X) \to K_0(\mathbb{P}(E)) \) is split injective and \( p^*(E) \) is a direct sum of line bundles.

We now introduce Chern classes and the Chern character, once again following Grothendieck’s point of view.

**Construction 4.3.** Let \( E \) be a rank \( r \) vector bundle on a smooth, quasi-projective variety \( X \) of dimension \( d \). Then \( CH^*(\mathbb{P}(E)) \) is a free module over \( CH^*(X) \) with generators \( 1, \zeta, \zeta^2, \ldots, \zeta^{r-1} \), where \( \zeta \in CH^1(\mathbb{P}(E)) \) denotes the divisor class associated to \( O_{\mathbb{P}(E)}(1) \).

We define the \( i \)-th Chern class \( c_i(E) \in CH^i(X) \) of \( E \) by the formula

\[
CH^*(\mathbb{P}(E)) = CH^*(X)[\zeta]/\sum_{i=0}^{r} (-1)^i p^*(c_i(E)) \cdot \zeta^{r-i}.
\]

We define the total Chern class \( c(E) \) by the formula

\[
c(E) = \sum_{i=0}^{r} c_i(E)
\]

and set \( c_i(E) = \sum_{i=0}^{r} c_i(E)t^i \). Then the Whitney sum formula asserts that \( c_t(E \oplus F) = c_t(E) \cdot c_t(F) \).

We define the Chern roots, \( \alpha_1, \ldots, \alpha_r \) of \( E \) by the formula

\[
c_t(E) = \prod_{i=1}^{r} (1 + \alpha_i t)
\]

where the factorization can be viewed either as purely formal or as occurring in \( \mathbb{P}(E) \). Observe that \( c_k(E) \) is the \( k \)-th elementary symmetric function of these Chern roots.
In other words, the Chern classes of the rank $r$ vector bundle $\mathcal{E}$ are given by the expression for $\zeta^r \in CH^*(\mathbb{P}(\mathcal{E}))$ in terms of the generators $1, \zeta, \ldots, \zeta^{r-1}$. Thus, the Chern classes depend critically on the identification of the first Chern class $\zeta$ of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ and the multiplicative structure on $CH^*(X)$. The necessary structure for such a definition of Chern classes is called an oriented multiplicative cohomology theory. The splitting principle guarantees that Chern classes are uniquely determined by the assignment of first Chern classes to line bundles.

We refer the interested reader to [4] for the definition of “operational Chern classes” defined for bundles on a non necessarily smooth variety.

**Construction 4.4.** Let $X$ be a smooth, quasi-projective variety, let $\mathcal{E}$ be a rank $r$ vector bundle over $X$, and let $\pi : F(\mathcal{E}) \to X$ be the associated bundle of flags of $\mathcal{E}$. Write $\pi^*(\mathcal{E}) = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r$, where each $\mathcal{L}_i$ is a line bundle on $F(\mathcal{E})$. Then $c_i(\pi^*(\mathcal{E})) = \prod_{i=1}^r (1 + c_1(\mathcal{L}_i))^i$.

We define the Chern character of $\mathcal{E}$ as

$$ \text{ch}(\mathcal{E}) = \sum_{i=1}^r \left( 1 + c_1(\mathcal{L}_i) + \frac{1}{2} c_1(\mathcal{L}_i)^2 + \frac{1}{3!} c_1(\mathcal{L}_i)^3 + \cdots \right) = \sum_{i=1}^r \exp(c_1(\mathcal{L}_i)),$$

where this expression is verified to lie in the image of the injective map $CH^*(X) \otimes \mathbb{Q} \to CH^*(\mathbb{P}(\mathcal{E})) \otimes \mathbb{Q}$. (Namely, one can identify $\text{ch}_k(\mathcal{E})$ as the $k$-th power sum of the Chern roots, and therefore expressible in terms of the Chern classes using Newton polynomials.)

Since $\pi^* : K_0(X) \to K_0(F(\mathcal{E})), \pi^* : CH^*(X) \to CH^*(F(\mathcal{E}))$ are ring homomorphisms, the splitting principle enables us to immediately verify that $\text{ch}$ is also a ring homomorphism (i.e., sends the direct sum of bundles to the sum in $CH^*(X)$ of Chern characters, sends the tensor product of bundles to the product in $CH^*(X)$ of Chern characters).

Grothendieck’s formulation of the Riemann-Roch theorem is an assertion of the behaviour of the Chern character $\text{ch}$ with respect to push-forward maps induced by a proper smooth map $f : X \to Y$ of smooth varieties. It is not the case that $\text{ch}$ commutes with these push-forward maps; one must modify the push forward map in K-theory by multiplication by the Todd class.

This modification of the Todd class is necessary even when consideration of the push-forward of a divisor. Indeed, the Todd class

$$ \text{td} : K_0(X) \to A^*(X)$$

(given explicitly for a vector bundle $E$ in terms of the Chern roots $\alpha_i$ of $E$ as $\prod_i \frac{\alpha_i}{1-e^{-\alpha_i}}$) is characterized by the properties that

- i. $\text{td}(L) = c_1(L)/(1 - exp(-c_1(L)));$
- ii. $\text{td}(E_1 \oplus E_2) = \text{td}(E_1) \cdot \text{td}(E_2)$; and
- iii. $\text{td} \circ f^* = f^* \circ \text{td}$.

The reader is recommended to consult [2] for a very nice overview of Grothendieck’s Riemann-Roch Theorem.

**Theorem 4.5.** (Grothendieck’s Riemann-Roch Theorem) Let $f : X \to Y$ be a projective map of smooth varieties. Then for any $x \in K_0(X)$, we have the equality

$$ \text{ch}(f_!(x)) \cdot \text{td}(T_Y) = f_*(\text{ch}(x) \cdot \text{td}(T_X))$$
where $T_X, T_Y$ are the tangent bundles of $X, Y$ and $td(T_X), td(T_Y)$ are their Todd classes.

Here, $f_1 : K_0(X) \rightarrow K_0(Y)$ is defined by identifying $K_0(X)$ with $K_0'(X)$, $K_0(Y)$ with $K_0'(Y)$, and defining $f_1 : K_0'(X) \rightarrow K_0(Y)$ by sending a coherent sheaf $F$ on $X$ to $\sum_i (-1)^i R^i f_* (F)$. The map $f_* : CH_*(X) \rightarrow CH_*(Y)$ is proper push-forward of cycles.

Just to make this more concrete and more familiar, let us consider a very special case in which $X$ is a projective, smooth curve, $Y$ is a point, and $x \in K_0(X)$ is the class of a line bundle $\mathcal{L}$. (Hirzebruch had earlier proved a version of Grothendieck’s theorem in which the target $Y$ was a point.)

**Example 4.6.** Let $C$ be a projective, smooth curve of genus $g$ and let $f : C \rightarrow \text{Spec} \mathbb{C}$ be the projection to a point. Let $\mathcal{L}$ be a line bundle on $C$ with first Chern class $D \in CH^1(C)$. Then

$$f_! ([\mathcal{L}]) = \dim H^0(C, \mathcal{L}) - \dim H^1(C, \mathcal{L}) \in \mathbb{Z},$$

and $ch : K_0(\text{Spec} \mathbb{C}) = \mathbb{Z} \rightarrow A^* (\text{Spec} \mathbb{C}) = \mathbb{Z}$ is an isomorphism. Let $K \in CH^1(C)$ denote the “canonical divisor”, the first Chern class of the line bundle $\Omega_C$, the dual of $T_C$. Then

$$td(T_C) = \frac{-K}{1 - (1 + K + \frac{1}{2} K^2)} = 1 - \frac{1}{2} K.$$  

Recall that $\deg(K) = 2g - 2$. Since $ch([\mathcal{L}]) = 1 + D$, we conclude that

$$f_* (ch([\mathcal{L}] \cdot td(T_C))) = f_* ((1 + D) \cdot (1 - \frac{1}{2} K)) = \deg(D) - \frac{1}{2} \deg(K).$$

(Note that $1 \in CH^0(X)$ is the fundamental class of $X$; that $f_* : CH^*(X) \rightarrow CH^*(\text{Spec} \mathbb{C})$ simply takes the 0-cycle component.) Thus, in this case, Riemann-Roch tell us that

$$\dim \mathcal{L}(C) - \dim H^1(C, \mathcal{L}) = \deg(D) + 1 - g.$$  

For our purpose, Riemann-Roch is especially important because of the following consequence.

**Theorem 4.7.** Let $X$ be a smooth quasi-projective variety. Then

$$ch_* : K_0(X) \otimes \mathbb{Q} \rightarrow CH^*(X) \otimes \mathbb{Q}$$

is a ring isomorphism.

**Proof.** The essential ingredient is the Riemann-Roch theorem. Namely, we have a natural map $CH^*(X) \rightarrow K_0'(X)$ sending an irreducible subvariety $Y$ to the class $[\mathcal{O}_Y]$ of the $\mathcal{O}_X$-module $\mathcal{O}_Y$. We put a filtration on $K_0'(X)$ using the dimension of support of a coherent sheaf $\mathcal{F} \in K_0'(X)$ and conclude using “localization” and “devissage” (see Lecture 5) that this natural map induces a surjection from $CH^*(X)$ to the associated graded group of $K_0'(X)$.

We show that the composition with the Chern character is an isomorphism on $CH^*(X) \otimes \mathbb{Q}$ by applying Grothendieck’s Riemann-Roch theorem to each closed immersion $Y \subset X$, for $Y$ an irreducible smooth subvariety of $X$. Namely, Riemann Roch implies that $ch_* ([\mathcal{O}_Y]) \in CH^*(X)$ has the form the sum of $Y$ and terms of lower dimension. Indeed, this argument applies to irreducible subvarieties $Y$ of $X$ with singularities, by observing that the contribution of singularities is also of higher codimension using a localization sequence and induction.
Thus, the associated graded map of $ch_*$ is an isomorphism, which implies that $ch_*$ is also an isomorphism.

\[ \square \]

**References**


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