3. Lecture 3

3.1. Freely generate qfh-sheaves. We recall that if $F$ is a homotopy invariant presheaf with transfers in the sense of the last lecture, then we have a well defined pairing

$$F(X) \otimes H_0(X/S) \to F(S)$$

given by associating to any irreducible $i : Y \subset X$ finite, surjective over $S$ the mapping $Tr_{Y/S} \circ i^* : F(X) \to F(Y) \to F(S)$. Moreover, recall that any qfh-sheaf admits transfers.

Consider the following presheaf, where $\mathbb{Z}[1/p]$ denotes localization of the residue characteristic of $k$ (so that if $k$ is of characteristic 0, then $\mathbb{Z}[1/p]$ should be read as $\mathbb{Z}$):

$$(Sch/k)^{op} \to Ab, \ Y \mapsto \mathbb{Z}[1/p]Hom_{(Sch/k)}(Y, X).$$

We denote by $F_X$ the associated qfh-sheaf.

The following result of Suslin-Voevodsky identifies this sheaf when applied to normal varieties and relates this to the Suslin complex $Sus_\ast(X)$ of $X$. One can interpret this theorem as saying that cycles in $X \times Y$ each component of which is finite and surjective over a normal variety $Y$ are locally in the qfh-topology a sum of graphs of morphisms from $Y$ to $X$.

**Theorem 3.1.** If $Y \in (Sch/k)$ is normal, then

$$F_X(Y) = C_0(X \times Y/Y) \otimes \mathbb{Z}[1/p].$$

In particular,

$$F_X(\Delta^\bullet) = Sus_\ast(X) \otimes \mathbb{Z}[1/p].$$

Sheafifying in the qfh-topology gives us transfers. The following proposition enables us to obtain presheaves which are homotopy invariant.

**Proposition 3.2.** Assume that $F$ is a presheaf with transfers. Let $(F_\ast)$ denote the complex of sheaves given in degree $q$ as the qfh-sheaf associated to the presheaf $Y \mapsto F(Y \times \Delta^q)$. For any $q \geq 0$, consider the presheaf

$$\mathcal{H}_q((F_\ast)) : (Sch/k)^{op} \to Ab$$

defined as the homology sheaves of the complex $(F_\ast)$ (or, equivalently, the sheaf associated to the presheaf $Y \mapsto H_q(F(Y \times \Delta^\bullet))$). Then $\mathcal{H}_q((F_\ast))$ is a homotopy invariant presheaf with transfers.

**Proof.** One first shows that evaluation at 0, 1 $\in \Delta^1$ determine chain homotopy maps

$$F(Y \times \Delta^\bullet \times \Delta^1) \to F(Y \times \Delta^\bullet).$$

This enables one to show that upon taking qfh-sheaves that $\mathcal{H}_q((F_\ast))$ is homotopy invariant. Moreover, the naturality properties of transfers on $F$ imply that $Tr_{Y/S}$ gives us a transfer map on complexes $F(X \times \Delta^\bullet) \to F(S \times \Delta^\bullet)$, so that taking associated qfh-sheaves gives us transfers on $\mathcal{H}_q((F_\ast))$. $\square$
3.2. Proof of Suslin-Voevodsky theorem. We shall sketch a proof of the following theorem.

**Theorem 3.3.** (Suslin-Voevodsky [2]) Let $F_X$ denote the qfh-sheaf associated to the presheaf

$$(\text{Sch/k})^{op} \rightarrow \text{Ab}, \quad Y \mapsto \mathbb{Z}[1/p] \text{Hom}_{(\text{Sch/k})}(Y, X).$$

Then for any positive integer $n$ invertible in $k$, the natural maps of complexes of qfh-sheaves

$$F_X(\Delta^\bullet) \rightarrow (F_X^\ast) \leftarrow F_X$$

induce isomorphisms of Ext-groups

$$(3.3.1) \quad \text{Ext}^q_{\text{qfh}}(\mathbb{Z}/n) \rightarrow \text{Ext}^q_{\text{qfh}}(\mathbb{Z}/n).$$

Moreover,

$$\text{Ext}^q_{\text{qfh}}(F_X, \mathbb{Z}/n) \simeq \text{Ext}^q(X, \mathbb{Z}/n).$$

**Proof.** The two isomorphisms of (3.3.1) are proved considering the two hypercohomology spectral sequences for $\text{Ext}^q_{\text{qfh}}(\mathbb{Z}/n)$. The first isomorphism follows by comparing the homology at each level of the map of complexes of presheaves with transfers

$$F_X(\Delta^\bullet) \otimes \mathbb{Z}/n \rightarrow (F_X^\ast) \otimes \mathbb{Z}/n,$$

where the left hand complex is viewed as a complex of constant presheaves. Since the homology presheaves of these complexes are presheaves with transfer which are annihilated by multiplication by $n$, we can apply the Suslin-Voevodsky theorem to conclude that the induced map on homology presheaves is an isomorphism.

The second isomorphism does not use the fact that $F_X$ is a presheaf with transfers, but is a general fact that $F_X \rightarrow (F_X(- \times \Delta^0))$ induces an isomorphism in $\text{Ext}^q_{\text{qfh}}(-, \mathbb{Z}/n)$.

The right hand side is almost by definition $H^\ast(X_{\text{qfh}}, \mathbb{Z}/n)$. The comparison with the etale cohomology of $X$ is achieved by realizing explicitly sufficiently fine qfh coverings together with “resolution of singularities” (in characteristic $p > 0$, one uses de Jong’s modifications rather than resolutions which are not known to exist).

3.3. Discussion of Chow groups. Recall that $X$ is said to be integral if $O_X(U)$ is an integral domain for all open subsets $U \subset X$. The field of fractions $K$ of such an integral variety is the field of fractions of $O_X(U)$ for any affine open subset $U$. If $O_X(U)$ is integrally closed in $K$ for every affine open subset $U$, then the stalk $O_{X, x}$ at any (scheme-theoretic point) $x \in X$ of codimension 1 is a discrete valuation ring.

**Definition 3.4.** Let $X$ be an integral variety regular in codimension 1 and let $K$ be its field of fractions. For any $0 \neq f \in K$, we define the principal divisor $(f)$ associated to $f$ to be the following formal sum of codimension 1, irreducible subvarieties

$$(f) = \sum_{x \in X^{(1)}} v_x(f) [x].$$

Here, $X^{(1)} \subset X$ consists of the scheme-theoretic points of codimension 1, $v_x : K^* \rightarrow \mathbb{Z}$ is the discrete valuation at $x \in X^{(1)}$, and $[x] \subset X$ is the codimension 1 irreducible subvariety of $X$ given as the closure of $x$.

A formal sum

$$D = \sum_{x \in X^{(1)}} n_x [x], \quad n_x \in \mathbb{Z}$$
with all but finitely many $n_x$ equal to 0 is said to be a locally principal divisor provided that for every $x \in X^{(1)}$ there exists some Zariski open neighborhood $x \in U_x \subset X$ and some $f_x \in K$ such that $D_{|U_x} = (f_x)|_{U_x}$.

**Definition 3.5.** Let $X$ be a quasi-projective algebraic variety. An algebraic $r$-cycle on $X$ if a formal sum

$$
\sum_Y n_Y[Y], \quad Y \text{ irreducible of dimension } r, \quad n_Y \in \mathbb{Z}
$$

with all but finitely many $n_Y$ equal to 0.

Equivalently, an algebraic $r$-cycle is a finite integer combination of points of $X$ of dimension $r$.

If $Y \subset X$ is a subvariety each of whose irreducible components $Y_1, \ldots, Y_m$ is $r$-dimensional, then the algebraic $r$-cycle

$$
Z = \sum_{i=1}^m [Y_i]
$$

is called the cycle associated to $Y$.

The group of (algebraic) $r$-cycles on $X$ will be denoted $Z^r(X)$.

Two $r$-cycles $Z, Z'$ on a quasi-projective variety $X$ if their difference lies in the subgroup $Z_{r, \text{rat}}(X) \subset Z^r(X)$ generated by cycles of the form $W_{|X \times \{p\}} - W_{|X \times \{q\}}$, where $U \subset \mathbb{P}^1$ is a Zariski open set containing points $p, q \in U$ and $W \subset X \times U$ is a cycle each of whose irreducible components maps surjectively onto $U$.

The Chow group $CH_r(X) = Z^r(X)/Z_{r, \text{rat}}(X)$ is the group of $r$-cycles modulo rational equivalence.

**Theorem 3.6.** (cf. [1]) Assume that $X$ is an integral variety regular in codimension 1. Let $\mathcal{D}(X)$ denote the group of locally principal divisors on $X$ modulo principal divisors. Then there is a natural isomorphism

$$
\text{Pic}(X) \sim \sim \mathcal{D}(X).
$$

If $\mathcal{L} \in \text{Pic}(X)$ has a non-zero global section $s \in \mathcal{L}(X) = \Gamma(X, \mathcal{L})$, then this isomorphism sends $\mathcal{L}$ to $\sum_{x \in X^{(1)}(\mathbb{Q})} e_x(s)[x]$.

Moreover, if $\mathcal{O}_{X,x}$ is a unique factorization domain for every $x \in X$, then

$$
\mathcal{D}(X) \sim CH^1(X),
$$

the Chow group of codimension 1 cycles on $X$ modulo rational equivalence.

**Remark 3.7.** Not only is this an example of relating bundles to cycles, but it is also an example of duality. Namely, $\text{Pic}(X)$ is contravariant, whereas $CH_r(\mathcal{L})$ is covariant for proper maps. This suggests that for $\text{Pic}(X)$ to be isomorphic to $CH^1(X)$, some smoothness condition on $X$ is required.

Observe that in the above definition we can replace the role of $r + 1$-cycles on $X \times \mathbb{P}^1$ and their geometric fibres over $0, \infty$ by $r + 1$-cycles on $X \times U$ for any non-empty Zariski open $U \subset X$ and geometric fibres over any two $k$-rational points $p, q \in U$.

**Remark 3.8.** Given some $r + 1$ dimensional irreducible subvariety $V \subset X$ together with some $f \in k(V)$, we may define $(f) = \sum_S \text{ord}_S(f)[S]$ where $S$ runs through the codimension 1 irreducible subvarieties of $V$. Here, $\text{ord}_S(\mathcal{L})$ is the valuation
centered on $S$ if $V$ is regular at the codimension 1 point corresponding to $S$; more generally, $\text{ord}_S(f)$ is defined to be the length of the $O_{V,S}$-module $O_{V,S}/(f)$.

We readily check that $(f)$ is rationally equivalent to $0$: namely, we associate to $(V,f)$ the closure $W = \Gamma_f \subset X \times P^1$ of the graph of the rational map $V \dasharrow P^1$ determined by $f$. Then $(f) = W_{|X \times \{0\}} - W_{X \times \{\infty\}}$.

Conversely, given an $r+1$-dimensional irreducible subvariety $W$ on $X \times P^1$ which maps onto $P^1$, the composition $W \subset X \times P^1 \dasharrow P^1$ determines $f \in \text{frac}(W)$ such that

$$(f) = W_{|X \times \{0\}} - W_{X \times \{\infty\}}.$$ 

Thus, the definition of rational equivalence on $r$-cycles of $X$ can be given in terms of the equivalence relation generated by

$$\{(f), f \in \text{frac}(W) ; W \text{ irreducible of dimension } r+1\}$$

In particular, we conclude that the subgroup of principal divisors inside the group of all locally principal divisors consists precisely of those locally principal divisors which are rationally equivalent to $0$.

**Example 3.9.** For essentially formal reasons, $\text{Pic}(X) \simeq H^1(X, \mathcal{O}_X^*)$. If $k$ is the complex field, we can use the exponential sequence of sheaves in the analytic topology

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O}_X \xrightarrow{exp} \mathcal{O}_X^* \rightarrow 0$$

to conclude that the kernel of $H^1(X^{an}, \mathcal{O}_X^*) \rightarrow H^2(X^{an}, \mathbb{Z})$ is the complex vector space $H^1(X^{an}, \mathcal{O}_X)$ modulo the discrete subgroup $\mathcal{H}^1(X^{an}, \mathbb{Z})$. For example, if $X = C$ is a smooth, projective curve of genus $g$, then $\mathcal{H}^1(C)$ fits in a short exact sequence

$$0 \rightarrow \mathbb{C}^g/\mathbb{Z}^{2g} = H^1(C, \mathcal{O}_C^*)/H^1(C^{an}, \mathbb{Z}) \rightarrow \mathcal{H}^1(C) \rightarrow \mathbb{Z} = H^2(C^{an}, \mathbb{Z}) \rightarrow 0.$$ 

**Example 3.10.** Let $X = \mathbb{A}^N$. Then any $N-1$-cycle (i.e., Weil divisor) $Z \in \mathcal{H}_{N-1}(\mathbb{A}^N)$ is principal, so that $\mathcal{H}_{N-1}(\mathbb{A}^N) = 0$.

More generally, consider the map $\mu : \mathbb{A}^N \times \mathbb{A}^1 \rightarrow \mathbb{P}^N \times \mathbb{A}^1$ which sends $(x_1, \ldots, x_n), t$ to $(t \cdot x_1, \ldots, t \cdot x_n, 1), t$. Consider an irreducible subvariety $Z \subset \mathbb{A}^N$ of dimension $r > N$ not containing the origin and $\mathbb{Z} \subset \mathbb{P}^N$ be its closure. Let $W = \mu^{-1}(\mathbb{Z} \times \mathbb{A}^1)$. Then $W[0] = \emptyset$ whereas $W[1] = Z$. Thus, $\mathcal{H}_r(\mathbb{A}^N) = 0$ for any $r < N$.

**Example 3.11.** Arguing in a similar geometric fashion, we see that the inclusion of a linear plane $P^{N-1} \subset \mathbb{P}^N$ induces an isomorphism $\mathcal{H}_r(P^{N-1}) = \mathcal{H}_r(\mathbb{P}^N)$ provided that $r < N$ and thus we conclude by induction that $\mathcal{H}_r(\mathbb{P}^N) = \mathbb{Z}$ if $r \leq N$. Namely, consider $\mu : \mathbb{P}^N \times \mathbb{A}^1 \rightarrow \mathbb{P}^N \times \mathbb{A}^1$ sending $(x_0, \ldots, x_N), t$ to $(x_0, \ldots, x_{N-1}, t \cdot x_N), t$ and set $W = \mu^{-1}(\mathbb{Z} \times \mathbb{A}^1)$ for any $Z$ not containing $(0, \ldots, 0, 1)$. Then $W[0] = pr_{N*}(Z), W[1] = Z$.

**Example 3.12.** Let $C$ be a smooth curve. Then $\text{Pic}(C) \simeq \mathcal{H}_0(X)$.

**Definition 3.13.** If $f : X \rightarrow Y$ is a proper map of quasi-projective varieties, then the proper push-forward of cycles determines a well defined homomorphism

$$f_* : \mathcal{H}_r(X) \rightarrow \mathcal{H}_r(Y), \quad r \geq 0.$$ 

Namely, if $Z \subset X$ is an irreducible subvariety of $X$ of dimension $r$, then $[Z]$ is sent to $d : [f(Z)] \subset \mathcal{H}_r(Y)$ where $[k(Z) : k(f(Z))] = d$ if $\text{dim} Z = \text{dim} f(Z)$ and is sent to $0$ otherwise.
If \( g : W \to X \) is a flat map of quasi-projective varieties of relative dimension \( e \), then the flat pull-back of cycles induces a well defined homomorphism

\[ g^* : CH_r(X) \to CH_{r+e}(W), \quad r \geq 0. \]

Namely, if \( Z \subset X \) is an irreducible subvariety of \( X \) of dimension \( r \), then \([Z]\) is sent to the cycle on \( W \) associated to \( Z \times_X W \subset W \).

**Proposition 3.14.** Let \( Y \) be a closed subvariety of \( X \) and let \( U = X \setminus Y \). Let \( i : Y \to X, j : U \to X \) be the inclusions. Then the sequence

\[ CH_r(Y) \xrightarrow{i_*} CH_r(X) \xrightarrow{j^*} CH_r(U) \to 0 \]

is exact for any \( r \geq 0 \).

*Proof.* If \( V \subset U \) is an irreducible subvariety of \( U \) of dimension \( r \), then the closure of \( V \) in \( X \), \( \overline{V} \subset X \), is an irreducible subvariety of \( X \) of dimension \( r \) with the property that \( j^*(\overline{V}) = [V] \). Thus, we have an exact sequence

\[ Z_r(Y) \xrightarrow{i_*} Z_r(X) \xrightarrow{j^*} Z_r(U) \to 0. \]

If \( Z = \sum n_i [Y_i] \) is a cycle on \( X \) with \( j^*(Z) = 0 \in CH_r(U) \), then \( j^*Z = \sum W, f (f) \) where each \( W \subset U \) is an irreducible subvarieties of \( U \) of dimension \( r + 1 \) and \( f \in k(W) \). Thus, \( Z' = \sum n_i [\overline{Y}_i] - \sum \pi f (f) \) is an \( r \)-cycle on \( Y \) with the property that \( i_*(Z') \) is rationally equivalent to \( Z \). Exactness of the asserted sequence of Chow groups is now clear.

\[ \square \]

**Corollary 3.15.** Let \( H \subset \mathbb{P}^N \) be a hypersurface of degree \( d \). Then \( CH_{N-1}(\mathbb{P}^N \setminus H) = \mathbb{Z}/d\mathbb{Z} \).

**Example 3.16.** Mumford shows that if \( S \) is a projective smooth surface with a non-zero global algebraic 2-form (i.e., \( H^0(S, \Lambda^2(\Omega_S)) \neq 0 \)), then \( CH_0(S) \) is not finite dimensional (i.e., must be very large).

Bloch’s Conjecture predicts that if \( S \) is a projective, smooth surface with geometric genus equal to 0 (i.e., \( H^0(S, \Lambda^2(\Omega_S)) = 0 \)), then the natural map from \( CH_0(S) \) to the (finite dimensional) Albanese variety is injective.

### 3.4 Intersection product

**Theorem 3.17.** Let \( X \) be a smooth quasi-projective variety of dimension \( d \). Then there exists a pairing

\[ CH_r(X) \otimes CH_s(X) \xrightarrow{\cdot} CH_{d-r-s}, \quad d \geq r + s, \]

with the property that if \( Z = [Y], Z' = [W] \) are irreducible cycles of dimension \( r, s \) respectively and if \( Y \cap W \) has dimension \( \leq d - r - s \), then \( Z \bullet Z' \) is a cycle which is a sum with positive coefficients indexed by the irreducible subvarieties of \( Y \cap W \) of dimension \( d - r - s \).

For notational purposes, we shall often write \( CH^*(X) \) for \( CH_{d-s}(X) \). With this indexing convention, the intersection pairing has the form

\[ CH^*(X) \otimes CH^*(X) \xrightarrow{\cdot} CH^{*+t}(X). \]
Proof. Classically, this was proved by showing the following geometric fact: given a codimension \( r \) cycle \( Z \) and a codimension \( s \) cycle \( W = \sum_j m_j R_j \) with \( r + s \leq d \), then there is another codimension \( r \) cycle \( Z' = \sum_i n_i Y_i \) rationally equivalent to \( Z \) (i.e., determining the same element in \( CH^r(X) \)) such that \( Z' \) meets \( W \) “properly”; in other words, every component \( C_{i,j,k} \) of each \( Y_i \cap R_j \) has codimension \( r + s \). One then defines

\[
Z' \cdot W = \sum_{i,j,k} n_i m_j \cdot \text{int}(Y_i \cap R_j, C_{i,j,k}) C_{i,j,k}
\]

where \( \text{int}(Y_i \cap R_j, C_{i,j,k}) \) is a positive integer determined using local commutative algebra, the intersection multiplicity. Furthermore, one shows that if one chooses a \( Z'' \) rationally equivalent to both \( Z, Z' \) and also intersecting \( W \) properly, then \( Z' \cdot W \) is rationally equivalent to \( Z'' \cdot W \).

To be continued next lecture . . .

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References


Department of Mathematics, Northwestern University, Evanston, IL 60208
E-mail address: eric@math.northwestern.edu